Prime Power Graph of Finite Cyclic Groups

Abstract: The square graph $\Gamma(G)$ of a finite group G is an undirected graph with vertex set G, in which two distinct vertices a and b are adjacent if and only if $a^2 = b$ or $b^2 = a$. For a finite group G and a prime p, the p-power graph $\Gamma_p(G)$ has vertex set G, with two distinct vertices a and b adjacent if and only if $a^p = b$ or $b^p = a$. This paper examines square graphs and p-power graphs of finite cyclic groups. For square graphs, conditions determining the maximum degree are established. In particular, the maximum degree is 2 for cyclic groups of odd order greater than 3 and for the cyclic group of order 6, while it is 3 for cyclic groups of even order other than 2 and 6. The connected components of the square graph of a cyclic group of odd order are shown to be cycles, K_2, or isolated vertices. Vertices of degree 3 are also characterised for cyclic groups of even order other than 6. For a positive integer n and a prime p, the maximum degree of $\Gamma_p(\mathbb{Z}_n)$ satisfies $\Delta(\Gamma_p(\mathbb{Z}_n)) \leq p+1$. Moreover, when $\gcd(p,n) = 1$ and $n > p^2$, the maximum degree is 2. These results describe degree behaviour and connected-component structure within the graph constructions considered.

Keywords: square graph; p-power graph; finite cyclic group; maximum degree; connected components; vertex degree; modular congruence; cyclic group order; graph structure; prime power

INTRODUCTION

Kelarev and Quinn (2002) introduced the directed power graph for semigroups. In this graph, each element of the semigroup is a vertex, and there is an arc from one element to another if the latter is a positive integral power of the former. Chakrabarty et al. (2009) later defined the undirected power graph for a semigroup. In this graph, the elements of the semigroup are vertices, and an edge joins two vertices if one element is a power of the other. Several studies have investigated properties of power graphs associated with finite groups. The power graph of every finite group is connected, and, for a finite cyclic group, the power graph is complete if the group has order 1 or p^i, where p is a prime and i is a positive integer (Chakrabarty et al., 2009). Curtin and Pourgholi (2014) demonstrated that, among all finite groups of a given order, the power graph of the cyclic group has the maximum number of edges. Recently, various graphs associated with algebraic structures have been studied (Rana, Sehgal, Bhatia, & P. Kumar, 2024; Rana, Sehgal, Bhatia, & V. Kumar, 2024; Siwach et al., 2024; Swathi & Sunitha, 2021). Recent studies have further examined graph constructions arising from group actions and power relations, including suborbital graphs of alternating-group actions, chordal power graphs, and structural relationships between power graphs and enhanced power graphs (Bošnjak et al., 2024; Brachter & Kaja, 2023; Kadedesya et al., 2023).

Inspired by the definition of power graphs for finite groups, a new graph called the square graph of a finite group was introduced by Swathi and Sunitha (2021), and upper bounds for the chromatic number and clique number of square graphs associated with finite groups were determined. Within the scope of this manuscript, a focused characterisation of degree behaviour and connected-component structure for square graphs of finite cyclic groups, together with the maximum-degree behaviour of the corresponding p-power graphs, remains to be developed. **Objective.** This paper investigates square graphs of some finite cyclic groups and identifies certain properties that they satisfy. The study also examines the maximum-degree properties of p-power graphs of finite cyclic groups.

PRELIMINARIES

A graph $\Gamma = (V, E)$ is an undirected simple graph with vertex set V and edge set E. The order of Γ is defined as $|V|$. The number of edges incident with a vertex x is the degree of x in Γ, denoted by $\deg_\Gamma(x)$ or simply $\deg(x)$. The minimum degree of the vertices in Γ is denoted by $\delta(\Gamma)$, and the maximum degree is denoted by $\Delta(\Gamma)$.

The definitions and results that follow are sourced from Fraleigh (2003). A group G is called cyclic if there exists an element $a \in G$ such that every element of G can be written as a power of a. In other words, G is cyclic if $G = \langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ for some $a \in G$. $\mathbb{Z}_4$, defined as the set $\{0, 1, 2, 3\}$, is a finite cyclic group under addition modulo 4. It can be represented as:

$$\mathbb{Z}_4 = \{0, 1, 2, 3\}.$$

This group is generated by any of its elements, for example, $\langle 1 \rangle = \{0, 1, 2, 3\}$. In general, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ is a cyclic group under addition modulo n, and $\langle 1 \rangle = \{0, 1, 2, \ldots, n-1\}$. Consider a group G with identity element e. The number of elements in G is referred to as the order of G, denoted by $o(G)$. The group G is finite if its order, $o(G)$, is finite. For an element $g \in G$, the order of g is defined as the smallest positive integer i such that $g^i = e$, and is denoted by $o(g)$.

