NUMERICAL RANGE AND ROBUSTNESS OF GRASSMANNIAN FRAMES

2020 *Mathematics Subject Classification.* Primary 46B15, 47A12, 42C15; Secondary 47A10, 47A99.

Key words and phrases. Grassmannian frames, frame operator, Gramian operator, erasure robustness, numerical range, numerical radius, Rayleigh map, complex projective space.

ABSTRACT. This paper establishes an operator-theoretic and geometric framework for the study of finite-dimensional Grassmannian frames through the lens of numerical range theory and complex projective geometry. We investigate the numerical ranges of the frame operator S and the Gramian operator G, demonstrating that both sets form compact real intervals precisely governed by their spectral extrema. By employing the Rayleigh map over complex projective space, we link the geometric structure of the numerical range to optimal equiangular frame configurations and prove that frame tightness corresponds exactly to the collapse of $W(S)$ to a single degenerate point. Furthermore, we provide a complete operator-theoretic characterisation of erasure resilience, proving that equal-norm Parseval frames are uniquely optimal for 1-erasure robustness and deriving sharp resilience bounds for α-tight Grassmannian configurations.

1. INTRODUCTION

Finite frame theory, originally introduced by Duffin and Schaeffer (1952) in non-harmonic Fourier series and later expanded by Daubechies et al. (1986) for wavelet expansions, provides the mathematical framework for redundant vector representations in finite-dimensional complex Hilbert spaces $H = \mathbb{C}^m$. Unlike orthonormal bases, which force unique expansions, frames consist of an overcomplete collection of vectors $\Phi = \{v_j\}_{j=1}^{M} \subset \mathbb{C}^m$ ($M \geq m$) offering superior resilience to noise and data erasures compared to traditional bases. Among all frame configurations, frames that minimise the maximum cross-correlation among distinct vectors (**Grassmannian frames**) are optimal for applications ranging from sparse recovery to codebook design in wireless communications.

Recent operator-theoretic work on frame operators has further examined how frame bounds, redundancy, tightness, and duality interact with norm-attaining behaviour in Hilbert spaces (Evans & Moraa, 2025).

This structural redundancy is invaluable across modern applied mathematics and engineering. In quantum information and quantum tomography, maximal equiangular tight frames consisting of d^2 vectors in $\mathbb{C}^d$ form Symmetric Informationally Complete Positive Operator-Valued Measures (SIC-POVMs) (Renes et al., 2004), enabling optimal quantum state reconstruction with minimal statistical variance. In 5G/6G wireless communications and multi-user MIMO beamforming, precoding codebooks are constructed as line packings over $\mathbb{C}^m$ (Love et al., 2003; Strohmer & Heath, 2003), where minimising maximal cross-correlation between codebook vectors directly suppresses multi-user interference and maximises channel capacity over fading channels.

Furthermore, in compressed sensing and sparse signal recovery, the performance of sparse recovery algorithms such as Orthogonal Matching Pursuit and Basis Pursuit is governed by the Restricted Isometry Property (RIP) and the mutual coherence parameter $\mu(\Phi)$ (Candès & Tao, 2006; Donoho, 2006; Tropp, 2004), where a small mutual coherence guarantees exact recovery of k-sparse signals whenever $\mu(\Phi) < 1/(2k-1)$.

A fundamental quest in frame theory is to find unit-norm frames $\Phi = \{v_j\}_{j=1}^{M} \subset C^m$ that minimise the maximal pairwise inner product magnitude, known as the *mutual coherence*:

$$\mu(\Phi) = \max_{1 \le j < k \le M} |\langle v_j, v_k \rangle_{C^m}|.$$

Configurations achieving this minimum are called *Grassmannian frames*. Geometrically, Grassmannian frames correspond to optimal packings of complex lines in the complex projective space CP^{m-1} equipped with the Fubini-Study metric. When a Grassmannian frame meets the fundamental Welch lower bound with equality, it forms an *Equiangular Tight Frame* (ETF). However, ETFs only exist for very restricted combinations of dimension m and number of vectors $M \le m^2$. For general parameters (M, m), Grassmannian frames serve as optimal generalisations, balancing mutual coherence and energy distribution.

