UNDER PEER REVIEW

Original Research Article

Generalised Saxena and Gupta Integral Transform

Abstract

In this paper, we introduce a new integral transform called the Generalised Saxena and Gupta Transform which is basically a generalisation of Saxena & Gupta Transform. We also investigate some important properties of this transform. Moreover, a comparison of the Generalised Saxena and Gupta transform with other integral transforms, such as the Laplace and Elzaki transforms, for some elementary functions is presented in tabular form.

Keywords: Saxena and Gupta Transform; Elzaki Transform; Laplace Transform

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1 Introduction

Integral transformations have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics, and engineering science. Historically, the origin of the integral transforms, including the Laplace and Fourier transforms(Debnath et al.,2006 ; Debnath et al.,2014) can be traced back to the celebrated work of P. S. Laplace (1749–1827) on probability theory in the 1780s and to the monumental treatise of Joseph Fourier (1768–1830) on La Théorie Analytique de la Chaleur published in 1822. There are many other integral transformations, including the Elzaki transform, the Aboodh transform, the Hilbert transform, and the Stieltjes transform, and the sumudu transform which are widely used to solve initial- and boundary-value problems involving ordinary and partial differential equations, as well as other problems in mathematics, science, and engineering(Gupta et al.,2025; Belgacem et al., 2006; Mohammed et al.,2017; Aboodh et al.,2013). A new integral transform, called the Saxena & Gupta transform (Saxena , 2023) is used to solve a system of ordinary differential equations. It provides a powerful and systematic method for solving

ordinary differential equations by transforming them into simpler algebraic equations, making them a valuable tool in various scientific and engineering applications. This method is widely used in various fields like engineering, physics, and applied mathematics. Here, we introduce a new integral transform that extends the applicability of the Saxena and Gupta transform.

2 Main Results

Definition 2.1. The generalised Saxena and Gupta transform for a piecewise continuous function $f(t)$ on $[0,\infty)$ is defined by

$$Z_{\alpha,\beta}\{f(t)\} = \frac{1}{\nu^{\alpha}}\int_0^{\infty} f(\nu^{\beta}t)e^{-t}\,dt, \qquad \alpha,\beta>0.$$

Special Case:- For $\alpha=\beta=1$,the generalized transform (1) reduces to

$$Z_{1,1}\{f(t)\} = \frac{1}{\nu}\int_0^{\infty} f(\nu t)e^{-t}\,dt,$$

which is exactly the classical Saxena and Gupta transform (Saxena, 2023).

Theorem 2.1 (**Linearity**). *Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[0,\infty)$ such that $z_{\alpha,\beta}\{f(t)\}$ and $z_{\alpha,\beta}\{g(t)\}$ exist, and let $a,b>0$. Then the generalised Saxena and Gupta transform satisfies*

$$Z_{\alpha,\beta}\{af(t)+bg(t)\} = aZ_{\alpha,\beta}\{f(t)\} + bZ_{\alpha,\beta}\{g(t)\}.$$

Proof. Using the definition of the generalised Saxena and Gupta transform,

$$Z_{\alpha,\beta}\{af(t)+bg(t)\} = \frac{1}{\nu^{\alpha}}\int_0^{\infty}\left[af(\nu^{\beta}t)+bg(\nu^{\beta}t)\right]e^{-t}\,dt.$$

By the linearity of integration,

$$Z_{\alpha,\beta}\{af(t)+bg(t)\} = a\frac{1}{\nu^{\alpha}}\int_0^{\infty} f(\nu^{\beta}t)e^{-t}\,dt \quad + b\frac{1}{\nu^{\alpha}}\int_0^{\infty} g(\nu^{\beta}t)e^{-t}\,dt.$$

Identifying each integral as the generalised Saxena and Gupta transform,

$$Z_{\alpha,\beta}\{af(t)+bg(t)\} = aZ_{\alpha,\beta}\{f(t)\} + bZ_{\alpha,\beta}\{g(t)\}.$$

Hence, the linearity property is proved.

