UNDER PEER REVIEW

Original Research Article

On Characterization of Order-Divisor Graphs of Γ_1-Nonderanged Permutation Groups

Abstract

For a prime $p \geq 5$, the Γ_1-nonderanged permutation group $G_p^{\Gamma_1} = \{\omega_1, \ldots, \omega_{p-1}\}$ is a cyclic group of order $p-1$ with multiplication $\omega_i\omega_j = \omega_{ij \bmod p}$. The order–divisor graph $\mathrm{OD}(G_p^{\Gamma_1})$ has the elements of the group as vertices, two distinct vertices being adjacent if and only if their orders are different and one order divides the other. In this paper we describe the structure of this graph completely: it is obtained from the divisibility-comparability graph on the divisors of $p-1$ by replacing each divisor d by an independent set of $\varphi(d)$ vertices. From this description we derive an explicit degree formula and edge count, and prove that the graph is connected, non-complete and non-regular, with diameter 2, radius 1, girth 3, maximum degree $p-2$ and minimum degree at least 2 (equal to 2 only for $p = 5$). The center consists of the identity ω_1 together with the unique involution ω_{p-1} when $p-1$ is a power of 2 (that is, when p is a Fermat prime), and is $\{\omega_1\}$ otherwise; the periphery is the complement of the center. We also show that $\mathrm{OD}(G_p^{\Gamma_1})$ is complete multipartite if and only if $p-1 = 2^k$ or $p = 2q+1$ with q an odd prime. The results are illustrated by adjacency matrices and drawings for $p = 5, 7, 11$ and confirmed by computer for every prime $5 \leq p < 600$.

Keywords: Order–divisor graph; Γ_1-nonderanged permutation group; cyclic group; eccentricity; central vertex; complete multipartite graph; graph invariants

2020 Mathematics Subject Classification: 05C25; 05C12; 20D60

1 Introduction

Group theory is a unifying language for symmetry, with applications in number theory, combinatorics, cryptography and mathematical physics. A permutation group is a collection of permutations of a set that is closed under composition (Beaumont and Peterson, 1955); permutation groups form one of the oldest and most fundamental areas of group theory and act as a bridge between algebra and combinatorics (Dixon and Mortimer, 1996). A fruitful way of studying a group is to attach a graph to it so that algebraic properties are translated into graph-theoretic ones. Power graphs, commuting graphs and several graphs defined through element orders are well-known examples.

The *order–divisor graph* $\mathrm{OD}(G)$ of a finite group G was introduced by Rehman et al. (2018): its vertices are the elements of G, and two vertices are adjacent when their orders are different and one order divides the other. Chalapathi and Kumar (2017) studied its density, girth and diameter and showed that two finite groups are isomorphic if and only if their order–divisor graphs are isomorphic. Related constructions include the generalized order divisor graph (Choosuwan et al., 2024), the order divisor–power graph (Ejima et al., 2025), the order prime divisor graph (Sen et al., 2021) and the order–divisor graph of the subgroups of a group (Garba et al., 2020).

The Γ_1-nonderanged permutation group $G_p^{\Gamma_1}$ has been studied from several directions, including its algebraic structure and construction (Suleiman et al., 2020; Yusuf and Ejima, 2023), representations and Young tableaux (Aminu et al., 2016; Garba et al., 2017), fuzzy and sum structures (Aremu et al., 2018), combinatorial statistics (Magami and Ibrahim, 2021; Ibrahim et al., 2022; Aremu et al., 2019) and orbit counting (Garba et al., 2022). Graphs attached to the group have also been investigated: permutation graphs (Jaiyeola et al., 2025), the inverse graph (Yusuf et al., 2025) and the power graph (Yusuf and Umar, 2025). The power graph, for example, is connected with maximum degree $p-2$ and girth 3, and its diameter is 1 for $p=5$ and 2 for $p>5$ (Yusuf and Umar, 2025). To the best of our knowledge, the order–divisor graph of $G_p^{\Gamma_1}$ has not been described.