The geometric, combinatorial, and algebraic aspects of line packings have been studied extensively (Haas & Casazza, 2017; Strohmer & Heath, 2003), but their operator-theoretic structure remains less systematically explored. Key operator-theoretic invariants of a frame Φ are embodied by its *analysis operator* $T : \mathbb{C}^m \to \mathbb{C}^M$, *synthesis operator* $T^* : \mathbb{C}^M \to \mathbb{C}^m$, *frame operator* $S = T^*T \in \mathcal{M}_m(\mathbb{C})$, and *Gramian operator* $G = TT^* \in \mathcal{M}_M(\mathbb{C})$. While the condition number $\kappa(S) = \beta/\alpha$ traditionally governs frame robustness, operator theory provides another powerful tool: the **Numerical Range** and the **Numerical Radius**. For a linear operator A acting on a complex Hilbert space H, its numerical range is defined as:

$$W(A) = \{\langle Av, v \rangle_H : v \in H, ||v||_H = 1\}.$$

Recent work on multiple summing multilinear operators also illustrates the continuing use of structural operator properties and operator ideals in functional-analytic settings (Yadav, 2025).

When signals transmitted via frame coefficients suffer vector deletions (erasures), the worst-case reconstruction error is dictated by the numerical range and operator norm of sub-collection frame operators $S_D = T^*DT$, where $D \in M_M(C)$ is a diagonal masking matrix modelling erasures over C^M. This paper bridges frame theory and operator theory by establishing how the geometric shape, boundary properties, and inclusion regions of the numerical range of the frame operator and its associated Gram matrix dictate the robustness and optimality of Grassmannian frames.

The recovery of signals from incomplete frame coefficients remains an active problem, including the construction of non-canonical dual frames that compensate for erasures (Arambašić & Stoeva, 2024).

Recent results also connect erasure robustness of finite frames with structured matrix properties and deterministic full-spark constructions (Li, 2026).

Organisation of the Paper. The main contributions and structure of this paper are organised as follows. First, we establish foundational concepts and operator relationships over the complex field C. We then prove structural theorems showing that the numerical range of both S and G are compact real intervals $[\lambda_{min}, \lambda_{max}]$ and $[0, \lambda_{max}]$ governed by their spectral bounds, prove that frame tightness corresponds to the collapse of $W(S) = \{\alpha\}$ to a single point, quantify non-tightness using the eccentricity of W(G), demonstrate non-uniqueness of Grassmannian frames via concrete matrix calculations, and study the Rayleigh quotient mapping on complex projective space CP^{m-1}. Then, we present an operator-theoretic analysis of erasure resilience, deriving novel erasure bounds via numerical range inequalities and providing complete self-contained proofs showing that equal-norm Parseval frames uniquely minimise worst-case 1-erasure reconstruction error. Finally, we formulate an optimisation framework over the Grassmannian manifold based on numerical range boundary ratios, accompanied by geometric visualisations and open research directions.

Research gap. Although the manuscript addresses Grassmannian frames from geometric, algebraic, and operator-theoretic perspectives, a unified treatment directly linking numerical-range geometry with frame tightness, projective structure, and erasure resilience remains insufficiently developed. The present study therefore focuses on these connections within one finite-dimensional operator framework.

Objective of the Study. This study aims to characterise the numerical ranges of the frame and Gramian operators of finite-dimensional Grassmannian frames, relate these ranges to tightness and the Rayleigh map on complex projective space, and analyse erasure resilience through numerical-range and operator-norm bounds.

Acknowledgements. The first author acknowledges institutional support from the National Defence University - Kenya, and the second author extends special gratitude to the International Science Programme (ISP) in Mathematics at Uppsala University, Sweden.

2. Preliminaries and Definitions

Throughout this paper, all vector spaces are finite-dimensional complex Hilbert spaces. We denote the m-dimensional complex Euclidean space by $\mathbb{C}^m$, equipped with the standard complex inner product:

$$\langle x, y\rangle = \sum_{n=1}^{m} x_n \overline{y_n} \quad \text{for all } x, y \in \mathbb{C}^m,$$

which is linear in the first component and conjugate-linear in the second. The induced Euclidean norm is $\|x\| = \sqrt{\langle x, x\rangle}$. We denote the vector space of sequence space coefficients by $\ell_2^M := \mathbb{C}^M$ equipped with the standard inner product $\langle c, d\rangle = \sum_{j=1}^{M} c_j \overline{d_j}$.

