Deferred Statistical Convergence in Paranormed Space

Abstract

Deferred statistical convergence extends statistical convergence by evaluating convergence over intervals determined by two sequences of non-negative integers. This work considers that framework in an arbitrary paranormed space (X, g). It extends deferred statistical convergence from scalar-valued sequences to X-valued sequences through the paranorm g and introduces deferred statistical g-boundedness. The study establishes basic inclusion relations among g-convergent, bounded, deferred statistically g-convergent, and deferred statistically g-bounded sequences. It also characterises deferred statistically g-convergent sequences in terms of subsequences that are g-convergent outside a set of zero deferred density, and shows that such sequences agree, for almost all indices deferred with respect to D, with a g-convergent sequence. A corresponding characterisation is given for deferred statistically g-bounded sequences. In addition, the concept of strongly deferred Cesàro g-summability is introduced. Strong deferred Cesàro g-summability is shown to imply deferred statistical g-convergence, while the reverse implication does not hold in general. For g-bounded sequences, however, deferred statistical g-convergence implies strong deferred Cesàro g-summability to the same limit. Further inclusion results are obtained under conditions on the sequences defining the deferred intervals. Overall, the manuscript places these convergence, boundedness, subsequence, and summability notions within a common paranormed-space setting, clarifies their stated relationships, and provides a unified basis for their comparison.

Keywords: Statistical convergence, deferred density, paranormed space, strong deferred Cesàro summability.

MSC: 40A35, 40A05, 46A45.

1 Introduction

Fast (1951) and Steinhaus (1951) generalised the usual notion of convergence of real-valued sequences by independently introducing the idea of statistical convergence in the same year, 1951. In 1953, Buck (1953) exemplified it as "convergence in density". Schoenberg (1959) studied statistical convergence as a summability method. This concept is also contained in a monograph by Zygmund (1979) as "almost convergence". Later, statistical convergence was investigated from the perspective of sequence spaces and its connection to summability methods by various authors, namely Gupta and Bhardwaj (2018), Connor (1988, 1989), Et and Şengül (2014), Fridy (1985), Işık (2010), Işık and Akbaş (2017), Mursaleen (2000), Rath and Tripathy (1994), Šalát (1980), Temizsu et al. (2016), Bhardwaj and Dhawan (2015), Sharma and Kumar (2025), and many others.

The concept of natural density is the main pillar of the notion of statistical convergence and is defined as follows.

Definition 1.1 (Niven and Zuckerman (1980)). *The natural density of a subset A of the set $\mathbb{N}$ of natural numbers is denoted by $\delta(A)$ and defined as*

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{m \leq n : m \in A\}\right|,$$

provided the limit exists. ***It is easy to verify that***

(i) $\delta(A) = 0$ ***for a finite subset*** A ***of*** $\mathbb{N}$***.***

(ii) $\delta(A) + \delta(\mathbb{N} - A) = 1$ *for any subset* A *of* $\mathbb{N}$*.*

Definition 1.2. ***A scalar-valued sequence*** $u = (u_m)$ ***is said to be statistically convergent to*** $\ell \in \mathbb{R}$ ***if, for every*** $\varepsilon > 0$***,***

$$\delta\big(\{m \in \mathbb{N} : |u_m - \ell| \geq \varepsilon\}\big) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n : |u_m - \ell| \geq \varepsilon\}| = 0.$$

In this case we write $u_m \to \ell(S)$*. The set of all statistically convergent sequences is denoted by* S*.*

Various extensions of statistical convergence, namely λ**-statistical convergence,** A**-statistical convergence,** μ**-statistical convergence,** I**-statistical convergence, lacunary statistical convergence,** f**-statistical convergence, and many others, have been introduced by several mathematicians. One may refer to Aizpuru et al. (2014), Bhardwaj and Gupta (2014), Çolak (2003), Et and Yilmazer (2020), Fridy and Orhan (1993), Nakano (1953), Nuray et al. (2022), and Sharma and Kumar (2025), where many more references can be found.**

Agnew (1932) introduced the concept of the deferred Cesàro mean of scalar-valued sequences $u = (u_m)$ **as follows:**

$$(D_{p,q}u)_n = \frac{1}{q(n) - p(n)} \sum_{m=p(n)+1}^{q(n)} u_m, \qquad n = 1, 2, 3, \ldots, \tag{1}$$

where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of non-negative integers satisfying

$$p(n) < q(n) \quad \text{and} \quad \lim_{n\to\infty} q(n) = \infty. \tag{2}$$

Besides being regular, Agnew (1932) showed that this method has many more important properties.