Problem and contributions. We determine the structure of $\mathrm{OD}(G_p^{\Gamma_1})$ for every prime $p \geq 5$ and answer the following questions: Is the graph connected, complete or regular? What are its diameter, radius and girth? What are its maximum and minimum degrees? Which vertices are central and which are peripheral? Our main contributions are:

(i) a structure theorem (Theorem 4.1) that expresses $\mathrm{OD}(G_p^{\Gamma_1})$ through the divisors of $p-1$, with an explicit degree formula and edge count;

(ii) exact values of all the classical invariants (Theorems 4.4–4.8), including a complete description of the center and periphery, which depends on whether

$p-1$ is a power of 2;

(iii) a classification of the primes for which the graph is complete multipartite (Proposition 4.10);

(iv) adjacency matrices and drawings for $p = 5, 7, 11$, and a computer verification of all results for every prime $5 \le p < 600$.

The paper is organised as follows. Section 2 recalls the definitions and the arithmetic of the group. Section 3 describes the construction and the algorithm. Section 4 contains the theorems, the worked examples and a discussion, and Section 5 concludes.

2 Preliminaries

Throughout, $p \ge 5$ is a prime, $n = p-1$ (so $n \ge 4$ and n is even), φ is Euler's totient function and $D(n)$ is the set of positive divisors of n. Graphs are finite, simple and undirected; we follow Bondy and Murty (2008) for graph terminology.

Definition 2.1 (Γ_1-nonderanged permutation group (Aminu et al., 2016))**.** For $1 \le i \le p-1$, let ω_i be the permutation of $\{1, \dots, p\}$ whose one-line form is

$$\omega_i = \big(1,\ 1+i,\ 1+2i,\ \dots,\ 1+(p-1)i\big) \pmod p,$$

with residues taken in $\{1, \dots, p\}$. Then $G_p^{\Gamma_1} = \{\omega_1, \dots, \omega_{p-1}\}$ is a group of order $p-1$ under the rule $\omega_i \omega_j = \omega_{ij \bmod p}$.

Example 2.2. For $p = 5$, in one-line notation, $\omega_1 = 12345$, $\omega_2 = 13524$, $\omega_3 = 14253$ and $\omega_4 = 15432$.

Definition 2.3 (Order–divisor graph (Rehman et al., 2018))**.** Let G be a finite group and $\mathrm{o}(a)$ the order of $a \in G$. The graph $\mathrm{OD}(G)$ has vertex set G, and distinct vertices a, b are adjacent if and only if $\mathrm{o}(a) \ne \mathrm{o}(b)$ and ($\mathrm{o}(a) \mid \mathrm{o}(b)$ or $\mathrm{o}(b) \mid \mathrm{o}(a)$).

Definition 2.4 (Graph invariants)**.** For a connected graph Γ the *degree* $\deg(v)$ is the number of edges at v; the *distance* $d(u, v)$ is the length of a shortest u–v path; the *eccentricity* is $e(v) = \max_u d(v, u)$; the *radius* and *diameter* are the minimum and maximum eccentricity; the *girth* is the length of a shortest cycle; a vertex is *central* if $e(v) = \mathrm{rad}(\Gamma)$ and *peripheral* if $e(v) = \mathrm{diam}(\Gamma)$; the sets of such vertices are the *center* and the *periphery*. A graph is k*-regular* if all degrees equal k, and *complete multipartite* $K_{n_1,\dots,n_r}$ if its vertex set splits into independent sets of sizes $n_1, \dots, n_r$ with every two vertices from different sets adjacent. The adjacency matrix is $A = (a_{ij})$ with $a_{ij} = 1$ if $i \sim j$ and 0 otherwise.

Lemma 2.5. *The map $\omega_i \mapsto i \bmod p$ is an isomorphism of $G_p^{\Gamma_1}$ onto the multiplicative group $(\mathbb{Z}_p^*, \cdot)$. Consequently:*

(a) $G_p^{\Gamma_1}$ is cyclic of order $n = p - 1$, and $\mathrm{o}(\omega_i)$ is the least $k \geq 1$ with $i^k \equiv 1 \pmod p$;

(b) for every $d \in D(n)$ the set $C_d = \{\omega \in G_p^{\Gamma_1} : \mathrm{o}(\omega) = d\}$ has exactly $\varphi(d)$ elements, and $\sum_{d|n} \varphi(d) = n$;

(c) $C_1 = \{\omega_1\}$, $C_2 = \{\omega_{p-1}\}$, and $|C_n| = \varphi(n) \geq 2$.