Definition 2.1. A family of vectors $\Phi = \{v_j\}_{j=1}^{M} \subset \mathbb{C}^m$ is said to be a **finite frame** for $\mathbb{C}^m$ if there exist positive constants $0 < \alpha \leq \beta < \infty$, called **frame bounds**, such that

$$\alpha\|v\|^2 \leq \sum_{j=1}^{M} |\langle v, v_j\rangle|^2 \leq \beta\|v\|^2 \quad \text{for every } v \in \mathbb{C}^m.$$

Remark 2.2.

(a) If $\alpha = \beta$ in Definition 2.1, then Φ is called an **α-tight frame**. In this case, $\sum_{j=1}^{M} |\langle v, v_j\rangle|^2 = \alpha\|v\|^2$ for all $v \in \mathbb{C}^m$.
(b) If $\alpha = \beta = 1$, then Φ is called a **Parseval frame** (or normalised tight frame).
(c) If there exists a constant $c > 0$ such that $\|v_j\| = c$ for all $j = 1, \dots, M$, then Φ is an **equal-norm frame**. If $c = 1$, then Φ is a **unit-norm frame**.

The fundamental linear operators associated with a frame $\Phi = \{v_j\}_{j=1}^M \subset \mathbb{C}^m$ are defined as follows (Casazza & Kutyniok, 2012, 2013):

Definition 2.3. The **analysis operator** $T : \mathbb{C}^m \to \mathbb{C}^M$ associated with Φ is defined by

$$Tv = \big(\langle v, v_j\rangle\big)_{j=1}^M \in \mathbb{C}^M \quad \text{for all } v \in \mathbb{C}^m.$$

Definition 2.4. The **synthesis operator** $T^* : \mathbb{C}^M \to \mathbb{C}^m$ is the linear adjoint operator of T, defined by

$$T^*(c) = \sum_{j=1}^{M} c_j v_j \quad \text{for all } c = (c_j)_{j=1}^M \in \mathbb{C}^M.$$

Definition 2.5. The **frame operator** $S : \mathbb{C}^m \to \mathbb{C}^m$ associated with Φ is defined by $S := T^*T$. That is,

$$Sv = T^*Tv = \sum_{j=1}^{M} \langle v, v_j\rangle v_j \quad \text{for all } v \in \mathbb{C}^m.$$

Notably, in matrix representation with respect to the standard basis, $S \in \mathscr{M}_m(\mathbb{C})$ is a self-adjoint (Hermitian) positive-definite matrix.

Definition 2.6. The **Gramian operator** $G : \mathbb{C}^M \to \mathbb{C}^M$ associated with Φ is defined by $G := TT^*$. That is,

$$Gc = TT^*c = \left(\sum_{j=1}^{M} c_j \langle v_j, v_k\rangle\right)_{k=1}^{M} \quad \text{for all } c = (c_j)_{j=1}^M \in \mathbb{C}^M.$$

The matrix representation $G \in \mathscr{M}_M(\mathbb{C})$ is Hermitian positive-semidefinite of rank m, with entries $G_{k,j} = \langle v_j, v_k\rangle$.

Theorem 2.7 (Njagi et al., 2018). *Let $\Phi = \{v_j\}_{j=1}^M$ be a frame for $\mathbb{C}^m$ with analysis operator T, frame operator S, and Gramian operator G. Then:*

(a) The non-zero eigenvalues of $S \in \mathscr{M}_m(\mathbb{C})$ and $G \in \mathscr{M}_M(\mathbb{C})$ coincide, including multiplicities.

(b) Φ *is an* α*-tight frame if and only if* $S = \alpha I_m$*. In particular, if* Φ *is a unit-norm* α*-tight frame, then* $\alpha = \frac{M}{m}$*.*

(c) Φ *is a Parseval frame if and only if* G *is an orthogonal projection of rank* m *in* $\mathbb{C}^M$*; that is,* $G^2 = G = G^*$*.*

(d) S *is invertible on* $\mathbb{C}^m$*, whereas* G *is invertible on* $\mathbb{C}^M$ *if and only if* $M = m$*.*

Definition 2.8. Let $\Phi = \{v_j\}_{j=1}^{M} \subset \mathbb{C}^m$ be a unit-norm frame.