It was Agnew's work that motivated Küçükaslan and Yılmaztürk (2016) and Yılmaztürk and Küçükaslan (2013) to introduce the concepts of deferred density and deferred statistical convergence as follows.

Definition 1.3 (Küçükaslan and Yılmaztürk (2016); Yılmaztürk and Küçükaslan (2013))**.** *Let* A *be a subset of* $\mathbb{N}$ *and denote the set* $\{m : p(n) < m \leq q(n),\ m \in A\}$ *by* $A_{p,q}(n)$*. The deferred density of* A *is defined as*

$$\delta_D(A) = \lim_{n\to\infty} \frac{1}{q(n) - p(n)} |A_{p,q}(n)|,$$

provided the limit exists. It is observed that

(i) if $q(n) = n$*,* $p(n) = 0$*, then deferred density* $\delta_D(A)$ *coincides with natural density of* A*, i.e.,* $\delta_D(A) = \delta(A)$*;*

(ii) if $A_1 \subset A_2$*, then* $\delta_D(A_1) \leq \delta_D(A_2)$*;*

(iii) $\delta_D(A) + \delta_D(\mathbb{N} - A) = 1$*.*

Definition 1.4 (Gupta and Bhardwaj (2018))**.** ***A property*** P ***holds for almost all*** m ***deferred with respect to*** D ***for the sequence*** $u = (u_m)$ ***if the set of indices*** m ***for which*** u_m ***does not satisfy property*** P ***has zero deferred density.*** *We write this as* u_m *satisfies property* P *as a.a.* m *w.r.t.* D*.*

Definition 1.5 (Küçükaslan and Yılmaztürk (2016); Yılmaztürk and Küçükaslan (2013))**.** ***A scalar-valued sequence*** $u = (u_m)$ ***is said to be deferred statistically convergent to some scalar*** $\ell \in \mathbb{R}$ ***if, for each*** $\varepsilon > 0$***,***

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} |\{p(n) < m \leq q(n) : |u_m - \ell| \geq \varepsilon\}| = 0,$$

i.e.,

$$|u_m - \ell| \leq \varepsilon \qquad a.a.\ m\ w.r.t.\ D.$$

In this case we write $u_m \to \ell(S_D)$. We denote the set of all deferred statistically convergent scalar sequences by S_D. If $q(n) = n$ and $p(n) = 0$, then deferred statistical convergence coincides with usual statistical convergence.

The notion of a paranorm is a generalisation of the absolute value function and is defined as follows.

Definition 1.6. ***A paranorm is a function $g : X \to \mathbb{R}$ defined on a linear space X such that, for all $z_1, z_2, z_3 \in X$,***

(p_1) $g(z) = 0$ *if* $z = \theta$,

(p_2) $g(-z) = g(z)$,

(p_3) $g(z_1 + z_2) \leq g(z_1) + g(z_2)$,

(p_4) *if* (α_n) *is a sequence of scalars with* $\alpha_n \to \alpha_0$ $(n \to \infty)$ *and* $z_n, z' \in X$ *with* $z_n \to z'$ $(n \to \infty)$ *in the sense that* $g(z_n - z') \to 0$ $(n \to \infty)$, *then* $\alpha_n z_n \to \alpha_0 z'$ $(n \to \infty)$ *in the sense that* $g(\alpha_n z_n - \alpha_0 z') \to 0$ $(n \to \infty)$.

A paranorm g for which $g(z) = 0$ implies $z = \theta$ is called a total paranorm on X, and the pair (X, g) is called a total paranormed space. We recall that each seminorm (norm) g on X is a paranorm (total paranorm), but the converse need not be true.

Alotaibi and Alroqi (2012) generalised the concept of statistical convergence and studied strong Cesàro summability and the statistical Cauchy property in a paranormed space. Alghamdi and Mursaleen (2013) and Ercan (2021) further generalised this concept by introducing λ-statistical convergence and the deferred Cesàro mean in a paranormed space. Let us recall some definitions in the paranormed space (X, g).