Theorem 2.2 (**Exponential Multiplication Property**). *Let*

$$Z_{\alpha,\beta}[f(t)] = \frac{1}{\nu^{\alpha}}\int_0^{\infty} f(\nu^{\beta}t)e^{-t}\,dt$$

Then

$$Z_{\alpha,\beta}[e^{at}f(t)] = \frac{1}{\nu^{\alpha}}\int_0^{\infty} f(\nu^{\beta}t)e^{-(1-a\nu^{\beta})t}\,dt,$$

provided that

$$1-a\nu^{\beta}>0.$$

Proof. From the definition of the generalised Saxena and Gupta transform,

$$Z_{\alpha,\beta}[e^{at}f(t)] = \frac{1}{\nu^{\alpha}}\int_0^{\infty} e^{a(\nu^{\beta}t)}f(\nu^{\beta}t)e^{-t}\,dt.$$

Combining the exponential terms,

$$e^{a\nu^{\beta}t}e^{-t} = e^{-(1-a\nu^{\beta})t},$$

we obtain

$$Z_{\alpha,\beta}[e^{at}f(t)] = \frac{1}{\nu^{\alpha}}\int_0^{\infty} f(\nu^{\beta}t)e^{-(1-a\nu^{\beta})t}\,dt.$$

Now let

$$u = (1-a\nu^{\beta})t.$$

Then

$$t = \frac{u}{1-a\nu^{\beta}}, \qquad dt = \frac{du}{1-a\nu^{\beta}}.$$

Substituting these into the above integral gives

$$\begin{aligned} Z_{\alpha,\beta}[e^{at}f(t)] &= \frac{1}{\nu^{\alpha}}\int_0^{\infty} f\left(\frac{\nu^{\beta}}{1-a\nu^{\beta}}u\right)e^{-u}\frac{du}{1-a\nu^{\beta}} \\ &= \frac{1}{\nu^{\alpha}(1-a\nu^{\beta})}\int_0^{\infty} f\left(\frac{\nu^{\beta}}{1-a\nu^{\beta}}u\right)e^{-u}\,du. \end{aligned}$$

Hence,

$$Z_{\alpha,\beta}[e^{at}f(t)] = \frac{1}{\nu^{\alpha}(1-a\nu^{\beta})}\int_0^{\infty} f\left(\frac{\nu^{\beta}}{1-a\nu^{\beta}}u\right)e^{-u}\,du.$$

If

$$Z_{\alpha,\beta}[f(t)] = F(\nu)$$

then

$$Z_{\alpha,\beta}\left[e^{at}f(t)\right] = \frac{1}{1-a\nu^{\beta}}\,F\left(\frac{\nu}{(1-a\nu^{\beta})^{1/\beta}}\right)$$

Theorem 2.3 (**Scaling Property**). *Let*

$$Z_{\alpha,\beta}\{f(t)\} = F(\nu),$$

where the generalised Saxena and Gupta transform is defined by

$$Z_{\alpha,\beta}\{f(t)\} = \frac{1}{\nu^{\alpha}}\int_0^{\infty} f(\nu^{\beta}t)e^{-t}\,dt.$$

Then, for any constant $a > 0$*,*

$$Z_{\alpha,\beta}\{f(at)\} = a^{\frac{\alpha}{\beta}}F\left(a^{\frac{1}{\beta}}\nu\right).$$

Proof. From the definition of the generalised Saxena & Gupta transform,

$$Z_{\alpha,\beta}\{f(at)\} = \frac{1}{\nu^{\alpha}} \int_0^{\infty} f(a\nu^{\beta}t)e^{-t}\,dt.$$

Now,

$$F\left(a^{\frac{1}{\beta}}\nu\right) = \frac{1}{\left(a^{\frac{1}{\beta}}\nu\right)^{\alpha}} \int_0^{\infty} f(a\nu^{\beta}t)e^{-t}\,dt.$$

Hence,

$$\int_0^{\infty} f(a\nu^{\beta}t)e^{-t}\,dt = \left(a^{\frac{1}{\beta}}\nu\right)^{\alpha} F\left(a^{\frac{1}{\beta}}\nu\right).$$

Substituting this into the transform gives

$$\begin{aligned} Z_{\alpha,\beta}\{f(at)\} &= \frac{1}{\nu^{\alpha}} \left(a^{\frac{1}{\beta}}\nu\right)^{\alpha} F\left(a^{\frac{1}{\beta}}\nu\right) \\ &= a^{\frac{\alpha}{\beta}} F\left(a^{\frac{1}{\beta}}\nu\right). \end{aligned}$$

Therefore,

$$Z_{\alpha,\beta}\{f(at)\} = a^{\frac{\alpha}{\beta}} F\left(a^{\frac{1}{\beta}}\nu\right).$$

Special Case: For $\alpha = \beta = 1$,

$$Z\{f(at)\} = aF(a\nu),$$

which is the scaling property of the Saxena & Gupta transform (Saxena, 2023).