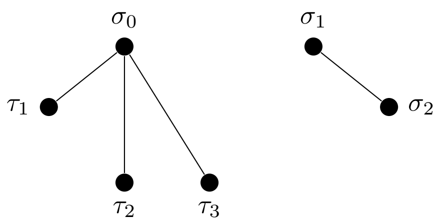
Theorem 1. *If G is a finite cyclic group of order n, then G is isomorphic to $\mathbb{Z}_n$.*

Definition 1. *Two integers a and b are said to be congruent modulo n, denoted by $a \equiv b \pmod{n}$, if n divides their difference; that is, if $a - b$ is divisible by n.*

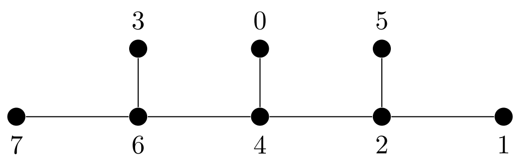
In this paper, the focus is solely on finite groups and finite graphs. The terminology and notation for graphs follow Chartrand and Zhang (2006), while those for groups follow Fraleigh (2003). The following definition and results are taken from Swathi and Sunitha (2021).

Definition 2. *For a finite group G, the square graph $\Gamma(G)$ is defined with vertex set G, and its edge set consists of pairs of vertices a and b if either $a^2 = b$ or $b^2 = a$, with $a \neq b$.*

Example 1. *The square graph of the symmetric group on 3 elements (S_3) can be visualised as $K_{1,3} \cup K_2$ since $\sigma_1^2 = \sigma_2$, $\sigma_2^2 = \sigma_1$, and $\tau_i^2 = \sigma_0$ for $i = 1, 2, 3$, where $\sigma_0 = e$, $\sigma_1 = (1\ 2\ 3)$, $\sigma_2 = (1\ 3\ 2)$, $\tau_1 = (1\ 2)$, $\tau_2 = (2\ 3)$, and $\tau_3 = (1\ 3)$.*



Example 2. *The square graph of the cyclic group $\mathbb{Z}_8$ is the following tree.*



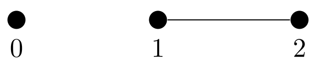
Example 3. *$K_{1,n}$ is the square graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times).*

Example 4. *The square graph of $\mathbb{Z}_2$ is K_2.*

Theorem 2. *For a finite cyclic group G, $\Delta(\Gamma(G)) \leq 3$.*

Proof. If G is cyclic and $o(G) = n$, then $G \simeq \mathbb{Z}_n$; let the vertices of $\Gamma(G)$ be the first n non-negative integers. If k is a vertex in $\Gamma(G)$, then k is adjacent to $2k$. All other adjacencies of k arise from the solutions of the congruence $2x \equiv k \pmod{n}$. This congruence has at most 2 solutions modulo n. Hence, k can be adjacent to at most 3 vertices. $\square$

Example 5. *The square graph of $\mathbb{Z}_3$ is as follows.*



So $\Delta(\Gamma(\mathbb{Z}_3)) = 1$.

From the above theorem, we obtain the following result.

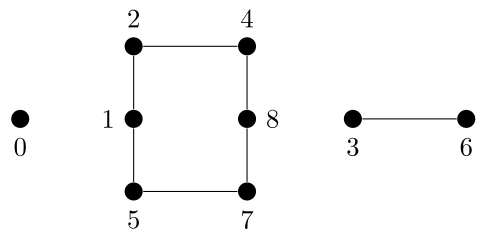
Properties of Square Graph of Finite Cyclic Group

In this section, we examine the maximum degree of the square graph of finite cyclic groups, as well as the connected components of the square graph of finite cyclic groups of odd order.