(a) The **maximal frame correlation** (or mutual coherence) $\mu(\Phi)$ is defined by

$$\mu(\Phi) = \max_{1 \leq j \neq k \leq M} |\langle v_j, v_k \rangle|.$$

(b) A unit-norm frame Φ is called a **Grassmannian frame** if it solves the optimisation problem of minimising mutual coherence among all unit-norm frames of M vectors in $\mathbb{C}^m$:

$$\mu(\Phi) = \min\{\mu(\Psi) : \Psi = \{w_j\}_{j=1}^{M} \subset \mathbb{C}^m, \|w_j\| = 1\}.$$

(c) For any unit-norm frame $\Phi = \{v_j\}_{j=}^{M} \subset \mathbb{C}^m$, the mutual coherence satisfies the **Welch bound** (Strohmer & Heath, 2003):

$$\mu(\Phi) \geq \sqrt{\frac{M - m}{m(M-1)}}.$$

Equality holds if and only if Φ is an **Equiangular Tight Frame** (ETF). If equality holds, Φ is called an **optimal Grassmannian frame**. The Welch bound can only be saturated over $\mathbb{C}$ if $M \leq m^2$.

Definition 2.9. Let $A \in \mathscr{M}_n(\mathbb{C})$ be a linear operator on $\mathbb{C}^n$.

(a) The **numerical range** (or field of values) of A is defined as

$$W(A) = \{\langle Av, v \rangle_{\mathbb{C}^n} : v \in \mathbb{C}^n, \|v\| = 1\}.$$

(b) The **numerical radius** of A is defined by

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\} = \sup_{\|v\|=1} |\langle Av, v \rangle|.$$

Proposition 2.10 (Williams, 1969). *Let* $A, B \in \mathcal{M}_n(\mathbb{C})$*. The following properties hold over* $\mathbb{C}$*:*

(1) $W(A^*) = \overline{W(A)} = \{\bar{\lambda} : \lambda \in W(A)\}$*.*

(2) The spectrum $\sigma(A)$ *is contained in* $W(A)$*.*

(3) $W(A)$ *is a compact, convex subset of* $\mathbb{C}$ *(Toeplitz-Hausdorff Theorem).*

(4) $W(A) \subseteq \mathbb{R}$ *if and only if* A *is self-adjoint (*$A = A^*$*).*

(5) If $U \in \mathscr{M}_n(\mathbb{C})$ *is unitary (*$U^*U = I_n$*), then* $W(U^*AU) = W(A)$*.*

(6) For any $s, t \in \mathbb{C}$*,* $W(sA + tI_n) = sW(A) + t$*.*

(7) $W(A+B) \subseteq W(A) + W(B)$.

(8) $\frac{1}{2}\|A\|_2 \leq w(A) \leq \|A\|_2$, *where* $\|A\|_2$ *denotes the spectral operator norm. If* A *is normal* ($A^*A = AA^*$)*, then* $w(A) = \|A\|_2 = \max_{\lambda \in \sigma(A)} |\lambda|$.

3. Numerical Range of Grassmannian Frame Operators

In this section, we investigate the numerical range and numerical radius of frame operators S and Gramian operators G associated with Grassmannian frames.

Definition 3.1. Let $S \in \mathscr{M}_m(\mathbb{C})$ be a frame operator. We define the class of operators similar to S by

$$\Psi(S) = \left\{ B^{-1}SB : B \in \mathrm{GL}(m, \mathbb{C}) \right\}.$$

We begin by establishing that optimal Grassmannian configurations for a given pair (M, m) are generally non-unique, even up to unitary equivalence.

Lemma 3.2. *Grassmannian frames in* $\mathbb{C}^m$ *are not unique up to unitary equivalence.*

Proof. Let $\overline{\Phi} = \{\Phi_1, \dots, \Phi_n\}$ be a finite collection of unit-norm frames $\Phi_i = \{v_{ij}\}_{j=1}^M \subset \mathbb{C}^m$. The Grassmannian frame optimisation problem seeks to minimise the mutual coherence $\mu(\Phi)$ over valid frame configurations. Since the objective function $\mu(\Phi) = \max_{j \neq k} |\langle v_j, v_k \rangle|$ is continuous and defined on the compact product manifold $(\mathbb{S}^{2m-1})^M$, the global minimum is attained.