Theorem 2.4 (**Convolution Property**). *Let*

$$Z_{\alpha,\beta}\{f(t)\} = F(\nu), \qquad Z_{\alpha,\beta}\{g(t)\} = G(\nu),$$

where

$$Z_{\alpha,\beta}\{f(t)\} = \frac{1}{\nu^{\alpha}} \int_0^{\infty} f(\nu^{\beta}t)e^{-t}\,dt.$$

If the convolution of f and g is defined by

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du,$$

then

$$Z_{\alpha,\beta}\{(f * g)(t)\} = \nu^{\alpha+\beta} F(\nu)G(\nu).$$

Proof. From the definition of the generalised Saxena and Gupta transform,

$$Z_{\alpha,\beta}\{(f * g)(t)\} = \frac{1}{\nu^{\alpha}} \int_0^{\infty} (f * g)(\nu^{\beta}t)e^{-t}\,dt.$$

Using the definition of convolution,

$$= \frac{1}{\nu^{\alpha}} \int_0^{\infty} \left[\int_0^{\nu^{\beta}t} f(x)\, g(\nu^{\beta}t - x)\,dx \right] e^{-t}\,dt.$$

Let

$$x = \nu^{\beta}u, \qquad dx = \nu^{\beta}du.$$

Then

$$(f*g)(\nu^\beta t) = \nu^\beta \int_0^t f(\nu^\beta u) g\Big(\nu^\beta (t-u)\Big)\, du.$$

Hence,

$$Z_{\alpha,\beta}\{(f*g)(t)\} = \frac{\nu^\beta}{\nu^\alpha} \int_0^\infty e^{-t} \int_0^t f(\nu^\beta u) g(\nu^\beta (t-u))\, du\, dt.$$

Interchanging the order of integration and letting

$$v = t - u,$$

we obtain

$$Z_{\alpha,\beta}\{(f*g)(t)\} = \frac{\nu^\beta}{\nu^\alpha} \left(\int_0^\infty f(\nu^\beta u) e^{-u}\, du\right) \left(\int_0^\infty g(\nu^\beta v) e^{-v}\, dv\right).$$

Since

$$F(\nu) = \frac{1}{\nu^\alpha} \int_0^\infty f(\nu^\beta u) e^{-u}\, du,$$

and

$$G(\nu) = \frac{1}{\nu^\alpha} \int_0^\infty g(\nu^\beta v) e^{-v}\, dv,$$

we have

$$\int_0^\infty f(\nu^\beta u) e^{-u}\, du = \nu^\alpha F(\nu),$$

and

$$\int_0^\infty g(\nu^\beta v) e^{-v}\, dv = \nu^\alpha G(\nu).$$

Therefore,

$$\begin{aligned} Z_{\alpha,\beta}\{(f*g)(t)\} &= \frac{\nu^\beta}{\nu^\alpha} \left(\nu^\alpha F(\nu)\right) \left(\nu^\alpha G(\nu)\right) \\ &= \nu^{\alpha+\beta} F(\nu) G(\nu). \end{aligned}$$

Hence,

$$Z_{\alpha,\beta}\{(f*g)(t)\} = \nu^{\alpha+\beta} F(\nu) G(\nu).$$

Special Case: For $\alpha = \beta = 1$,

$$Z\{(f*g)(t)\} = \nu^2 F(\nu) G(\nu).$$

Theorem 2.5 (**Derivative Property**). *Let $f(t)$ be piecewise continuously differentiable on $[0,\infty]$ and of exponential order a, with $a < \frac{1}{\nu^\beta}$. Assume further that*

$$Z_{\alpha,\beta}\{f(t)\} = F(\nu),$$

where

$$Z_{\alpha,\beta}\{f(t)\} = \frac{1}{\nu^\alpha} \int_0^\infty f(\nu^\beta t) e^{-t}\, dt.$$