Definition 1.7. ***Let (u_m) be a sequence in (X, g) and $u_0 \in X$. If, for a given $\varepsilon > 0$, there exists a positive integer m_0 such that $g(u_m - u_0) < \varepsilon$ for all $m \geq m_0$, then we say that (u_m) is convergent to u_0 and write***

$$g\text{-}\lim_m u_m = u_0.$$

We shall denote the class of all convergent sequences by $c(g)$.

Definition 1.8. *A sequence (u_m) in (X, g) is said to be bounded if there exists some $M > 0$ such that $g(u_m) < M$ for all $m \geq 1$. We write ℓ_∞^g as the class of all bounded sequences.*

In the present paper, we extend the notion of deferred statistical convergence of scalar-valued sequences introduced by Küçükaslan and Yılmaztürk (2016) and Yılmaztürk and Küçükaslan (2013) to that of X-valued sequences through the paranorm g on X, where X is any non-empty set. It is shown that the terms of a deferred statistically g-convergent sequence (u_m) coincide with those of a g-convergent sequence for almost all m deferred with respect to D. Finally, we conclude the paper by introducing the concept of strongly deferred Cesàro g-summable sequences, and it is shown that a g-bounded sequence that is deferred statistically g-convergent is strongly deferred Cesàro g-summable to the same limit.

Throughout the paper, we consider the sequences of non-negative integers $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ satisfying $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$.

Recent work has extended statistical and deferred statistical convergence to n-normed spaces, generalised difference sequences of order alpha, and L-fuzzy normed spaces (Khan & Shatarah, 2020; León-Saavedra et al., 2025; Rahaman & Mursaleen, 2024).

Related developments in summability theory have also considered generalised matrix summability for infinite series using power-increasing sequences and Cesàro means (Ozarslan, 2026).

Research gap. The manuscript brings deferred density, g-convergence, g-boundedness, deferred statistically dense subsequences, and strong deferred g-Cesàro summability into a common paranormed-space framework; the relationships among these notions therefore require explicit formulation and comparison within the setting adopted here.

Objective. The objective of this study is to define deferred statistical g-convergence and deferred statistical g-boundedness in a paranormed space, characterise deferred statistically

g-convergent sequences through deferred statistically dense g-convergent subsequences, and examine the relation between deferred statistical g-convergence and strong deferred g-Cesàro summability for g-bounded sequences.

2 Deferred g-statistical convergence

We start this section by introducing the concepts of deferred statistical g-convergence and deferred statistical g-boundedness. In addition, various inclusion relations between these concepts are established.

Definition 2.1. *Let $u = (u_m)$ be a sequence in (X, g).* ***The sequence (u_m) is said to be deferred statistically g-convergent to some $u_0 \in X$ if, for a given $\varepsilon > 0$,***

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : p(n) + 1 \leq m \leq q(n),\ g(u_m - u_0) \geq \varepsilon\}\right| = 0,$$

provided the limit exists, and we write

$$u_m \xrightarrow{S_D^g(c)} u_0.$$

We denote by $S_D^g(c)$ the set of all deferred statistically g-convergent sequences from (X, g).

Definition 2.2. *A sequence $u = (u_m)$ in the paranormed space (X, g) is said to be deferred statistically g-bounded if there exists $M > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : p(n) + 1 \leq m \leq q(n),\ g(u_m) > M\}\right| = 0,$$

provided the limit exists. ***We denote the set of all deferred statistically g-bounded sequences from (X, g) by $S_D^g(b)$.***

Remark 2.3. ***It is observed that we have the following special cases.***

(i) ***For $q(n) = n$ and $p(n) = 0$, Definition 2.1 coincides with the definition of statistical convergence in a paranormed space defined and studied by Alotaibi and Alroqi (2012).***

(ii) ***For $p(n) = n - \lambda_n + 1$ and $q(n) = n$, Definitions 2.1 and 2.2 reduce to those of λ-statistical convergence and λ-statistical boundedness in a paranormed space, which were studied and defined by Alghamdi and Mursaleen (2013).***

(iii) ***For a paranorm g defined on $X = \mathbb{R}$ as $g(z) = |z|$, $z \in \mathbb{R}$, and $q(n) = m_n$ and $p(n) = m_{n-1}$, where $\theta = (m_n)$ is a lacunary sequence, Definitions 2.1 and 2.2 turn out to be the definitions of lacunary statistical convergence and lacunary statistical boundedness defined by Fridy and Orhan (1993).***

(iv) ***For a paranorm g defined on $X = \mathbb{R}$ as $g(z) = |z|$, $z \in \mathbb{R}$, $q(n) = n$, and $p(n) = 0$, Definition 2.1 coincides with the definition of statistical convergence defined and studied by Šalát (1980).***

