# Prime Power Graph of Finite Cyclic Groups

**Abstract:** The Square graph, denoted as $\Gamma(G)$, is introduced as an undirected graph associated with a finite group G. In this graph, the vertex set is defined to be G, and two distinct vertices, $a$ and $b$, are considered adjacent if and only if either $a^2 = b$ or $b^2 = a$ and $a \neq b$. For a finite group $G$ and a prime number $p$, the $p$-power graph $\Gamma_p(G)$ is defined with a vertex set $G$, and its edge set consists of pairs of vertices $a$ and $b$ if either $a^p = b$ or $b^p = a$, and $a$ is not equal to $b$. In this paper, authors discussed the properties of square and $p$-power graph of finite cyclic groups. Necessary conditions for maximum degree of square graph of finite cyclic groups to be 2 and 3 are identified. It has been proved that connected component of a square graph of cyclic group of odd order is a cycle, $K_2$ or an isolated vertex. Also, vertices with degree 3 are identified in the square graph derived from a cyclic group of even order not equal to 6. For a positive integer $n$ and a prime $p$, the maximum degree of the $p$-power graph $\Gamma_p(\mathbb{Z}_n)$ satisfies $\Delta(\Gamma_p(\mathbb{Z}_n)) \leq p + 1$. If $\gcd(p, n) = 1$ and $n > p^2$, then $\Delta(\Gamma_p(\mathbb{Z}_n)) = 2$.

**Keywords: Square graph, $p$-power graph, maximum degree, connected components**

## INTRODUCTION

Kelarev and Quinn [5] introduced the directed power graph in 2002 for a mathematical structure called a semigroup. In this graph, each element in the semigroup is a point (vertex), and there is an arrow (arc) from one element to another if the second element is a power (raised to some natural number) of the first. Chakrabarty et al. [1] and others later defined the undirected power graph for a semigroup. In this graph, elements of the semigroup are points, and there is a line (edge) between two points if one element is a power of the other. Many researchers have found interesting things about special graphs called power graphs related to finite groups. In one study, it was shown that the power graphs of finite groups are always connected. They also found that for a finite cyclic group, the power graph is complete if the group has an order of 1 or $p^i$, where $p$ is a prime number and $i$ is a positive integer. Curtin and Pourgholi [3] demonstrated that among all finite groups of a given order, power graphs of cyclic groups possess maximum number of edges. In recent times many graphs associated with algebraic structures have been studied recently [6-9].

Inspired by the definition of power graphs for finite groups, a new type of graph called the square graph of finite groups has been introduced in [9] and upper limits for the chromatic number and clique number of square graphs associated with finite groups has been determined. In this paper, studies on square graphs of some finite cyclic groups are conducted and certain properties satisfied by them are identified.

# PRELIMINARIES

A graph $\Gamma = (V, E)$ is an undirected simple graph with a vertex set $V$ and an edge set $E$. The order of $\Gamma$ is defined as $|V|$. The number of edges incident with a vertex $x$ is the degree of $x$ in $\Gamma$, indicated by $\deg \Gamma(x)$ or simply $\deg(x)$. $\delta(\Gamma)$ denotes the minimum degree of vertices in $\Gamma$, and the maximum degree is denoted by $\Delta(\Gamma)$.
The definitions and results that follow are sourced from [4]. A group $G$ is called cyclic if there exists an element $a \in G$ such that every element of $G$ can be written as a power of $a$. In other words, $G$ is cyclic if $G = \langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ for some $a$ in $G$. $\mathbb{Z}_4$, is defined as the set $\{0, 1, 2, 3\}$, is a finite cyclic group under addition modulo 4. It can be represented as:

$$\mathbb{Z}_4 = \{0, 1, 2, 3\}$$

This group is generated by any of its elements, for example, $\langle 1 \rangle = \{0, 1, 2, 3\}$. In general, $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ is a cyclic group under addition modulo $n$ and $\langle 1 \rangle = \{0, 1, 2, ..., n-1\}$. Consider a group $G$ with an identity element $e$. The number of elements in $G$ is referred to as the order of $G$, denoted by $o(G)$. $G$ is considered finite if its order, $o(G)$, is finite. For an element $g \in G$, the order of $g$ is defined as the smallest positive integer $i$ such that raising $g$ to the power of $i$ results in the identity element $e$, and this is denoted by $o(g)$.