Then,

$$Z_{\alpha,\beta}\{f'(t)\} = \frac{F(\nu)}{\nu^\beta} - \frac{f(0)}{\nu^{\alpha+\beta}}.$$

Proof. From the definition of the generalised Saxena and Gupta transform,

$$Z_{\alpha,\beta}\{f'(t)\} = \frac{1}{\nu^\alpha} \int_0^\infty f'(\nu^\beta t) e^{-t}\, dt.$$

Let

$$x = \nu^\beta t, \qquad dt = \frac{dx}{\nu^\beta}.$$

Then,

$$Z_{\alpha,\beta}\{f'(t)\} = \frac{1}{\nu^{\alpha+\beta}} \int_0^\infty f'(x) e^{-x/\nu^\beta}\, dx.$$

Integrating by parts,

$$\int_0^\infty f'(x) e^{-x/\nu^\beta}\, dx = \left[f(x) e^{-x/\nu^\beta} \right]_0^\infty + \frac{1}{\nu^\beta} \int_0^\infty f(x) e^{-x/\nu^\beta}\, dx.$$

Suppose that

$$\lim_{x\to\infty} f(x) e^{-x/\nu^\beta} = 0,$$

we obtain

$$\int_0^\infty f'(x) e^{-x/\nu^\beta}\, dx = -f(0) + \frac{1}{\nu^\beta} \int_0^\infty f(x) e^{-x/\nu^\beta}\, dx.$$

Since

$$F(\nu) = \frac{1}{\nu^{\alpha+\beta}} \int_0^\infty f(x) e^{-x/\nu^\beta}\, dx,$$

it follows that

$$\int_0^\infty f(x) e^{-x/\nu^\beta}\, dx = \nu^{\alpha+\beta} F(\nu).$$

Therefore,

$$\begin{aligned} Z_{\alpha,\beta}\{f'(t)\} &= \frac{1}{\nu^{\alpha+\beta}} \left[-f(0) + \frac{\nu^{\alpha+\beta}}{\nu^\beta} F(\nu) \right] \\ &= \frac{F(\nu)}{\nu^\beta} - \frac{f(0)}{\nu^{\alpha+\beta}}. \end{aligned}$$

Hence,

$$Z_{\alpha,\beta}\{f'(t)\} = \frac{F(\nu)}{\nu^\beta} - \frac{f(0)}{\nu^{\alpha+\beta}}.$$

Special Case: For $\alpha = \beta = 1$,

$$Z\{f'(t)\} = \frac{F(\nu)}{\nu} - \frac{f(0)}{\nu^2},$$

which is the derivative property of the classical Saxena and Gupta transform (Saxena, 2023).

Theorem 2.6 (**Second Derivative Property**). *Let*

$$Z_{\alpha,\beta}\{f(t)\} = F(\nu),$$

where

$$Z_{\alpha,\beta}\{f(t)\} = \frac{1}{\nu^\alpha} \int_0^\infty f(\nu^\beta t) e^{-t}\, dt.$$

Then,

$$Z_{\alpha,\beta}\{f''(t)\} = \frac{F(\nu)}{\nu^{2\beta}} - \frac{f(0)}{\nu^{\alpha+2\beta}} - \frac{f'(0)}{\nu^{\alpha+\beta}}.$$

Proof. Using the first derivative property,

$$Z_{\alpha,\beta}\{f''(t)\} = \frac{Z_{\alpha,\beta}\{f'(t)\}}{\nu^\beta} - \frac{f'(0)}{\nu^{\alpha+\beta}}.$$

Since

$$Z_{\alpha,\beta}\{f'(t)\} = \frac{F(\nu)}{\nu^{\beta}} - \frac{f(0)}{\nu^{\alpha+\beta}},$$

we obtain

$$\begin{aligned} Z_{\alpha,\beta}\{f''(t)\} &= \frac{1}{\nu^{\beta}}\left(\frac{F(\nu)}{\nu^{\beta}} - \frac{f(0)}{\nu^{\alpha+\beta}}\right) - \frac{f'(0)}{\nu^{\alpha+\beta}} \\ &= \frac{F(\nu)}{\nu^{2\beta}} - \frac{f(0)}{\nu^{\alpha+2\beta}} - \frac{f'(0)}{\nu^{\alpha+\beta}}. \end{aligned}$$