Theorem 3. *If n is odd and $n > 3$, then $\Delta(\Gamma(\mathbb{Z}_n)) = 2$.*

Proof. Let k be a vertex of $\Gamma(\mathbb{Z}_n)$. If n is odd, then $\gcd(2, n) = 1$ and $1 \mid k$, so the congruence $2x \equiv k \pmod{n}$ has a unique solution. Thus, $\Delta(\Gamma(\mathbb{Z}_n)) \leq 2$. Since n is odd and $n > 3$, vertex 4 and the solution of $2x \equiv 2 \pmod{n}$ are adjacent to vertex 2. Therefore, the degree of vertex 2 is 2. Hence, $\Delta(\Gamma(\mathbb{Z}_n)) = 2$. $\square$

Example 6. *The square graph of $\mathbb{Z}_9$ is given below.*

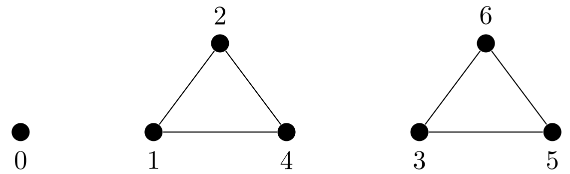


So $\Delta(\Gamma(\mathbb{Z}_9)) = 2$.

Theorem 4. *If $n > 1$ is odd and relatively prime to 3, then a connected component of $\Gamma(\mathbb{Z}_n)$ is a cycle or an isolated vertex.*

Proof. Since n is odd, vertex 0 is isolated in $\Gamma(\mathbb{Z}_n)$. Let k be a vertex other than vertex 0. Then vertex k is adjacent to vertex $2k$ and to the vertex x such that $x \equiv \left(\frac{n-1}{2} + 1\right) k \pmod{n}$. These two vertices are distinct because n is relatively prime to 3. Since $\Delta(\Gamma(\mathbb{Z}_n)) \leq 2$, the degree of vertex k is 2. Thus, every vertex other than vertex 0 has degree 2, so all other connected components of $\Gamma(\mathbb{Z}_n)$ are cycles. □

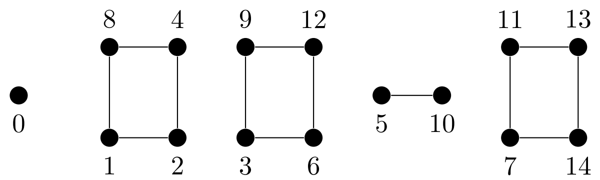
Example 7. *The square graph of $\mathbb{Z}_7$ is given below.*



Theorem 5. *If $n > 1$ is odd and $\gcd(n, 3) = 3$, then a connected component of $\Gamma(\mathbb{Z}_n)$ is a cycle, K_2, or an isolated vertex.*

Proof. Since n is odd, vertex 0 is isolated in $\Gamma(\mathbb{Z}_n)$. Let k be a vertex other than vertex 0. Then vertex k is adjacent to vertex $2k$ and to the vertex corresponding to the solution of the congruence $2x \equiv k \pmod{n}$. These vertices can be identical if $4k \equiv k \pmod{n}$. This happens if and only if $k = \frac{n}{3}$ or $k = \frac{2n}{3}$. In that case, vertex k and vertex $2k$ form K_2 in $\Gamma(\mathbb{Z}_n)$. The remaining vertices have degree 2, so they form cycles in $\Gamma(\mathbb{Z}_n)$. □

Example 8. *Consider $\mathbb{Z}_{15}$. The square graph of $\mathbb{Z}_{15}$ is given below.*

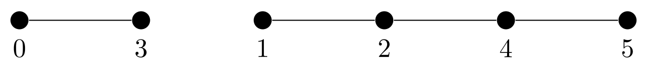


The above theorem is also illustrated in Example 6 for $\mathbb{Z}_9$.

Theorem 6. *If n is an even number other than 2 or 6, then $\Delta(\Gamma(\mathbb{Z}_n)) = 3$.*

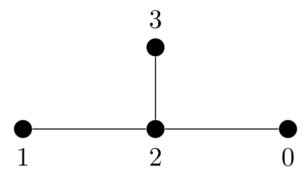
Proof. If $n = 4$, vertex 2 is adjacent to vertices 0, 1, and 3. Thus, $\Delta(\Gamma(\mathbb{Z}_4)) = 3$. Now let n be an even number greater than 6. Then vertex 2 is adjacent to vertices 1, 4, and $\left(\frac{n}{2} + 1\right)$. Hence, $\Delta(\Gamma(\mathbb{Z}_n)) = 3$. □

Example 9. *Consider $\mathbb{Z}_6$. The square graph of $\mathbb{Z}_6$ is*



So $\Delta(\Gamma(\mathbb{Z}_6)) = 2$.