Proof. The map is a bijection and a homomorphism by the multiplication rule. The group $\mathbb{Z}_p^*$ is cyclic because p has a primitive root, which gives (a). A cyclic group of order n has exactly one subgroup of each order $d \mid n$, and it contains exactly $\varphi(d)$ generators, giving (b). For (c), $\varphi(1) = \varphi(2) = 1$, the only involution of $\mathbb{Z}_p^*$ is $-1 \equiv p - 1$, and $\varphi(m) \geq 2$ for all $m \geq 3$ while $n \geq 4$. □

3 Materials and Methods

The approach is theoretical and is supported by computation for selected primes.

3.1 Construction of the graph

Let $p \geq 5$ be prime. The vertex set of $\mathrm{OD}(G_p^{\Gamma_1})$ is $V = G_p^{\Gamma_1}$. By Lemma 2.5(a), the order of ω_i is the multiplicative order of i modulo p, which is computed by modular arithmetic. Distinct vertices ω_i, ω_j are adjacent if and only if

$$\mathrm{o}(\omega_i) \neq \mathrm{o}(\omega_j) \quad \text{and} \quad \big(\mathrm{o}(\omega_i) \mid \mathrm{o}(\omega_j) \ \text{ or } \ \mathrm{o}(\omega_j) \mid \mathrm{o}(\omega_i)\big).$$

3.2 Algorithm

Algorithm 3.1. *Construction of $\mathrm{OD}(G_p^{\Gamma_1})$.*

1. Input a prime $p \geq 5$ and generate ω_i for $1 \leq i \leq p - 1$.
2. Compute $\mathrm{o}(\omega_i)$, the least k with $i^k \equiv 1 \pmod p$.
3. For every pair $i \neq j$: if $\mathrm{o}(\omega_i) \neq \mathrm{o}(\omega_j)$ and one order divides the other, add the edge $\omega_i\omega_j$.
4. Output the graph and its adjacency matrix A.

3.3 Invariants and tools

Connectivity and distances are obtained from breadth-first search on A; the degree sequence from the row sums of A; eccentricities, radius, diameter, center and periphery from the distance matrix; and the girth from the shortest cycle. All results in Section 4 are proved by hand and, in addition, were checked by a computer program (Python with NetworkX) that builds $\mathrm{OD}(G_p^{\Gamma_1})$ with Algorithm 3.1 for every prime $5 \le p < 600$ (107 primes) and compares the computed invariants, degree formula, edge count, center and periphery with the theorems below. No discrepancy was found.

4 Results and Discussion

4.1 Structure of the graph

Theorem 4.1 (Structure)**.** *Let $p \ge 5$ be prime, $n = p - 1$, and let C_d ($d \in D(n)$) be as in Lemma 2.5. Then:*

(a) each C_d is an independent set of size $\varphi(d)$, and the sets C_d partition the vertex set;

(b) for $d \ne e$, either every vertex of C_d is adjacent to every vertex of C_e (when $d \mid e$ or $e \mid d$), or there is no edge between them;

(c) if $\omega \in C_d$ then

$$\deg(\omega) = \sum_{\substack{e \in D(n),\, e \ne d \\ e \mid d \text{ or } d \mid e}} \varphi(e) = (d - \varphi(d)) + \sum_{\substack{d \mid e \mid n \\ e > d}} \varphi(e); \tag{4.1}$$

(d) $|E(\mathrm{OD}(G_p^{\Gamma_1}))| = \sum_{d \mid e,\ d < e,\ e \mid n} \varphi(d)\varphi(e)$.

Proof. Adjacency depends only on the orders of the two vertices, and vertices of the same order are never adjacent; this gives (a) and (b), with sizes from Lemma 2.5(b). Summing the sizes of the classes joined to C_d gives the first expression in (c); the second follows from $\sum_{e \mid d} \varphi(e) = d$. For (d) count each pair of comparable classes C_d, C_e once; it contributes $|C_d||C_e|$ edges. □

Corollary 4.2. *Let m be the largest odd divisor of n. Then $\deg(\omega_1) = n - 1 = p - 2$, the $\varphi(n)$ generators have degree $n - \varphi(n)$, and $\deg(\omega_{p-1}) = n - m$.*

Proof. Take $d = 1$, $d = n$ and $d = 2$ in (4.1). For $d = 2$, ω_{p-1} is adjacent to the identity and to every element whose order is even and larger than 2. Since $\sum_{e \mid m} \varphi(e) = m$, the even divisors of n carry $n - m$ elements, one of which is ω_{p-1} itself, so $\deg(\omega_{p-1}) = 1 + (n - m - 1) = n - m$. □

4.2 Connectedness, completeness and regularity

Proposition 4.3 (Connectedness). $\mathrm{OD}(G_p^{\Gamma_1})$ *is connected for every prime* $p \geq 5$.