Hence,

$$Z_{\alpha,\beta}\{f''(t)\} = \frac{F(\nu)}{\nu^{2\beta}} - \frac{f(0)}{\nu^{\alpha+2\beta}} - \frac{f'(0)}{\nu^{\alpha+\beta}}.$$

Special Case: For $\alpha = \beta = 1$,

$$Z\{f''(t)\} = \frac{F(\nu)}{\nu^{2}} - \frac{f(0)}{\nu^{3}} - \frac{f'(0)}{\nu^{2}}.$$

Theorem 2.7 (**General nth Derivative Property**). *Let*

$$Z_{\alpha,\beta}\{f(t)\} = F(\nu),$$

where

$$Z_{\alpha,\beta}\{f(t)\} = \frac{1}{\nu^{\alpha}}\int_{0}^{\infty} f(\nu^{\beta}t)e^{-t}\,dt.$$

Then, for any positive integer n,

$$Z_{\alpha,\beta}\{f^{(n)}(t)\} = \frac{F(\nu)}{\nu^{n\beta}} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\nu^{\alpha+(n-k)\beta}}.$$

Proof. Using the recursive relation

$$Z_{\alpha,\beta}\{f^{(n)}(t)\} = \frac{1}{\nu^{\beta}}Z_{\alpha,\beta}\{f^{(n-1)}(t)\} - \frac{f^{(n-1)}(0)}{\nu^{\alpha+\beta}},$$

and applying it repeatedly, we obtain

$$\begin{aligned} Z_{\alpha,\beta}\{f^{(n)}(t)\} &= \frac{F(\nu)}{\nu^{n\beta}} - \frac{f(0)}{\nu^{\alpha+n\beta}} - \frac{f'(0)}{\nu^{\alpha+(n-1)\beta}} - \cdots - \frac{f^{(n-1)}(0)}{\nu^{\alpha+\beta}} \\ &= \frac{F(\nu)}{\nu^{n\beta}} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\nu^{\alpha+(n-k)\beta}}. \end{aligned}$$

Hence,

$$Z_{\alpha,\beta}\{f^{(n)}(t)\} = \frac{F(\nu)}{\nu^{n\beta}} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\nu^{\alpha+(n-k)\beta}}.$$

Special Case: For $\alpha = \beta = 1$,

$$Z\{f^{(n)}(t)\} = \frac{F(\nu)}{\nu^{n}} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\nu^{\,n-k+1}}.$$

Theorem 2.8 (**Inverse Transform**). *Let*

$$Z_{\alpha,\beta}\{f(t)\} = F(\nu),$$

where the generalised Saxena and Gupta transform is defined by

$$Z_{\alpha,\beta}\{f(t)\} = \frac{1}{\nu^{\alpha}} \int_0^{\infty} f(\nu^{\beta} t) e^{-t}\, dt, \qquad \alpha, \beta > 0.$$

Then the inverse transform is given by

$$f(t) = \mathcal{L}^{-1}\left\{ s^{-\frac{\alpha+\beta}{\beta}} F\left(s^{-1/\beta}\right)\right\},$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

Proof. From the definition of the generalised Saxena and Gupta transform,

$$F(\nu) = \frac{1}{\nu^{\alpha}} \int_0^{\infty} f(\nu^{\beta} t) e^{-t}\, dt.$$

Let

$$x = \nu^{\beta} t, \qquad dt = \frac{dx}{\nu^{\beta}}.$$

Then

$$F(\nu) = \frac{1}{\nu^{\alpha+\beta}} \int_0^{\infty} f(x) e^{-x/\nu^{\beta}}\, dx.$$