Example 10. *The square graph of $\mathbb{Z}_4$ is*



So $\Delta(\Gamma(\mathbb{Z}_4)) = 3$.

Example 2 also illustrates the above theorem.

Theorem 7. *If n is even and $n \neq 6$, then the following are true for $\Gamma(\mathbb{Z}_n)$:*

1. *Degree of vertex 0 is 1.*

2. *If k is odd, then degree of vertex k is 1.*

3. *If k is even, is not equal to zero, and $n \mid \frac{3k}{2}$ or $n \mid \left(\frac{3k}{2} - \frac{n}{2}\right)$, then degree of vertex k is 2. Otherwise, the degree of vertex k is 3.*

Proof.

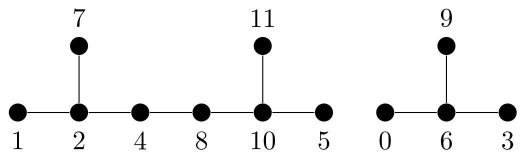
1. Since n is even, vertex $\frac{n}{2}$ is the only vertex adjacent to vertex 0. Thus, the degree of vertex 0 is 1.

2. If k is odd and n is even, the congruence $2x \equiv k \pmod{n}$ has no solution. However, vertex k is adjacent to vertex $2k$. Thus, the degree of vertex k is 1.

3. If k and n are even, the congruence $2x \equiv k \pmod{n}$ has exactly two solutions, namely $\frac{k}{2}$ and $\frac{k}{2} + \frac{n}{2}$. Thus, vertex k is adjacent to vertices $\frac{k}{2}$, $\frac{k}{2} + \frac{n}{2}$, and $2k$. If $n \mid \frac{3k}{2}$, then vertex $\frac{k}{2}$ and vertex $2k$ are identical. If $n \mid \left(\frac{3k}{2} - \frac{n}{2}\right)$, then vertex $\left(\frac{3k}{2} - \frac{n}{2}\right)$ and vertex $2k$ are identical. Thus, in both cases, the degree of vertex k is 2. Otherwise, the degree of vertex k is 3.

□

Example 11. *The square graph of $\mathbb{Z}_{12}$ is*



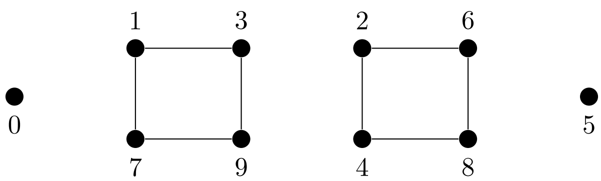
Example 8 also illustrates the above theorem.

Properties of p-Power graph of Finite Cyclic Group

In this section, we examine the maximum degree of the p-power graph of finite cyclic groups.

Definition 3. *For a finite group G and a prime number p, the p-power graph $\Gamma_p(G)$ is defined with vertex set G, and its edge set consists of pairs of vertices a and b if either $a^p = b$ or $b^p = a$, with $a \neq b$.*

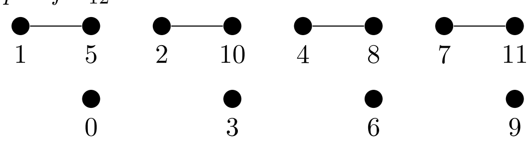
Example 12. *The 3-power graph of $\mathbb{Z}_{10}$ is*



Theorem 8. *For a finite cyclic group G and a prime number p, $\Delta(\Gamma_p(G)) \leq p + 1$.*

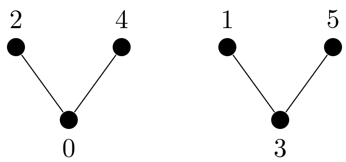
Proof. If G is cyclic and $o(G) = n$, then $G \simeq \mathbb{Z}_n$; let the vertices of $\Gamma_p(G)$ be the first n non-negative integers. If k is a vertex in $\Gamma_p(G)$, then k is adjacent to pk. All other adjacencies of k arise from the solutions of the congruence $px \equiv k \pmod{n}$. This congruence has at most p solutions modulo n. Hence, k can be adjacent to at most $p + 1$ vertices. □

Example 13. *The 5-power graph of $\mathbb{Z}_{12}$ is*



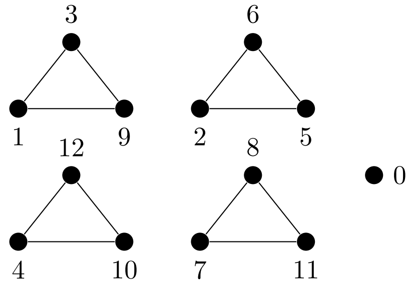
and $\Delta(\Gamma_5(\mathbb{Z}_{12})) = 1$.