However, distinct configurations Φ_{t_1} and Φ_{t_2} that cannot be mapped to one another via any unitary matrix $U \in \mathcal{U}(m)$ can achieve the identical minimal maximal pairwise coherence value:

$$\mu(\Phi_{t_1}) = \mu(\Phi_{t_2}) = \min_{\Phi} \mu(\Phi).$$

Thus, optimal Grassmannian frames are non-unique up to unitary equivalence. □

Example 3.3 (Mercedes-Benz frame)**.** Consider the collection $\overline{\Phi} = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}$ of unit-norm frames in $\mathbb{C}^2$ ($M = 3, m = 2$) given explicitly by:

$$\Phi_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \right\}, \quad \Phi_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \right\},$$

$$\Phi_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \right\}, \quad \text{and} \quad \Phi_4 = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \right\}.$$

We evaluate the exact pairwise frame correlations for each configuration:

- **For** $\Phi_1 = \{v_{11}, v_{12}, v_{13}\}$:

$$|\langle v_{11}, v_{12}\rangle| = \frac{1}{2}, \quad |\langle v_{11}, v_{13}\rangle| = \frac{1}{2}, \quad |\langle v_{12}, v_{13}\rangle| = \left|\frac{1}{4} - \frac{3}{4}\right| = \frac{1}{2}.$$

  Hence, $\mu(\Phi_1) = \frac{1}{2}$.

- **For** $\Phi_2 = \{v_{21}, v_{22}, v_{23}\}$:

$$|\langle v_{21}, v_{22}\rangle| = 0, \quad |\langle v_{21}, v_{23}\rangle| = \frac{3}{5}, \quad |\langle v_{22}, v_{23}\rangle| = \frac{4}{5}.$$

  Hence, $\mu(\Phi_2) = \frac{4}{5}$.

- **For** $\Phi_3 = \{v_{31}, v_{32}, v_{33}\}$:

$$|\langle v_{31}, v_{32}\rangle| = 0, \quad |\langle v_{31}, v_{33}\rangle| = \frac{4}{5}, \quad |\langle v_{32}, v_{33}\rangle| = \frac{3}{5}.$$

  Hence, $\mu(\Phi_3) = \frac{4}{5}$.

- **For** $\Phi_4 = \{v_{41}, v_{42}, v_{43}\}$:

$$|\langle v_{41}, v_{42}\rangle| = \frac{1}{2}, \quad |\langle v_{41}, v_{43}\rangle| = \frac{1}{2}, \quad |\langle v_{42}, v_{43}\rangle| = \left|\frac{1}{4} - \frac{3}{4}\right| = \frac{1}{2}.$$

  Hence, $\mu(\Phi_4) = \frac{1}{2}$.

Comparing the mutual coherence values across the collection $\overline{\Phi}$, we obtain:

$$\min_{\overline{\Phi}} \mu(\Phi_i) = \mu(\Phi_1) = \mu(\Phi_4) = \frac{1}{2}.$$

Notice that this minimum value satisfies the Welch bound equality:

$$\sqrt{\frac{3-2}{2(3-1)}} = \frac{1}{2}.$$

Thus, both Φ_1 and Φ_4 achieve absolute optimality as Equiangular Tight Frames (Grassmannian frames) in $\mathbb{C}^2$, explicitly confirming their non-uniqueness.

We now state and prove the main structural theorem characterising the numerical range and numerical radius of frame and Gramian operators.

Theorem 3.4. *Let* $\Phi = \{v_j\}_{j=1}^{M}$ *be a frame for* $\mathbb{C}^m$ $(M > m)$ *with frame operator* $S \in \mathscr{M}_m(\mathbb{C})$ *and Gramian operator* $G \in \mathscr{M}_M(\mathbb{C})$. *Let* $0 < \lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m = \lambda_{\max}$ *denote the eigenvalues of* S.