Identifying the integral as the Laplace transform of $f(t)$,

$$\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} f(t) e^{-st}\, dt,$$

and letting

$$s = \frac{1}{\nu^{\beta}},$$

we obtain

$$F(\nu) = \frac{1}{\nu^{\alpha+\beta}} \mathcal{L}\{f(t)\}\left(\frac{1}{\nu^{\beta}}\right).$$

Since

$$\nu = s^{-1/\beta},$$

it follows that

$$F\left(s^{-1/\beta}\right) = s^{\frac{\alpha+\beta}{\beta}} \mathcal{L}\{f(t)\}(s).$$

Hence,

$$\mathcal{L}\{f(t)\}(s) = s^{-\frac{\alpha+\beta}{\beta}} F\left(s^{-1/\beta}\right).$$

Applying the inverse Laplace transform to both sides yields

$$f(t) = \mathcal{L}^{-1}\left\{ s^{-\frac{\alpha+\beta}{\beta}} F\left(s^{-1/\beta}\right)\right\}.$$

Special Case. For $\alpha = \beta = 1$,

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{F(1/s)}{s^2} \right\},$$

which is the inversion formula corresponding to the classical Saxena and Gupta transform.

3 Transforms of Some Elementary Functions

1. When $f(t) = 1$. Then,

$$Z_{\alpha,\beta}\{1\} = \frac{1}{\nu^\alpha}\int_0^\infty e^{-t}\,dt$$
$$= \frac{1}{\nu^\alpha}.$$

Hence,

$$Z_{\alpha,\beta}\{1\} = \frac{1}{\nu^\alpha}.$$

2. When

$$f(t) = t^n, \qquad n > -1.$$

Then,

$$Z_{\alpha,\beta}\{t^n\} = \frac{1}{\nu^\alpha}\int_0^\infty (\nu^\beta t)^n e^{-t}\,dt$$
$$= \nu^{\,n\beta-\alpha}\int_0^\infty t^n e^{-t}\,dt.$$

Using

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\,dt,$$

we obtain

$$Z_{\alpha,\beta}\{t^n\} = \nu^{\,n\beta-\alpha}\Gamma(n+1).$$

3. Let

$$f(t) = e^{at}.$$

Then,

$$Z_{\alpha,\beta}\{e^{at}\} = \frac{1}{\nu^\alpha}\int_0^\infty e^{a\nu^\beta t}e^{-t}\,dt$$
$$= \frac{1}{\nu^\alpha}\int_0^\infty e^{-(1-a\nu^\beta)t}\,dt.$$

Hence,

$$Z_{\alpha,\beta}\{e^{at}\} = \frac{1}{\nu^\alpha(1-a\nu^\beta)}, \qquad a\nu^\beta < 1.$$

4. Let

$$f(t) = \sin(at).$$

Then

$$Z_{\alpha,\beta}\{\sin(at)\} = \frac{1}{\nu^\alpha}\int_0^\infty \sin(a\nu^\beta t)e^{-t}\,dt$$

Since, we know that

$$\int_0^\infty e^{-t}\sin(bt)\,dt = \frac{b}{1+b^2},$$

taking

$$b = a\nu^\beta,$$

gives

$$Z_{\alpha,\beta}\{\sin(at)\} = \frac{a\nu^\beta}{\nu^\alpha\,(1+a^2\nu^{2\beta})}.$$

5. Let

$$f(t) = \cos(at).$$

Then

$$Z_{\alpha,\beta}\{\cos(at)\} = \frac{1}{\nu^{\alpha}} \int_0^{\infty} \cos(a\nu^{\beta}t) e^{-t}\, dt$$

Since, we know that

$$\int_0^{\infty} e^{-t} \cos(bt)\, dt = \frac{1}{1+b^2},$$

taking

$$b = a\nu^{\beta},$$

gives

$$Z_{\alpha,\beta}\{\cos(at)\} = \frac{1}{\nu^{\alpha}\left(1 + a^2\nu^{2\beta}\right)}.$$

6. Let

$$f(t) = \sinh(at).$$

Then

$$Z_{\alpha,\beta}\{\sinh(at)\} = \frac{1}{\nu^{\alpha}} \int_0^{\infty} \sinh(a\nu^{\beta}t) e^{-t}\, dt$$