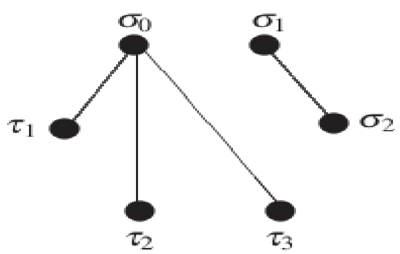
**Theorem 1.** *If $G$ is a finite cyclic group with order $n$, then $G$ is isomorphic to $\mathbb{Z}_n$.*

**Definition 1.** *Two integers $a$ and $b$ are said to be congruent modulo $n$, denoted as $a \equiv b \pmod{n}$, if $n$ divides their difference, i.e., if $a - b$ is divisible by $n$.*

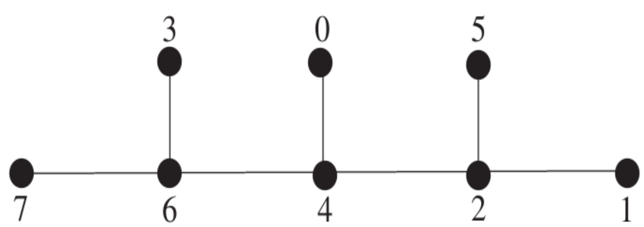
In this paper, our focus is solely on finite groups and finite graphs. The terminology and notation for graphs [2] and for groups [4] are adhered to in this context.
The following definition and results have been extracted from [9].

**Definition 2.** *For a finite group $G$, the square graph $\Gamma(G)$ is defined with a vertex set $G$, and its edge set consists of pairs of vertices $a$ and $b$ if either $a^2 = b$ or $b^2 = a$, and $a$ is not equal to $b$.*

**Example 1.** *The square graph of the symmetric group on 3 elements ($S_3$) can be visualized as $K_{1,3} \cup K_2$ since $\sigma_1^2 = \sigma_2$, $\sigma_2^2 = \sigma_1$, and $\tau_i^2 = \sigma_0$ for $i = 1, 2, 3$, where $\sigma_0 = e$, $\sigma_1 = (1\,2\,3), \sigma_2 = (1\,3\,2), \tau_1 = (1\,2), \tau_2 = (2\,3)$ and $\tau_3 = (1\,3)$.*




**Example 2.** *Square graph of the cyclic group $\mathbb{Z}_8$ is the following tree*



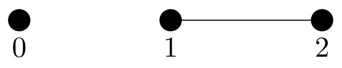
**Example 3.** *$K_{1,n}$ is the square graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ ($n$ times).*

**Example 4.** *Square graph of $\mathbb{Z}_2$ is $K_2$.*

**Theorem 2.** *For a finite cyclic group $G, \Delta(\Gamma(G)) \leq 3$.*

*Proof.* If $G$ is cyclic and $o(G) = n$, then $G \backsimeq \mathbb{Z}_n$ and let the vertices of $\Gamma(G)$ be the first $n$ non-negative integers. If k is a vertex in $\Gamma(G)$ then k is adjacent to 2k. All other adjacencies of $k$ comes from the solutions of the congruence relation $2x \equiv k (\mathrm{mod}\, n)$. But this congruence has at most 2 solutions modulo n. Hence $k$ can be adjacent to at most 3 vertices. □

**Example 5.** *Square graph of $\mathbb{Z}_3$ is the following*




*So* $\Delta(\Gamma(\mathbb{Z}_3)) = 1$.

From the above theorem we can arrive at the following result.

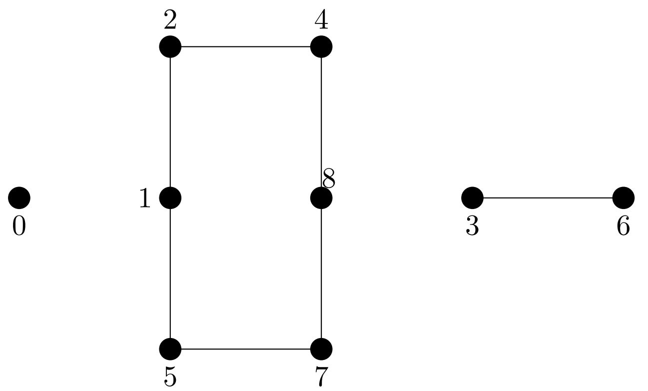
## Properties of Square Graph of Finite Cyclic Group

In this section, we examine the maximum degree of the square graph of finite cyclic groups, as well as the connected components of the square graph of finite cyclic groups with odd orders.