Proof. The identity has order 1, which divides every order, so ω_1 is adjacent to every other vertex. Any two vertices are therefore joined by a path of length at most two through ω_1. □

Theorem 4.4 (Universal vertices and maximum degree). *A vertex of* $\mathrm{OD}(G_p^{\Gamma_1})$ *is adjacent to all other vertices if and only if it is* ω_1*, or it is* ω_{p-1} *and* $p-1$ *is a power of* 2*. Consequently* $\Delta(\mathrm{OD}(G_p^{\Gamma_1})) = p-2$*, attained at* ω_1*, and also at* ω_{p-1} *exactly when* $p-1$ *is a power of* 2*.*

Proof. ω_1 is universal. If $\omega \in C_d$ with $d \geq 3$, then $|C_d| = \varphi(d) \geq 2$ and a second vertex of C_d is not adjacent to ω. The remaining vertex is ω_{p-1}, which by Corollary 4.2 has degree $n-m$; this equals $n-1$ if and only if $m=1$, that is, n is a power of 2. Since no vertex has degree larger than $n-1$, $\Delta = p-2$. □

Proposition 4.5 (Non-completeness). $\mathrm{OD}(G_p^{\Gamma_1})$ *is not complete.*

Proof. By Lemma 2.5(c), $|C_n| = \varphi(n) \geq 2$. Two distinct generators have the same order, hence are not adjacent. □

Theorem 4.6 (Non-regularity and minimum degree). $\mathrm{OD}(G_p^{\Gamma_1})$ *is not regular. Moreover* $\delta(\mathrm{OD}(G_p^{\Gamma_1})) \geq 2$ *for all* $p \geq 5$*, with* $\delta = 2$ *if and only if* $p = 5$*; for* $p \geq 7$ *one has* $\delta \geq 3$*.*

Proof. By Corollary 4.2, $\deg(\omega_1) = n-1$ whereas a generator has degree $n-\varphi(n) \leq n-2 < n-1$, so the degrees are not all equal. For the minimum degree, a vertex of order d with $1 < d < n$ is adjacent to ω_1 and to all $\varphi(n) \geq 2$ generators, so its degree is at least 3. A generator has degree $n-\varphi(n) \geq n/2$, because $\varphi(n) \leq n/2$ for even n. For $n \geq 6$ this is at least 3, while for $n = 4$ (that is, $p = 5$) it equals 2. □

4.3 Distance invariants

Theorem 4.7 (Diameter, radius and girth). *For every prime* $p \geq 5$*,* $\mathrm{diam}(\mathrm{OD}(G_p^{\Gamma_1})) = 2$*,* $\mathrm{rad}(\mathrm{OD}(G_p^{\Gamma_1})) = 1$ *and* $\mathrm{girth}(\mathrm{OD}(G_p^{\Gamma_1})) = 3$*.*

Proof. Diameter. Every two vertices are at distance at most 2 via ω_1 (Proposition 4.3), and the graph is not complete (Proposition 4.5), so some pair is at distance exactly 2. *Radius.* $e(\omega_1) = 1$, and $e(v) \geq 1$ for every vertex of a graph with at least two vertices. *Girth.* Let $\omega \in C_n$ be a generator. The vertices $\omega_1, \omega_{p-1}, \omega$ have the pairwise different orders $1, 2, n$ (as $n \geq 4$) and $1 \mid 2 \mid n$, so they are pairwise adjacent and form a triangle. A simple graph has no cycle shorter than 3. □

4.4 Central and peripheral vertices

Theorem 4.8 (Center and periphery)**.** *Let $p \geq 5$ be prime.*

(a) If $p-1$ is not a power of 2, then $\mathrm{Center}(\mathrm{OD}(G_p^{\Gamma_1})) = \{\omega_1\}$ and $\mathrm{Per}(\mathrm{OD}(G_p^{\Gamma_1})) = G_p^{\Gamma_1} \setminus \{\omega_1\}$.