Theorem 2.4. *Every g-convergent sequence is deferred statistically g-convergent, i.e., $c(g) \subset S_D^g(c)$.* ***The converse is not true in general.***

Proof. **The result follows from the fact that a finite set has zero deferred density.** For proper inclusion, consider a sequence $u = (u_m)$ in the paranormed space $X = \mathbb{R}$ with paranorm defined as $g(\xi) = |\xi|$ for all $\xi \in \mathbb{R}$ as

$$u_m = \begin{cases} q(n), & \text{if } m = q(n), \\ 0, & \text{if } m \neq q(n). \end{cases}$$

Then, for given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{m \in \mathbb{N} : p(n) + 1 \leq m \leq q(n),\ g(u_m - 0) \geq \varepsilon\}\right| = \lim_{n\to\infty} \frac{1}{q(n) - p(n)} = 0.$$

Hence (u_m) is deferred statistically g-convergent to 0. **Suppose, if possible, that (u_m) is g-convergent to 0. Then, for a given $\varepsilon > 0$,**

$$g(u_m - 0) < \varepsilon \quad \text{for all } m \geq m_0,$$

i.e.,

$$q(m) < \varepsilon \quad \text{for all } m \geq m_0, \quad m = q(n), \quad n \in \mathbb{N},$$

contrary to the fact that $q(m) \to \infty$ as $m \to \infty$. □

Theorem 2.5. *Every bounded sequence is deferred statistically g-bounded.* ***The converse is not true in general.***

Proof. **The proof is obvious because the empty set has zero deferred density, i.e., $\ell_\infty^g \subset S_D^g(b)$. The inclusion is proper in view of the example cited in Theorem 2.4.** □

Theorem 2.6. *Every deferred statistically g-convergent sequence is deferred statistically g-bounded, i.e., $S_D^g(c) \subset S_D^g(b)$; the inclusion is strict.*

Proof. **The proof is an immediate consequence of the fact that every finite set has zero deferred density.** For proper inclusion, consider $\mathbb{R}$ with paranorm defined as $g(\xi) = |\xi|$ for all $\xi \in \mathbb{R}$. Then the sequence (u_m) defined as

$$u_m = \begin{cases} 1, & \text{if } m = 2n, \\ -1, & \text{otherwise} \end{cases}$$

is deferred statistically g-bounded but not deferred statistically g-convergent. □

Theorem 2.7. *Every g-convergent sequence is deferred statistically g-bounded.*

Proof. **The proof follows immediately from Theorems 2.4 and 2.6.** □

Theorem 2.8 (Gupta and Bhardwaj (2018))**.** *Let f be an unbounded modulus. Then*

$$S_D^f\text{-}\lim u_m = \ell$$

if and only if there exists $T \subset \mathbb{N}$ such that $\delta_D^f(T) = 0$ and

$$\lim_{m \in \mathbb{N} - T} u_m = \ell.$$

By taking $f(z) = z$ in the above theorem, we have the following.

Theorem 2.9. ***A real-valued sequence (u_m) is deferred statistically convergent to $\ell \in \mathbb{R}$ if and only if there exists $T \subset \mathbb{N}$ such that $\delta_D(T) = 0$ and***

$$\lim_{m \in \mathbb{N} - T} u_m = \ell.$$

The following theorem shows that a deferred statistically g-convergent sequence in a paranormed space (X, g) has a structure similar to that in Theorem 2.9 for a real-valued deferred statistically convergent sequence.

Theorem 2.10. *A sequence (u_m) is deferred statistically g-convergent to $u_0 \in X$ if and only if there exists $T \subset \mathbb{N}$ such that $\delta_D(T) = 0$ and*

$$(u_m)_{m \in \mathbb{N} - T} \xrightarrow{g} u_0.$$

Proof. **Since (u_m) is deferred statistically g-convergent to $u_0 \in X$, for a given $\varepsilon > 0$,**

$$\lim_{n \to \infty} \frac{1}{q(n) - p(n)} \left| \{ m \in \mathbb{N} : p(n) + 1 \leq m \leq q(n),\ g(u_m - u_0) \geq \varepsilon \} \right| = 0,$$