**Theorem 3.** *If $n$ is odd and $n > 3$, then $\Delta(\Gamma(\mathbb{Z}_n)) = 2$.*

*Proof.* Let $k$ be a vertex of $\Gamma(\mathbb{Z}_n)$. If n is odd, then $\gcd(2, n) = 1$ and $1 \mid k$, so the congruence $2x \equiv k (\mathrm{mod}\, n)$ has a unique solution. Thus $\Delta(\Gamma(\mathbb{Z}_n)) \leq 2$. Since $n$ is odd and $n > 3$, vertex 4 and the solution of $2x \equiv 2 (\mathrm{mod}\, n)$ are adjacent to vertex 2. Therefore degree of vertex 2 is equal to 2. Hence $\Delta(\Gamma(\mathbb{Z}_n)) = 2$. □

**Example 6.** *Square graph of $\mathbb{Z}_9$ is given below*

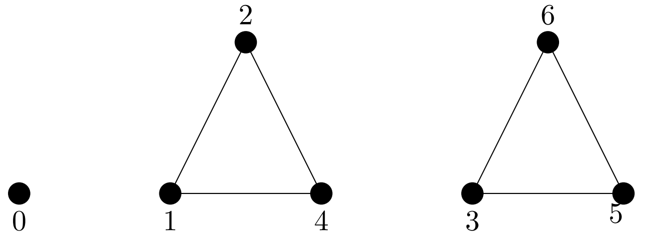



*So* $\Delta(\Gamma(\mathbb{Z}_9)) = 2$.

**Theorem 4.** *If $n > 1$ is odd and relatively prime to 3 , then a connected component of $\Gamma(\mathbb{Z}_n)$ is a cycle or an isolated vertex.*

*Proof.* Since $n$ is odd, vertex 0 is isolated in $\Gamma(\mathbb{Z}_n)$. Let vertex $k$ be a vertex other than vertex 0. Then vertex $k$ is adjacent to verxtex $2k$ and to the vertex $x$ such that $x \equiv ((\frac{(n-1)}{2} + 1))k (\mathrm{mod}\, n)$. Indeed these two vertices are distinct as $n$ is relatively prime to 3. Since $\Delta(\Gamma(\mathbb{Z}_n)) \leq 2$, degree of vertex $k$ is 2. Thus, every vertex other than vertex 0 has degree 2. So all other connected components of $\Gamma(\mathbb{Z}_n)$ are cycles. □

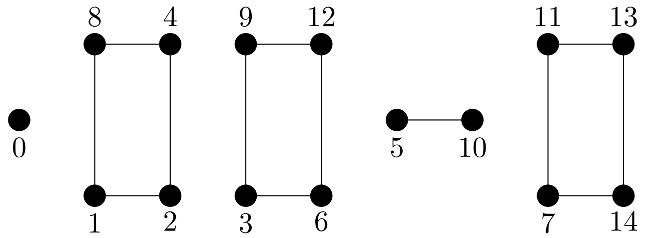
**Example 7.** *Square graph of $\mathbb{Z}_7$ is given below*




**Theorem 5.** *If $n > 1$ is odd and $\gcd(n, 3) = 3$, then a connected component of $\Gamma(\mathbb{Z}_n)$ is a cycle, $K_2$ or an isolated vertex.*

*Proof.* Since $n$ is odd, vertex 0 is isolated in $\Gamma(\mathbb{Z}_n)$. Let vertex $k$ be a vertex other than vertex 0. Then vertex $k$ is adjacent to vertex $2k$ and the vertex corresponding to the solution of the congruence $2x \equiv k (\mathrm{mod}\, n)$. These vertices can be identical if $4k \equiv k (\mathrm{mod}\, n)$. This happens if and only if $k = \frac{n}{3}$ or $k = \frac{2n}{3}$. In that case vertex $k$ and vertex $2k$ form $K_2$ in $\Gamma(\mathbb{Z}_n)$. The remaining vertices have degree 2. So they form cycles in $\Gamma(\mathbb{Z}_n)$. □

**Example 8.** *Consider $\mathbb{Z}_{15}$. Square graph of $\mathbb{Z}_{15}$ is given below*

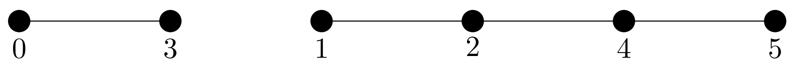



The above theorem is also illustrated in Example 6 for $\mathbb{Z}_9$.