Using the identity

$$\sinh x = \frac{e^{x} - e^{-x}}{2},$$

we have

$$\begin{aligned} Z_{\alpha,\beta}\{\sinh(at)\} &= \frac{1}{2\nu^{\alpha}} \int_0^{\infty} \left(e^{a\nu^{\beta}t} - e^{-a\nu^{\beta}t}\right) e^{-t}\, dt \\ &= \frac{1}{2\nu^{\alpha}} \left[\int_0^{\infty} e^{-(1-a\nu^{\beta})t}\, dt - \int_0^{\infty} e^{-(1+a\nu^{\beta})t}\, dt\right]. \end{aligned}$$

Using

$$\int_0^{\infty} e^{-ct}\, dt = \frac{1}{c}, \qquad c > 0,$$

we obtain

$$\begin{aligned} Z_{\alpha,\beta}\{\sinh(at)\} &= \frac{1}{2\nu^{\alpha}} \left[\frac{1}{1-a\nu^{\beta}} - \frac{1}{1+a\nu^{\beta}}\right] \\ &= \frac{1}{2\nu^{\alpha}} \left[\frac{(1+a\nu^{\beta}) - (1-a\nu^{\beta})}{(1-a\nu^{\beta})(1+a\nu^{\beta})}\right] \\ &= \frac{1}{2\nu^{\alpha}} \left[\frac{2a\nu^{\beta}}{1-a^2\nu^{2\beta}}\right] \\ &= \frac{a\nu^{\beta}}{\nu^{\alpha}\left(1-a^2\nu^{2\beta}\right)}. \end{aligned}$$

Hence,

$$Z_{\alpha,\beta}\{\sinh(at)\} = \frac{a\nu^{\beta}}{\nu^{\alpha}\left(1-a^2\nu^{2\beta}\right)}, \qquad a\nu^{\beta} < 1.$$

7. Let

$$f(t) = \cosh(at).$$

Then

$$Z_{\alpha,\beta}\{\cosh(at)\} = \frac{1}{\nu^{\alpha}} \int_0^{\infty} \cosh(a\nu^{\beta}t) e^{-t}\, dt$$

Using the identity

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

we have

$$\begin{aligned} Z_{\alpha,\beta}\{\cosh(at)\} &= \frac{1}{2\nu^\alpha}\int_0^\infty \left(e^{a\nu^\beta t} + e^{-a\nu^\beta t}\right) e^{-t}\, dt \\ &= \frac{1}{2\nu^\alpha}\left[\int_0^\infty e^{-(1-a\nu^\beta)t}\, dt + \int_0^\infty e^{-(1+a\nu^\beta)t}\, dt\right]. \end{aligned}$$

Using

$$\int_0^\infty e^{-ct}\, dt = \frac{1}{c}, \qquad c > 0,$$

we obtain

$$\begin{aligned} Z_{\alpha,\beta}\{\cosh(at)\} &= \frac{1}{2\nu^\alpha}\left[\frac{1}{1-a\nu^\beta} + \frac{1}{1+a\nu^\beta}\right] \\ &= \frac{1}{2\nu^\alpha}\left[\frac{(1+a\nu^\beta)+(1-a\nu^\beta)}{(1-a\nu^\beta)(1+a\nu^\beta)}\right] \\ &= \frac{1}{2\nu^\alpha}\left[\frac{2}{1-a^2\nu^{2\beta}}\right] \\ &= \frac{1}{\nu^\alpha\,(1-a^2\nu^{2\beta})}. \end{aligned}$$

Hence,

$$Z_{\alpha,\beta}\{\cosh(at)\} = \frac{1}{\nu^\alpha\,(1-a^2\nu^{2\beta})}, \qquad a\nu^\beta < 1.$$

4 Tables

Table 1: Generalised Saxena and Gupta transforms of some elementary functions.