(a). The numerical range of S *is the compact real line interval:*

$$W(S) = [\lambda_{\min}, \lambda_{\max}] \subset \mathbb{R}, \textit{and its numerical radius is } w(S) = \lambda_{\max} = \|S\|_2.$$

(b). The numerical range of G *is the compact real line interval:*

$$W(G) = [0, \lambda_{\max}] \subset \mathbb{R}, \textit{and its numerical radius is } w(G) = \lambda_{\max} = \|G\|_2.$$

(c). Φ *is an* α*-tight frame if and only if* $W(S) = \{\alpha\}$ *degenerates to a single point. In this case,* $W(G) = [0, \alpha]$*. For a unit-norm frame,* $W(S) = \left\{\frac{M}{m}\right\}$*,* $w(S) = \frac{M}{m}$*,* $W(G) = \left[0, \frac{M}{m}\right]$*, and* $w(G) = \frac{M}{m}$*.*

Proof.

(a). Since $S = T^*T \in \mathscr{M}_m(\mathbb{C})$ is Hermitian ($S = S^*$), Proposition 2.10(4) implies that $W(S) \subset \mathbb{R}$. By the Spectral Theorem for Hermitian matrices over $\mathbb{C}$, there exists an orthonormal basis $\{e_k\}_{k=1}^m$ of $\mathbb{C}^m$ consisting of eigenvectors of S such that $Se_k = \lambda_k e_k$ for $k = 1, \ldots, m$.

For any unit vector $v \in \mathbb{C}^m$ ($\|v\| = 1$), expanding $v = \sum_{k=1}^m c_k e_k$ with $\sum_{k=1}^m |c_k|^2 = 1$ yields:

$$\langle Sv, v\rangle = \left\langle \sum_{k=1}^m \lambda_k c_k e_k, \sum_{j=1}^m c_j e_j \right\rangle = \sum_{k=1}^m \lambda_k |c_k|^2.$$

By the Rayleigh-Ritz theorem:

$$\lambda_{\min} = \lambda_{\min} \sum_{k=1}^m |c_k|^2 \leq \sum_{k=1}^m \lambda_k |c_k|^2 \leq \lambda_{\max} \sum_{k=1}^m |c_k|^2 = \lambda_{\max}.$$

This establishes that $W(S) \subseteq [\lambda_{\min}, \lambda_{\max}]$. Conversely, choosing $v = e_1$ yields $\langle Se_1, e_1\rangle = \lambda_1 = \lambda_{\min} \in W(S)$, and choosing $v = e_m$ yields $\langle Se_m, e_m\rangle = \lambda_m = \lambda_{\max} \in W(S)$. Since the unit sphere $\mathbb{S}^{2m-1} \subset \mathbb{C}^m$ is connected and the quadratic form $v \mapsto \langle Sv, v\rangle$ is continuous, the image $W(S)$ is connected. By the Intermediate Value Theorem (or the Toeplitz-Hausdorff Theorem), $W(S) = [\lambda_{\min}, \lambda_{\max}]$. The numerical radius of S is $w(S) = \sup_{\lambda \in W(S)} |\lambda| = \max(|\lambda_{\min}|, |\lambda_{\max}|) = \lambda_{\max} = \|S\|_2$.

(b). The Gramian operator $G = TT^* \in \mathscr{M}_M(\mathbb{C})$ is Hermitian positive-semidefinite ($G = G^* \geq 0$). By Theorem 2.7(a), the non-zero eigenvalues of G coincide with those of S. Since $M > m$, $\mathrm{rank}(G) = m < M$, so 0 is an eigenvalue of G with multiplicity $M - m$. Thus, the spectrum of G is $\sigma(G) = \{\lambda_1, \ldots, \lambda_m, 0, \ldots, 0\}$. Applying the spectral argument to G over $\mathbb{C}^M$ yields $W(G) = [\lambda_{\min}(G), \lambda_{\max}(G)] = [0, \lambda_{\max}]$, and the numerical radius is $w(G) = \lambda_{\max} = \|G\|_2$.

(c). Φ is an α-tight frame $\iff S = \alpha I_m \iff \lambda_1 = \cdots = \lambda_m = \alpha \iff W(S) = [\alpha, \alpha] = \{\alpha\}$. For a unit-norm α-tight frame, taking the trace yields $\mathrm{Tr}(S) = \sum_{j=1}^M \|v_j\|^2 = M = \mathrm{Tr}(\alpha I_m) = m\alpha \implies \alpha = \frac{M}{m}$. Hence $W(S) = \left\{\frac{M}{m}\right\}$, $w(S) = \frac{M}{m}$, $W(G) = \left[0, \frac{M}{m}\right]$, and $w(G) = \frac{M}{m}$.