(b) If $p-1$ is a power of 2 (i.e. p is a Fermat prime), then $\mathrm{Center}(\mathrm{OD}(G_p^{\Gamma_1})) = \{\omega_1, \omega_{p-1}\}$ and $\mathrm{Per}(\mathrm{OD}(G_p^{\Gamma_1})) = G_p^{\Gamma_1} \setminus \{\omega_1, \omega_{p-1}\}$.

Proof. Since the radius is 1, a vertex is central if and only if it has eccentricity 1, i.e. if and only if it is adjacent to all other vertices. By Theorem 4.4 these are exactly the vertices listed in the center. Every other vertex is not universal, so its eccentricity is at least 2, and at most $\mathrm{diam} = 2$ by Theorem 4.7; hence it has eccentricity $2 = \mathrm{diam}$ and is peripheral. The central vertices have eccentricity $1 \neq 2$ and are not peripheral. □

Remark 4.9. The involution ω_{p-1} is central only in the arithmetically special case $p - 1 = 2^k$; the primes of this form currently known are $p = 5, 17, 257, 65537$. For all other primes the identity is the unique central vertex. The case $p = 5$ is visible in Example 4.11: there $\omega_4 = \omega_{p-1}$ is adjacent to $\omega_1, \omega_2, \omega_3$.

4.5 When is the graph complete multipartite?

Proposition 4.10. *$\mathrm{OD}(G_p^{\Gamma_1})$ is complete multipartite if and only if $p - 1 = 2^k$ for some $k \geq 2$, or $p - 1 = 2q$ for an odd prime q. Precisely,*

$$\mathrm{OD}(G_p^{\Gamma_1}) \cong K_{1,1,2,4,\ldots,2^{k-1}} \ \ (p-1=2^k), \qquad \mathrm{OD}(G_p^{\Gamma_1}) \cong K_{1,\,q-1,\,q} \ \ (p-1=2q).$$

Proof. Sufficiency. If $n = 2^k$, the divisors $1 \mid 2 \mid 4 \mid \cdots \mid 2^k$ form a chain, so any two classes C_d, C_e with $d \neq e$ are completely joined and each class is independent: the graph is $K_{\varphi(1),\varphi(2),\ldots,\varphi(2^k)} = K_{1,1,2,\ldots,2^{k-1}}$. If $n = 2q$, then $D(n) = \{1, 2, q, 2q\}$; the classes C_2 and C_q are not joined (since $2 \nmid q$ and $q \nmid 2$), while every other pair of classes is joined. So $C_2 \cup C_q$ (of size $1 + (q-1) = q$), C_{2q} (of size $q-1$) and C_1 are independent sets, and all edges between different sets are present.

Necessity. In a complete multipartite graph, if $x \not\sim y$ and $y \not\sim z$ for distinct x, y, z then $x \not\sim z$. Write $n = 2^a m$ with m odd; $a \geq 1$. Suppose $m > 1$ and let q be an odd prime divisor of m. If $a \geq 2$, pick $x \in C_2$, $y \in C_q$, $z \in C_4$: then $2, q$ and $q, 4$ are incomparable, so $x \not\sim y$ and $y \not\sim z$, but $2 \mid 4$ gives $x \sim z$. If $a = 1$ and m is not prime, choose a prime $q \mid m$ with $q < m$ and pick $x \in C_q$, $y \in C_2$, $z \in C_m$: the orders $q, 2$ and $2, m$ are incomparable, but $q \mid m$ gives $x \sim z$. In both cases the graph is not complete multipartite. The remaining possibilities are $m = 1$ ($n = 2^k$) and $a = 1$ with m prime ($n = 2q$). □

4.6 Illustrative examples

We now apply the algorithm to $p = 5, 7, 11$. The orders are obtained from the least k with $i^k \equiv 1 \pmod{p}$ and are listed in Table 1; the adjacency matrices are written with the vertices in the order $\omega_1, \omega_2, \ldots, \omega_{p-1}$.