$f(t)$	$Z_{\alpha,\beta}\{f(t)\}$
1	$\dfrac{1}{\nu^\alpha}$
t^n	$\nu^{\,n\beta-\alpha}\Gamma(n+1)$
e^{at}	$\dfrac{1}{\nu^\alpha(1-a\nu^\beta)}, \quad a\nu^\beta < 1$
$\sin(at)$	$\dfrac{a\nu^\beta}{\nu^\alpha(1+a^2\nu^{2\beta})}$
$\cos(at)$	$\dfrac{1}{\nu^\alpha(1+a^2\nu^{2\beta})}$
$\sinh(at)$	$\dfrac{a\nu^\beta}{\nu^\alpha(1-a^2\nu^{2\beta})}, \quad \lvert a\rvert\nu^\beta < 1$
$\cosh(at)$	$\dfrac{1}{\nu^\alpha(1-a^2\nu^{2\beta})}, \quad \lvert a\rvert\nu^\beta < 1$

Table 2: Generalised Saxena and Gupta transforms of some elementary functions.

$f(t)$	$Z_{\alpha,\beta}\{f(t)\}$
1	$\frac{1}{\nu^{\alpha}}$
$t^n, \quad n>-1$	$\nu^{n\beta-\alpha}\Gamma(n+1)$
e^{at}	$\frac{1}{\nu^{\alpha}(1-a\nu^{\beta})}, \quad a\nu^{\beta}<1$
$\sin(at)$	$\frac{a\nu^{\beta}}{\nu^{\alpha}(1+a^2\nu^{2\beta})}$
$\cos(at)$	$\frac{1}{\nu^{\alpha}(1+a^2\nu^{2\beta})}$
$\sinh(at)$	$\frac{a\nu^{\beta}}{\nu^{\alpha}(1-a^2\nu^{2\beta})}, \quad \lvert a\rvert\nu^{\beta}<1$
$\cosh(at)$	$\frac{1}{\nu^{\alpha}(1-a^2\nu^{2\beta})}, \quad \lvert a\rvert\nu^{\beta}<1$

Table 3: Laplace transform and the Generalised Saxena and Gupta transform of some elementary functions.

$f(t)$	$\mathcal{L}\{f(t)\}$	$Z_{\alpha,\beta}\{f(t)\}$
1	$\frac{1}{s}$	$\frac{1}{\nu^{\alpha}}$
$t^n, \ n>-1$	$\frac{\Gamma(n+1)}{s^{n+1}}$	$\nu^{n\beta-\alpha}\Gamma(n+1)$
e^{at}	$\frac{1}{s-a}$	$\frac{1}{\nu^{\alpha}(1-a\nu^{\beta})}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\frac{a\nu^{\beta}}{\nu^{\alpha}(1+a^2\nu^{2\beta})}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\frac{1}{\nu^{\alpha}(1+a^2\nu^{2\beta})}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$\frac{a\nu^{\beta}}{\nu^{\alpha}(1-a^2\nu^{2\beta})}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$\frac{1}{\nu^{\alpha}(1-a^2\nu^{2\beta})}$

Table 4: Elzaki transform and Generalised Saxena and Gupta transform of some elementary functions.

$f(t)$	$E\{f(t)\}$	$Z_{\alpha,\beta}\{f(t)\}$
1	v^2	$\frac{1}{\nu^\alpha}$
$t^n,\ n>-1$	$\Gamma(n+1)v^{n+2}$	$\Gamma(n+1)\nu^{n\beta-\alpha}$
e^{at}	$\frac{v^2}{1-av}$	$\frac{1}{\nu^\alpha(1-a\nu^\beta)}$
$\sin(at)$	$\frac{av^3}{1+a^2v^2}$	$\frac{a\nu^{\beta-\alpha}}{1+a^2\nu^{2\beta}}$
$\cos(at)$	$\frac{v^2}{1+a^2v^2}$	$\frac{\nu^{-\alpha}}{1+a^2\nu^{2\beta}}$
$\sinh(at)$	$\frac{av^3}{1-a^2v^2}$	$\frac{a\nu^{\beta-\alpha}}{1-a^2\nu^{2\beta}}$
$\cosh(at)$	$\frac{v^2}{1-a^2v^2}$	$\frac{\nu^{-\alpha}}{1-a^2\nu^{2\beta}}$

5 Conclusion

The Generalised Saxena and Gupta Transform is a generalisation of classical Saxena and Gupta Transform that broadens its applicability. It also provides an efficient and systematic method for solving nonlinear and fractional differential equations (FDEs) by transforming them into simpler algebraic equations, making it a valuable tool in various scientific and engineering applications.

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