i.e.,

$$\lim_{n \to \infty} \frac{1}{q(n) - p(n)} \left| \{ m \in \mathbb{N} : p(n) + 1 \leq m \leq q(n),\ z_m \geq \varepsilon \} \right| = 0,$$

where $z_m = g(u_m - u_0)$. **This implies that (z_m) is a deferred statistically convergent sequence of real numbers converging to zero. Using Theorem 2.9, there exists $T \subset \mathbb{N}$ such that $\delta_D(T) = 0$ and**

$$\lim_{m \in \mathbb{N} - T} z_m = 0,$$

i.e.,

$$\lim_{m \in \mathbb{N} - T} g(u_m - u_0) = 0.$$

Conversely, for a given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $g(u_m - u_0) < \varepsilon$ for all $m \in \mathbb{N} - T$ with $m \geq m_0$. The proof now follows immediately from the inclusion relation

$$\{m \in \mathbb{N} : g(u_m - u_0) \geq \varepsilon\} \subset T \cup \{1, 2, 3, \ldots, m_0 - 1\}.$$

□

The next theorem indicates that the terms of a deferred statistically g-convergent sequence coincide with those of a convergent sequence for almost all m deferred with respect to D.

Theorem 2.11. *A sequence (u_m) is deferred statistically g-convergent to $u_0 \in X$ if and only if there exists a sequence (v_m), g-convergent to u_0, such that $v_m = u_m$ a.a. m deferred w.r.t. D.*

Proof. **Let (u_m) be deferred statistically g-convergent to $u_0 \in X$; then we have $\delta_D(A) = 0$, where**

$$A = \{m \in \mathbb{N} : g(u_m - u_0) \geq \varepsilon\}.$$

Take

$$v_m = \begin{cases} u_0, & \text{if } m \in A, \\ u_m, & \text{if } m \in \mathbb{N} - A. \end{cases}$$

Now $\{m \in \mathbb{N} : v_m \neq u_m\} \subseteq A$ and so $v_m = u_m$ a.a.m. deferred w.r.t. D. Also

$$g(y_m - z_0) = \begin{cases} g(u_m - u_0), & \text{for } m \in \mathbb{N} - A, \\ 0, & \text{for } m \in A, \end{cases}$$

implies $g(v_m - u_0) < \varepsilon$ for all $m \geq 1$. Thus (v_m) is convergent to u_0 and $v_m = u_m$ a.a.m. deferred w.r.t. D.

Conversely, for a given $\varepsilon > 0$, there exists a positive integer m_0 such that $g(v_m - u_0) < \varepsilon$ for all $m \geq m_0$. The proof follows in view of the following inclusion:

$$\{m \in \mathbb{N} : g(u_m - u_0) > \varepsilon\} \subset \{m \in \mathbb{N} : u_m \neq v_m\} \cup \{1, 2, 3, \ldots, m_0 - 1\}.$$

□

Following similar lines, we have the following.

Theorem 2.12. *A sequence (u_m) is deferred statistically g-bounded if and only if there exists a sequence (v_m), g-bounded, such that $v_m = u_m$ a.a. m deferred w.r.t. D.*

3 Strongly deferred Cesàro g-summable sequences

We begin this section by introducing the notion of strongly deferred Cesàro g-summable sequences and observe that bounded deferred statistically g-convergent sequences are strongly deferred Cesàro g-summable.

Definition 3.1. *Let (X, g) be a paranormed space. A sequence (u_m) in (X, g) is said to be strongly deferred Cesàro g-summable to some $u_0 \in X$ if*

$$\lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{m=p(n)+1}^{q(n)} g(u_m - u_0) = 0.$$

We write it as $u_m \to u_0(w_D^g)$, and by $w_D^g(c)$ we mean the set of all strongly deferred Cesàro g-summable sequences in the paranormed space (X, g).

Theorem 3.2. *Every strongly deferred Cesàro g-summable sequence is deferred statistically g-convergent to the same limit, i.e., $w_D^g(c) \subset S_D^g(c)$.* ***The reverse implication need not be true.***

Proof. For any $\varepsilon > 0$ and $u_0 \in X$, we have

$$\frac{1}{q(n)-p(n)} \sum_{m=p(n)+1}^{q(n)} g(u_m - u_0) \geq \frac{1}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)\geq\varepsilon}}^{q(n)} g(u_m - u_0)$$
$$\geq \varepsilon \frac{1}{q(n)-p(n)} \left|\{m : p(n) < m \leq q(n),\ g(u_m - u_0) \geq \varepsilon\}\right|,$$

and hence the result follows.