Example 14. *The 3-power graph of $\mathbb{Z}_8$ is*



and $\Delta(\Gamma_3(\mathbb{Z}_6)) = 2$.

Example 15. *The 3-power graph of $\mathbb{Z}_{13}$ is*



and $\Delta(\Gamma_3(\mathbb{Z}_6)) = 2$.

Theorem 9. *If n is a positive integer and p is a prime number such that $\gcd(p, n) = 1$ and $n > p^2$, then $\Delta(\Gamma_p(\mathbb{Z}_n)) = 2$.*

Proof. Let k be a vertex of $\Gamma_p(\mathbb{Z}_n)$. If $\gcd(p, n) = 1$ and $1 \mid k$, then the congruence $px \equiv k \pmod{n}$ has a unique solution. Thus, $\Delta(\Gamma_p(\mathbb{Z}_n)) \leq 2$. Since $n > p^2$, vertex 1 and vertex p^2 are adjacent to vertex p. Therefore, the degree of vertex p is 2. Hence, $\Delta(\Gamma_p(\mathbb{Z}_n)) = 2$. □

Note: If $\gcd(p, n) = 1$, then the degree of vertex 0 is 0 in $\Gamma_p(\mathbb{Z}_n)$.

Example 16. *The 5-power graph of $\mathbb{Z}_{26}$ is given in Figure 1.*

From the figure, it is clear that $\Delta(\Gamma_5(\mathbb{Z}_{26})) = 2$.

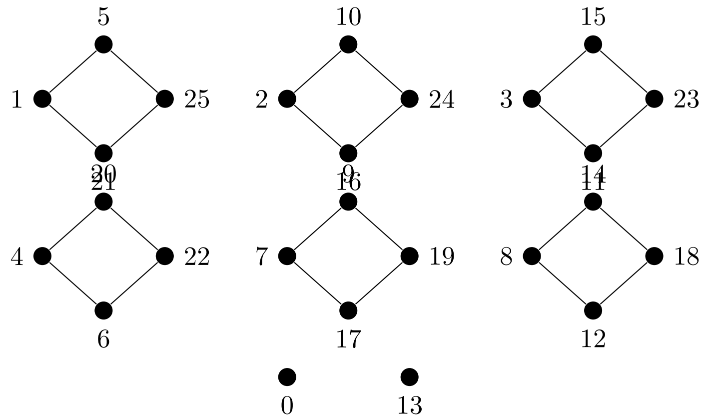


Figure 1: The 5-power graph of $\mathbb{Z}_{26}$: six disjoint 4-cycles and two isolated vertices.

CONCLUSION

This study examines square graphs and p-power graphs associated with finite cyclic groups, with emphasis on vertex degree and connected-component structure. For square graphs, the maximum degree is at most 3. When the cyclic group has odd order greater than 3, the maximum degree is 2; for the cyclic group of order 6, it is also 2, whereas for cyclic groups of even order other than 2 and 6, it is 3. For cyclic groups of odd order, the connected components are described as cycles, K_2 components, or isolated vertices, according to the divisibility condition involving 3. The degree behaviour of vertices in square graphs of cyclic groups of even order is also characterised. For p-power graphs, where p is prime, the maximum degree of $\Gamma_p(\mathbb{Z}_n)$ is bounded above by $p + 1$. In addition, if $\gcd(p, n) = 1$ and $n > p^2$, then the maximum degree is 2. These results provide a concise description of the degree properties established in the manuscript for the specified families of finite cyclic groups.

LIMITATIONS

The analysis is restricted to finite cyclic groups and to square graphs and p-power graphs defined by the specific adjacency relations used in this manuscript. The results focus mainly on maximum degree and connected-component structure and do not address other graph invariants or non-cyclic finite groups. The stated p-power maximum-degree equality is established under $\gcd(p, n) = 1$ and $n > p^2$; cases outside these conditions are not characterised here.

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