**Theorem 6.** *If $n$ is an even number other than 2 or 6 , then $\Delta(\Gamma(\mathbb{Z}_n)) = 3$.*

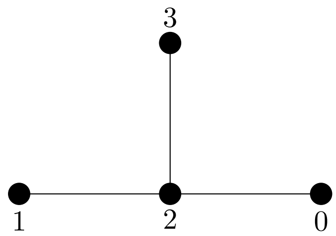
*Proof.* If $n = 4$, vertex 2 is adjacent to vertex 0, vertex 1 and vertex 3. So $\Delta(\Gamma(\mathbb{Z}_4)) = 3$. Now let $n$ be an even number greater than 6. Then vertex 2 is adjacent to vertex 1, vertex 4 and vertex $\left(\frac{n}{2} + 1\right)$. Hence $\Delta(\Gamma(\mathbb{Z}_n)) = 3$. □

**Example 9.** *Consider $\mathbb{Z}_6$. Square graph of $\mathbb{Z}_6$ is*




*So* $\Delta(\Gamma(\mathbb{Z}_6)) = 2$.

**Example 10.** *Square graph of $\mathbb{Z}_4$ is*




*So* $\Delta(\Gamma(\mathbb{Z}_4)) = 3$.

Example 2 also illustrates the above theorem

**Theorem 7.** *If $n$ is even and $n \neq 6$, then the following are true for $\Gamma(\mathbb{Z}_n)$:*

1. *Degree of vertex $0$ is 1.*
2. *If $k$ is odd, then degree of vertex $k$ is $1$.*
3. *If $k$ is even, not equal to zero and $n \mid \frac{3k}{2}$ or $n \mid \left(\frac{3k}{2} - \frac{n}{2}\right)$, then degree of vertex $k$ is $2$. Otherwise, the degree of vertex $k$ is $3$.*

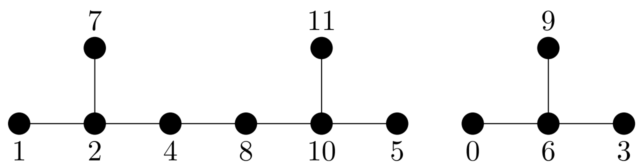
*Proof.* 1. Since $n$ is an even number, vertex $\frac{n}{2}$ is the only adjacent vertex of vertex $0$. So degree of vertex $0$ is $1$.

2. If $k$ is odd and $n$ is even, the congruence $2x \equiv k \pmod{n}$ has no solution. But vertex $k$ is adjacent to vertex $2k$. So degree of vertex $k$ is $1$.

3. If $k$ is even and $n$ is even, the congruence $2x \equiv k \pmod{n}$ has only two solutions, precisely $\frac{k}{2}$ and $\frac{k}{2} + \frac{n}{2}$. So vertex $k$ is adjacent to vertices $\frac{k}{2}$, $\frac{k}{2} + \frac{n}{2}$ and $2k$. If $n \mid \frac{3k}{2}$, then vertex $\frac{k}{2}$ and vertex $2k$ are identical. If $n \mid \left(\frac{3k}{2} - \frac{n}{2}\right)$, then vertex $\left(\frac{3k}{2} - \frac{n}{2}\right)$ and vertex $2k$ are identical. So in both of these cases degree of the vertex $k$ is $2$. Otherwise, the degree of vertex $k$ is $3$

$\square$

**Example 11.** *Square graph of $\mathbb{Z}_{12}$ is*



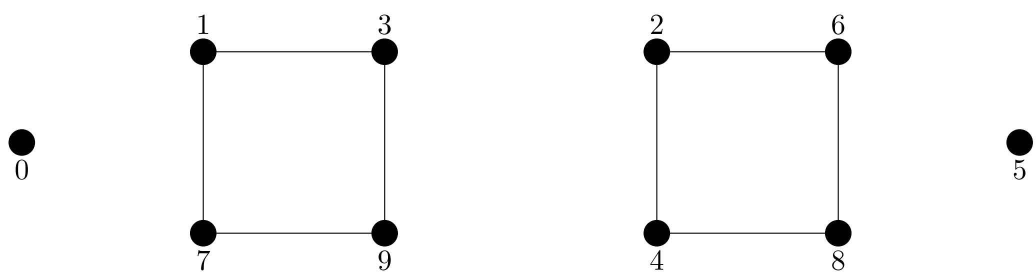

Example 8 also illustrates the above theorem.