Table 1: Orders of the elements ω_i of $G_p^{\Gamma_1}$ for $p = 5, 7, 11$.

p	Orders $\mathrm{o}(\omega_i)$ and classes C_d
5	$\mathrm{o}(\omega_1) = 1$, $\mathrm{o}(\omega_2) = \mathrm{o}(\omega_3) = 4$, $\mathrm{o}(\omega_4) = 2$ $C_1 = \{\omega_1\}$, $C_2 = \{\omega_4\}$, $C_4 = \{\omega_2, \omega_3\}$
7	$\mathrm{o}(\omega_1) = 1$, $\mathrm{o}(\omega_2) = \mathrm{o}(\omega_4) = 3$, $\mathrm{o}(\omega_3) = \mathrm{o}(\omega_5) = 6$, $\mathrm{o}(\omega_6) = 2$ $C_1 = \{\omega_1\}$, $C_2 = \{\omega_6\}$, $C_3 = \{\omega_2, \omega_4\}$, $C_6 = \{\omega_3, \omega_5\}$
11	$\mathrm{o}(\omega_1) = 1$, $\mathrm{o}(\omega_{10}) = 2$, $\mathrm{o}(\omega_3) = \mathrm{o}(\omega_4) = \mathrm{o}(\omega_5) = \mathrm{o}(\omega_9) = 5$, $\mathrm{o}(\omega_2) = \mathrm{o}(\omega_6) = \mathrm{o}(\omega_7) = \mathrm{o}(\omega_8) = 10$

Example 4.11 ($p = 5$)**.** Here $n = 4$ and $D(4) = \{1, 2, 4\}$. By Theorem 4.1, ω_1 is adjacent to $\omega_2, \omega_3, \omega_4$; ω_4 (order 2) is adjacent to ω_2, ω_3 (order 4); and $\omega_2 \not\sim \omega_3$. Thus $E = \{\omega_1\omega_2, \omega_1\omega_3, \omega_1\omega_4, \omega_4\omega_2, \omega_4\omega_3\}$, $|E| = 5$, and

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The graph (Figure 1) is K_4 minus an edge, i.e. $K_{1,1,2}$. The degrees are $3, 2, 2, 3$, so $\Delta = 3 = p - 2$ and $\delta = 2$; $\mathrm{diam} = 2$, $\mathrm{rad} = 1$, girth 3. Both ω_1 and ω_4 are universal, so $\mathrm{Center} = \{\omega_1, \omega_4\}$ and $\mathrm{Per} = \{\omega_2, \omega_3\}$, in agreement with Theorem 4.8(b).

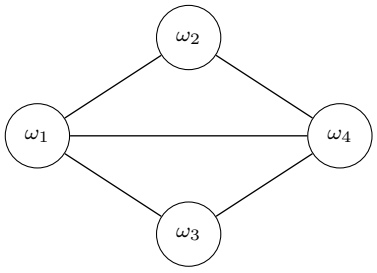


Figure 1: The order–divisor graph of $G_5^{\Gamma_1}$ ($K_{1,1,2}$).

Example 4.12 ($p = 7$)**.** Here $n = 6$ and $D(6) = \{1, 2, 3, 6\}$. The edges are $\omega_1\omega_i$ ($i = 2, \ldots, 6$); $\omega_6\omega_3$, $\omega_6\omega_5$ (since $2 \mid 6$); and $\omega_2\omega_3$, $\omega_2\omega_5$, $\omega_4\omega_3$, $\omega_4\omega_5$ (since $3 \mid 6$). The

elements ω_6 (order 2) and ω_2, ω_4 (order 3) are not adjacent because $2 \nmid 3$. Hence $|E| = 11$ and

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

The degrees are $(5, 3, 4, 3, 4, 3)$, so $\Delta = 5 = p - 2$ and $\delta = 3$. The graph (Figure 2) is $K_{1,2,3}$ with parts $\{\omega_1\}$, $\{\omega_3, \omega_5\}$ and $\{\omega_2, \omega_4, \omega_6\}$ (Proposition 4.10 with $q = 3$). Since 6 is not a power of 2, $\mathrm{Center} = \{\omega_1\}$ and $\mathrm{Per} = \{\omega_2, \ldots, \omega_6\}$.