To show that the inclusion is proper, consider the following. Let $X = \mathbb{R}$ with paranorm g defined as $g(\xi) = |\xi|$. Take a sequence (u_m) as

$$u_m = \begin{cases} (q(n)-p(n))^2, & \text{if } m = q(n),\ n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{q(n)-p(n)} \left|\{m : p(n) < m \leq q(n),\ g(u_m - 0) \geq \varepsilon\}\right| \leq \frac{1}{q(n)-p(n)} \to 0 \quad \text{as } n \to \infty$$

implies $(u_m) \in S_D^g(c)$. However, $(u_m) \notin w_D^g(c)$ as

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{m=p(n)+1}^{q(n)} g(u_m - 0) = \frac{(q(n)-p(n))^2}{q(n)-p(n)} \to \infty.$$

However, if we take g-bounded sequences, then the converse of Theorem 3.2 holds. □

Theorem 3.3. *If $(u_m) \in \ell_\infty^g$, then $S_D^g(c) \subseteq w_D^g(c)$.*

Proof. **Since $(u_m) \in \ell_\infty^g$, there exists $M > 0$ such that $g(u_m) \leq M$ for all $m \geq 1$.** Then, for $u_0 \in X$, we have

$$g(u_m - u_0) \leq g(u_m) + g(u_0) \leq M + g(u_0) \leq K,$$

where $K = M + g(u_0)$ (say). Hence

$$\frac{1}{q(n)-p(n)} \sum_{m=p(n)+1}^{q(n)} g(u_m - u_0) = \frac{1}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)\geq\varepsilon}}^{q(n)} g(u_m - u_0)$$
$$+ \frac{1}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)<\varepsilon}}^{q(n)} g(u_m - u_0)$$
$$\leq \frac{K}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)\geq\varepsilon}}^{q(n)} 1 + \frac{\varepsilon}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)<\varepsilon}}^{q(n)} 1$$
$$\leq \frac{K}{q(n)-p(n)} \left|\{m : p(n) < m \leq q(n),\ g(u_m - u_0) \geq \varepsilon\}\right| + \varepsilon,$$

and hence the result follows. □

In view of Theorems 3.2 and 3.3, we have the following.

Corollary 3.4. $\ell_\infty^g \cap S_D^g(c) = \ell_\infty^g \cap w_D^g(c)$.

Theorem 3.5. *Let $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}} \in \ell_\infty$. If $u_m \xrightarrow{S^g(c)} u_0$, then $u_m \xrightarrow{S_D^g(c)} u_0$, where ℓ_∞ is the space of all bounded scalar sequences.*

Proof. Let $(u_m) \in S^g(c)$. Then, for a given $\varepsilon > 0$, we have

$$
\begin{aligned}
&|\{m : p(n)+1 \le m \le q(n) : g(u_m - u_0) \ge \varepsilon\}| \\
&= |\{m : 1 \le m \le q(n) : g(u_m - u_0) \ge \varepsilon\}| - |\{m : 1 \le m \le p(n) : g(u_m - u_0) \ge \varepsilon\}|,
\end{aligned}
$$

and so

$$
\begin{aligned}
&\frac{1}{q(n)-p(n)} |\{m : p(n)+1 \le m \le q(n) : g(u_m - u_0) \ge \varepsilon\}| \\
&= \frac{q(n)}{q(n)-p(n)} \frac{1}{q(n)} |\{m : 1 \le m \le q(n) : g(u_m - u_0) \ge \varepsilon\}| \\
&\quad - \frac{p(n)}{q(n)-p(n)} \frac{1}{p(n)} |\{m : 1 \le m \le p(n) : g(u_m - u_0) \ge \varepsilon\}|.
\end{aligned}
$$

The result follows from the boundedness of the following two sequences:

$$
\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}} \quad \text{and} \quad \left\{1+\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}.
$$