## Properties of $p$-Power graph of Finite Cyclic Group

In this section, we examine the maximum degree of the $p$-power graph of finite cyclic groups.

**Definition 3.** *For a finite group $G$ and a prime number $p$, the $p$-power graph $\Gamma_p(G)$ is defined with a vertex set $G$, and its edge set consists of pairs of vertices $a$ and $b$ if either $a^p = b$ or $b^p = a$, and $a$ is not equal to $b$.*

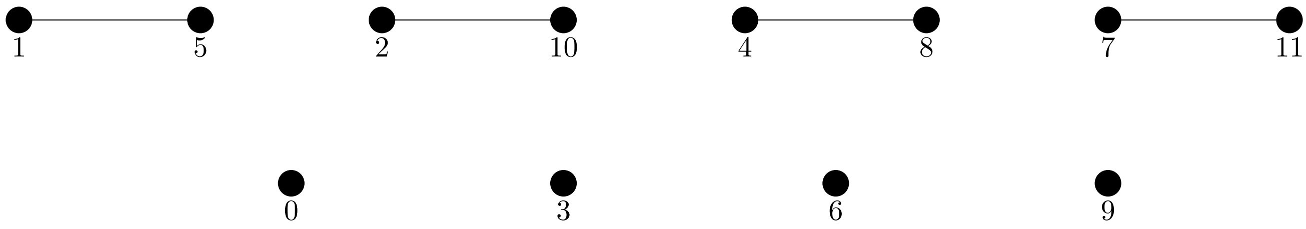
**Example 12.** *The $3$-power graph of $\mathbb{Z}_{10}$ is*




**Theorem 8.** *For a finite cyclic group $G$ and a prime number $p$, $\Delta(\Gamma_p(G)) \leq p + 1$.*

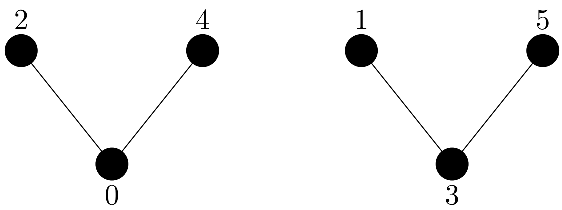
*Proof.* If $G$ is cyclic and $o(G) = n$, then $G \backsimeq \mathbb{Z}_n$ and let the vertices of $\Gamma_p(G)$ be the first $n$ non-negative integers. If k is a vertex in $\Gamma_p(G)$ then k is adjacent to $pk$. All other adjacencies of $k$ comes from the solutions of the congruence relation $px \equiv k (\mathrm{mod}\, n)$. But this congruence has at most $p$ solutions modulo n. Hence $k$ can be adjacent to at most $p + 1$ vertices. $\square$

**Example 13.** *The* 5*-power graph of* $\mathbb{Z}_{12}$ *is*



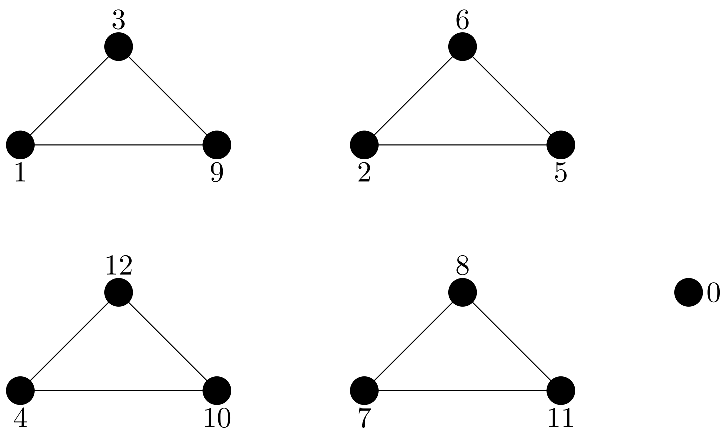

*and* $\Delta(\Gamma_5(\mathbb{Z}_{12})) = 1$.

**Example 14.** *The* 3*-power graph of* $\mathbb{Z}_8$ *is*




*and* $\Delta(\Gamma_3(\mathbb{Z}_6)) = 2$.