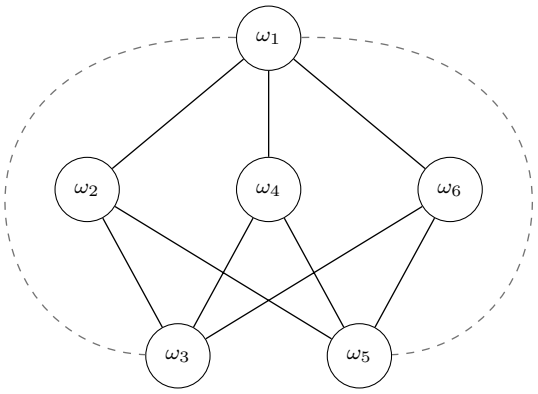


Figure 2: The order–divisor graph of $G_7^{\Gamma_1}$ ($K_{1,2,3}$); dashed edges join the identity ω_1 to the elements of order 6.

Example 4.13 ($p = 11$)**.** Here $n = 10$ and $D(10) = \{1, 2, 5, 10\}$. The identity is adjacent to all nine other vertices; ω_{10} (order 2) and $\omega_3, \omega_4, \omega_5, \omega_9$ (order 5) are each adjacent to all of $\omega_2, \omega_6, \omega_7, \omega_8$ (order 10); there are no other edges. So $|E| = 9 + 20 = 29$, in agreement with Theorem 4.1(d): $\varphi(1)\varphi(2) + \varphi(1)\varphi(5) + \varphi(1)\varphi(10) + \varphi(2)\varphi(10) + \varphi(5)\varphi(10) = 1 + 4 + 4 + 4 + 16 = 29$. In the vertex order $\omega_1, \ldots, \omega_{10}$,

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The degrees are 9 for ω_1, 6 for the elements of order 10, and 5 for the elements of orders 2 and 5, so $\Delta = 9 = p - 2$ and $\delta = 5$. The graph (Figure 3) is $K_{1,4,5}$, with parts $\{\omega_1\}$, $\{\omega_2, \omega_6, \omega_7, \omega_8\}$ and $\{\omega_{10}, \omega_3, \omega_4, \omega_5, \omega_9\}$ (Proposition 4.10 with $q = 5$). Again $\mathrm{Center} = \{\omega_1\}$.

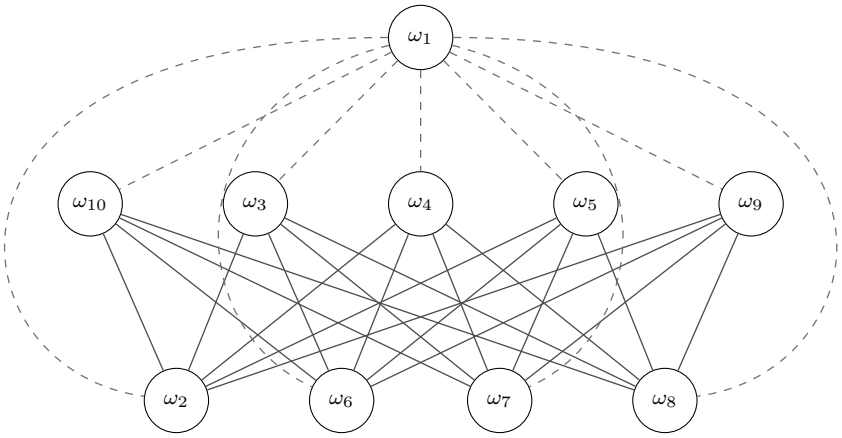


Figure 3: The order–divisor graph of $G_{11}^{\Gamma_1}$ ($K_{1,4,5}$); the top row consists of the elements of orders 2 and 5, the bottom row of the elements of order 10. Dashed edges are incident with the identity ω_1.

4.7 Summary of the results and computational verification

Table 2 collects the general results of this section and Table 3 lists the invariants computed by the algorithm for several primes, including the Fermat primes $5, 17, 257$ (where ω_{p-1} is also central) and primes such as $13, 19, 29$ for which the graph is not complete multipartite.

4.8 Discussion

The order–divisor graph of $G_p^{\Gamma_1}$ is determined by the divisor lattice of $p - 1$ alone: every group element contributes only through its order, and Theorem 4.1 shows that the graph is the divisibility-comparability graph of $D(p - 1)$ in which each divisor d is blown up into an independent set of $\varphi(d)$ vertices. This explains why the diameter, radius and girth are the same for all primes, since they are governed by the universal identity vertex, whereas the degrees, the number of edges and the finer structure depend on the factorisation of $p - 1$.