□

Remark 3.6. ***The converse of Theorem 3.5 need not be true even if*** $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ ***is bounded, i.e.,*** $S_D^g(c) \nsubseteq S^g(c)$***. Let us demonstrate this with the following example.*** *Let* $X = \mathbb{R}$ *with paranorm* g *defined on it as* $g(\xi) = |\xi|$ *for* $\xi \in X$*. Take* $p(n) = n$*.* ***Choose a positive integer*** $q(1)$ ***such that*** $q(1) > p(1) = 1$***. Next, choose a positive integer*** $q(2)$ ***with*** $q(2) > 2p(2) = 2^2$***, and so on; i.e., positive integers*** $q(n)$ ***are chosen such that*** $q(n) > np(n) = n^2$ ***for each positive integer*** $n \in \mathbb{N}$***.*** *Clearly, the sequence* $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ *is bounded. Take the sequence* $\{u_m\}$ *defined as*

$$
u_m = \begin{cases} 1, & \textit{if } p(n) < m \le 2p(n), \\ 0, & \textit{otherwise}, \end{cases} \qquad \textit{on } (p(n), q(n)].
$$

For $n \ge 2$*, we have* $q(n) > np(n) \ge 2p(n) > p(n)$*, and so*

$$
\begin{aligned}
\frac{1}{q(n)-p(n)} |\{m : p(n)+1 < m \le q(n) : g(u_m - 0) \ge \varepsilon\}| &= \frac{p(n)}{q(n)-p(n)} \\
&< \frac{\frac{1}{n}q(n)}{q(n)-p(n)} = \frac{\frac{1}{n}}{1-\frac{p(n)}{q(n)}} \to 0 \quad \textit{as } n \to \infty.
\end{aligned}
$$

Thus the sequence $u_m \to 0$ $(S_D^g(c))$*; however,*

$$
\frac{1}{2p(n)} |\{m \le 2p(n) : g(u_m - 0) \ge \varepsilon\}| \ge \frac{2p(n)-p(n)}{2p(n)} = \frac{1}{2} \quad \textit{for each } n \in \mathbb{N},
$$

which implies $u_m \nrightarrow 0$ $(S^g(c))$*.*

We now demonstrate that the converse of Theorem 3.5 for a particular choice of the sequence $q(n)$**, i.e., for** $q(n) = n$**, holds in the form of the following theorem.**

Theorem 3.7. *Let* $q(n) = n$ *and* $p(n)$ *be an arbitrary sequence satisfying (2). Then* $u_m \xrightarrow{S_D^g(c)} u_0$ *implies* $u_m \xrightarrow{S^g} u_0$*.*

Proof. **Let** $n \in \mathbb{N}$ **be arbitrary but fixed. Then, for this** n**, we have** $n = q(n)$**. Set** $n^{(1)} = p(n)$**. Now** $n^{(1)} = q(n^{(1)})$**; set** $n^{(2)} = p(n^{(1)})$**, and so on. Thus, we have a decreasing sequence**

$$
n = q(n) > p(n) = n^{(1)} = q(n^{(1)}) > p(n^{(1)}) = n^{(2)} = q(n^{(2)}) > p(n^{(2)}) = n^{(3)} > \cdots.
$$

Moreover,

$$\begin{aligned}\{m \le n : g(u_m - u_0) \ge \varepsilon\} &= \{m \le n^{(1)} : g(u_m - u_0) \ge \varepsilon\} \cup \{n^{(1)} < m \le n : g(u_m - u_0) \ge \varepsilon\} \\ &= \{m \le n^{(2)} : g(u_m - u_0) \ge \varepsilon\} \cup \{n^{(2)} < m \le n^{(1)} : g(u_m - u_0) \ge \varepsilon\} \\ &\quad \cup \{n^{(1)} < m \le n : g(u_m - u_0) \ge \varepsilon\} \\ &\vdots \\ &= \{m \le n^{(h)} : g(u_m - u_0) \ge \varepsilon\} \cup \{n^{(h)} < m \le n^{(h-1)} : g(u_m - u_0) \ge \varepsilon\} \\ &\quad \cup \cdots \cup \{n^{(2)} < m \le n^{(1)} : g(u_m - u_0) \ge \varepsilon\} \\ &\quad \cup \{n^{(1)} < m \le n : g(u_m - u_0) \ge \varepsilon\}\end{aligned}$$

for a certain positive integer h depending on n such that $n^{(h)} \ge 1$ and $n^{(h+1)} = 0$. Thus