**Example 15.** *The* 3*-power graph of* $\mathbb{Z}_{13}$ *is*




*and* $\Delta(\Gamma_3(\mathbb{Z}_6)) = 2$.

**Theorem 9.** *If $n$ is a positive integer and $p$ is a prime number such that $gcd(p, n) = 1$ and $n > p^2$, then $\Delta(\Gamma_p(\mathbb{Z}_n)) = 2$.*

*Proof.* Let $k$ be a vertex of $\Gamma_p(\mathbb{Z}_n)$. If $\gcd(p, n) = 1$ and $1 \mid k$, then the congruence $px \equiv k(\mathrm{mod}\, n)$ has a unique solution. Thus, $\Delta(\Gamma_p(\mathbb{Z}_n)) \leq 2$. Since $n > p^2$, vertex 1 and vertex $p^2$ are adjacent to vertex $p$. Therefore, degree of vertex $p$ is equal to 2. Hence $\Delta(\Gamma_p(\mathbb{Z}_n)) = 2$. $\square$

**Note:** If $\gcd(p, n) = 1$ then the degree of vertex 0 is 0 in $\Gamma_p(\mathbb{Z}_n)$.

**Example 16.** *The 5-power graph of $\mathbb{Z}_{26}$ is given in Figure 1.*
*From the figure it is clear that $\Delta(\Gamma_5(\mathbb{Z}_{26})) = 2$.*

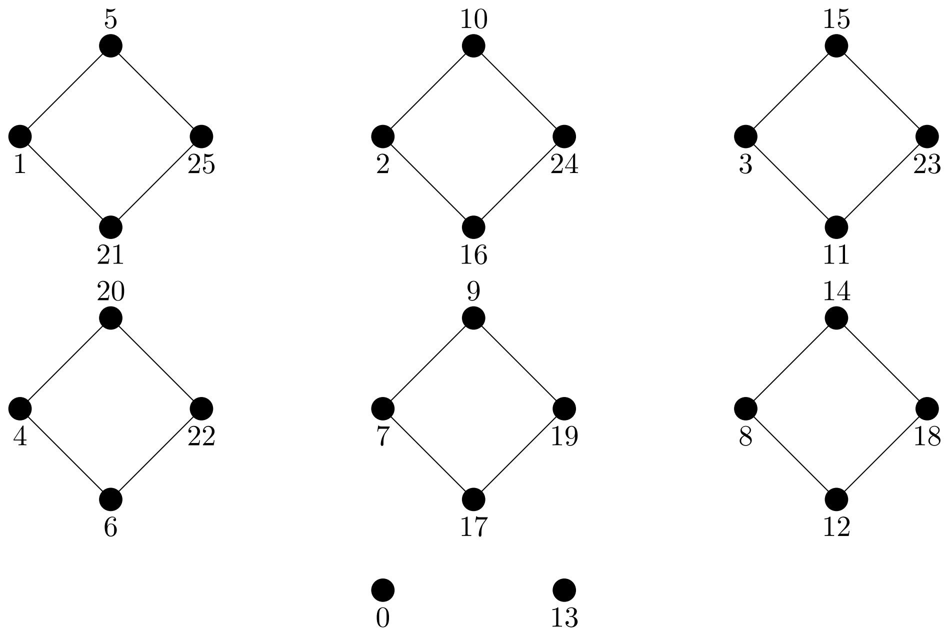



Figure 1: The 5-power graph of $\mathbb{Z}_{26}$: six disjoint 4-cycles and two isolated vertices.

# CONCLUSION

For a finite group $G$, the square graph of $G$ is constructed with a vertex set $G$, and its edge set consists of pairs of distinct vertices $a$ and $b$ if $a^2 = b$ or $b^2 = a$. Maximum degree of square graph of cyclic group of odd order is 2, square graph of cyclic group of order 6 is 2 and all other square graph of cyclic groups of even order is 3. Also connected component of a square graph of cyclic group of odd order is a cycle, $K_2$ or an isolated vertex. For a positive integer $n$ and a prime $p$, the maximum degree of the $p$-power graph $\Gamma_p(\mathbb{Z}_n)$ satisfies $\Delta(\Gamma_p(\mathbb{Z}_n)) \leq p + 1$. If $\gcd(p, n) = 1$ and $n > p^2$, then $\Delta(\Gamma_p(\mathbb{Z}_n)) = 2$.

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