Compared with the power graph of the same group (Yusuf and Umar, 2025), which has the same maximum degree $p - 2$ and girth 3, the order–divisor graph is never complete, so its diameter is 2 even for $p = 5$. Unlike the identity, the

Table 2: Properties of $\mathrm{OD}(G_p^{\Gamma_1})$ for a prime $p \geq 5$, $n = p - 1$.

Property	Value / description
Number of vertices	$p - 1$
Number of edges	$\sum_{d\mid e\mid n,\ d<e} \varphi(d)\varphi(e)$
Connected / complete / regular	Yes / No / No
Diameter, radius, girth	2, 1, 3
Maximum degree	$\Delta = p - 2$ (at ω_1; also at ω_{p-1} iff $p - 1 = 2^k$)
Minimum degree	$\delta = 2$ if $p = 5$; $\delta \geq 3$ if $p \geq 7$ (formula (4.1))
Degree of ω_1, ω_{p-1}, generators	$p - 2,\ p - 1 - m,\ n - \varphi(n)$
Center	$\{\omega_1\}$, or $\{\omega_1, \omega_{p-1}\}$ if $p - 1 = 2^k$
Periphery	$G_p^{\Gamma_1} \setminus \mathrm{Center}$
Eccentricities	1 on the center, 2 elsewhere
Complete multipartite	iff $p - 1 = 2^k$ or $p - 1 = 2q$, q an odd prime

involution ω_{p-1} is universal only when $p - 1$ is a power of 2. This is why a careful statement of the center requires distinguishing Fermat primes from all other primes (Theorem 4.8); a statement that the identity is always the only central vertex is true for $p \neq 5, 17, 257, 65537, \ldots$ but fails for $p = 5$, where ω_4 is adjacent to every other vertex.

5 Conclusions

We have given a complete description of the order–divisor graph of the Γ_1-nonderanged permutation group for every prime $p \geq 5$. The main findings are:

a $\mathrm{OD}(G_p^{\Gamma_1})$ is the divisibility-comparability graph of the divisors of $p - 1$ with each divisor d replaced by an independent set of $\varphi(d)$ vertices; degrees and edge numbers are given by explicit formulas (Theorem 4.1).

b The graph is connected, not complete and not regular, with diameter 2, radius 1 and girth 3 for all $p \geq 5$.

c The maximum degree is $p - 2$; the minimum degree is 2 for $p = 5$ and at least 3 for $p \geq 7$.

d The center is $\{\omega_1\}$, except when p is a Fermat prime, in which case it is $\{\omega_1, \omega_{p-1}\}$; the periphery consists of all remaining vertices.

e The graph is complete multipartite exactly when $p - 1 = 2^k$ or $p - 1 = 2q$ with q an odd prime, and then it is $K_{1,1,2,\ldots,2^{k-1}}$ or $K_{1,q-1,q}$ respectively.

Table 3: Invariants of $\mathrm{OD}(G_p^{\Gamma_1})$ computed for selected primes (all agree with the theorems; $|Z|$ is the size of the center).

p	$\lvert V\rvert$	$\lvert E\rvert$	δ	Δ	diam	rad	girth	$\lvert Z\rvert$	Structure
5	4	5	2	3	2	1	3	2	$K_{1,1,2}$
7	6	11	3	5	2	1	3	1	$K_{1,2,3}$
11	10	29	5	9	2	1	3	1	$K_{1,4,5}$
13	12	47	6	11	2	1	3	1	not multipartite
17	16	85	8	15	2	1	3	2	$K_{1,1,2,4,8}$
19	18	101	9	17	2	1	3	1	not multipartite
23	22	131	11	21	2	1	3	1	$K_{1,10,11}$
29	28	251	14	27	2	1	3	1	not multipartite
31	30	271	12	29	2	1	3	1	not multipartite
37	36	425	18	35	2	1	3	1	not multipartite
257	256	21845	128	255	2	1	3	2	$K_{1,1,2,\ldots,128}$

Possible directions for further work are the spectrum of the adjacency matrix, the case of composite moduli, other graphs on the same group (directed, weighted or hybrid graphs such as the order divisor–power graph), and the automorphism group of $\mathrm{OD}(G_p^{\Gamma_1})$, which by Theorem 4.1 contains the direct product of the symmetric groups on the classes C_d.

Competing Interests

The author declares that no competing interests exist.

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