$$\begin{aligned}\frac{1}{n}\left|\{m \le n : g(u_m - u_0) \ge \varepsilon\}\right| &= \sum_{m=0}^{h} \frac{1}{n}\left|\{n^{(m+1)} < m \le n^{(m)} : g(u_m - u_0) \ge \varepsilon\}\right| \\ &= \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n(n^{(m)} - n^{(m+1)})}\left|\{n^{(m+1)} < m \le n^{(m)} : g(u_m - u_0) \ge \varepsilon\}\right| \\ &= \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n(q(n^{(m)}) - p(n^{(m)}))}\left|\{p(n^{(m)}) < m \le q(n^{(m)}) : g(u_m - u_0) \ge \varepsilon\}\right| \\ &= \sum_{m=0}^{h} a_{n,m} t_m,\end{aligned}$$

where $n^{(0)} = n$,

$$a_{n,m} = \begin{cases} \dfrac{n^{(m)} - n^{(m+1)}}{n}, & \text{if } m = 0, 1, 2, \ldots, h, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$t_m = \begin{cases} \dfrac{1}{q(n^{(m)}) - p(n^{(m)})}\left|\{p(n^{(m)}) < m \le q(n^{(m)}) : g(u_m - u_0) \ge \varepsilon\}\right|, & \text{if } m = 0, 1, 2, \ldots, h, \\ 0, & \text{otherwise.} \end{cases}$$

Since (t_m) is a null sequence, the result follows by the Silverman–Toeplitz Theorem. □

Corollary 3.8. ***If we consider*** $q(n) = n$ ***and*** $p(n)$ ***to be an arbitrary sequence satisfying (2) such that*** $\left\{\dfrac{p(n)}{q(n) - p(n)}\right\}_{n \in \mathbb{N}} \in \ell_\infty$***, then*** $S^g(c) = S^g_D(c)$.

Proof. **The proof is an immediate consequence of Theorems 3.5 and 3.7.** □

Proceeding along the same lines as in Theorems 2.3.1 and 2.3.2 of Küçükaslan and Yılmaztürk (2016), we have the following.

Theorem 3.9. *Let $\{p(n)\}$, $\{q(n)\}$, $\{p'(n)\}$, and $\{q'(n)\}$ be sequences of natural numbers satisfying*

$$p(n) \le p'(n) < q'(n) \le q(n), \tag{3}$$

$$\lim_{n\to\infty} q(n) - p(n) = \infty = \lim_{n\to\infty} q'(n) - p'(n), \tag{4}$$

and both sequences $\{p'(n) - p(n)\}_{n\in\mathbb{N}}$ and $\{q'(n) - q(n)\}_{n\in\mathbb{N}}$ are in ℓ_∞. Then $S^g_{p',q'}(c) \subset S^g_{p,q}(c)$.

Theorem 3.10. *Let $\{p(n)\}$, $\{q(n)\}$, $\{p'(n)\}$, and $\{q'(n)\}$ be sequences of natural numbers satisfying conditions (2) and (3) of Theorem 3.9 such that*

$$\liminf_{n} \frac{q(n) - p(n)}{q'(n) - p'(n)} > 0.$$

Then $S^g_{p,q}(c) \subset S^g_{p',q'}(c)$.

Conclusion

This study extends deferred statistical convergence from scalar-valued sequences to sequences in an arbitrary paranormed space through a paranorm g. Within this framework, deferred statistical g-convergence and deferred statistical g-boundedness are defined, and their principal inclusion relations are established. The results show that every g-convergent sequence is deferred statistically g-convergent and that every deferred statistically g-convergent sequence is deferred statistically g-bounded, while the corresponding converses need not hold in general. Deferred statistically g-convergent sequences are further characterised by g-convergent subsequences after excluding a set of zero deferred density and by agreement with a g-convergent sequence for almost all indices deferred with respect to D. The analogous boundedness characterisation is also obtained. The manuscript additionally introduces strong deferred Cesàro g-summability and establishes its relation to deferred statistical g-convergence. Strong deferred Cesàro g-summability implies deferred statistical g-convergence, and for g-bounded sequences the converse also holds. Further inclusion relations are obtained under the stated conditions on the deferred-index sequences, completing the comparison of the convergence and summability concepts considered.

Limitations

This study is theoretical and is limited to sequences in paranormed spaces under the specified conditions imposed on the deferred-index sequences $p(n)$ and $q(n)$. Several converse results do not hold without additional assumptions, while some equivalences require g-boundedness or further boundedness conditions involving $p(n)$ and $q(n)$. The analysis is confined to deferred statistical g-convergence, g-boundedness, subsequence characterisations, and strong deferred Cesàro g-summability. Future studies may examine whether the established results remain valid under weaker conditions or in broader generalised convergence frameworks.

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