$\square$

Example 3.5 (Optimal Tight Grassmannian Frame in $\mathbb{C}^2$)**.** Consider the optimal $(3,2)$ Grassmannian frame Φ_1 in $\mathbb{C}^2$ from Example 3.3 (the Mercedes-Benz frame). The corresponding Hermitian frame operator $S = T^*T \in \mathscr{M}_2(\mathbb{C})$ and the Gramian operator $G = TT^* \in \mathscr{M}_3(\mathbb{C})$ are computed from the analysis operator T and synthesis operator T^* below

$$\mathbb{C}^2 \xrightarrow{T=\begin{bmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}} \mathbb{C}^3 \xrightarrow{T^*=\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}} \mathbb{C}^2.$$

That is, the frame operator $S = T^*T \in \mathscr{M}_2(\mathbb{C})$ and Gramian operator $G = TT^* \in \mathscr{M}_3(\mathbb{C})$ evaluate to:

$$S = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} = \frac{3}{2} I_2 \quad \text{and} \quad G = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

The spectrum of S is $\sigma(S) = \{1.5, 1.5\}$, yielding the degenerate numerical range $W(S) = \{1.5\} \subset \mathbb{R}$ and numerical radius $w(S) = 1.5$. The spectrum of G is $\sigma(G) = \{0, 1.5, 1.5\}$, producing $W(G) = [0, 1.5] \subset \mathbb{R}$ and numerical radius $w(G) = 1.5$.

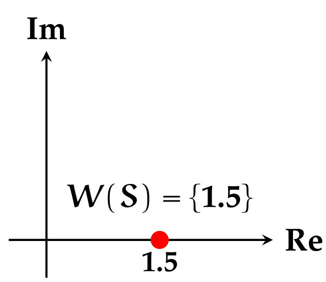

Figure 1. Numerical range $W(S) = \{1.5\}$ for the optimal $(3,2)$ Grassmannian frame.

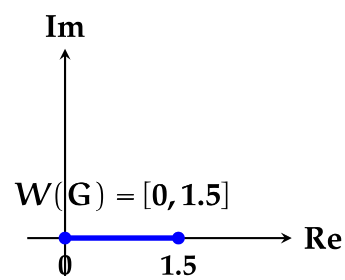

Figure 2. Numerical range $W(G) = [0, 1.5]$ corresponding to the same configuration.

Example 3.6 (Non-Tight Frame Calculation in $\mathbb{C}^2$)**.** Consider the non-tight unit-norm frame Φ_2 in $\mathbb{C}^2$ from Example 3.3:

$$\Phi_2 = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \right\}.$$

The associated analysis operator $T : \mathbb{C}^2 \to \mathbb{C}^3$ is:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

The synthesis operator $T^* : \mathbb{C}^3 \to \mathbb{C}^2$ is represented by the matrix:

$$T^* = \begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{4}{5} \end{bmatrix}.$$

The frame operator $S = T^*T \in \mathscr{M}_2(\mathbb{C})$ is explicitly evaluated as:

$$S = T^*T = \begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 1+0+\frac{9}{25} & 0+0+\frac{12}{25} \\ 0+0+\frac{12}{25} & 0+1+\frac{16}{25} \end{bmatrix} = \begin{bmatrix} \frac{34}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{41}{25} \end{bmatrix}.$$

The trace and determinant of S are $\mathbf{Tr}(S) = 3$ and $\mathbf{det}(S) = 2$. The characteristic equation $\mathbf{det}(S - \lambda I) = \lambda^2 - 3\lambda + 2 = 0$ yields eigenvalues:

$$\lambda_1 = \lambda_{min} = 1 \quad \text{and} \quad \lambda_2 = \lambda_{max} = 2.$$

Thus, the frame bounds are $\alpha = 1$ and $\beta = 2$, giving an exact condition number $\kappa(S) = \frac{\beta}{\alpha} = 2$.

Applying Theorem 3.4, the numerical range and numerical radius of S and G are determined exactly without any decimal or square-root approximations:

$$W(S) = [1, 2], \quad w(S) = 2,$$
$$W(G) = [0, 2], \quad w(G) = 2.$$

Proposition 3.7. *Let Φ be a frame for $\mathbb{C}^m$ with frame bounds $\alpha \leq \beta$ and Gramian matrix $G \in \mathscr{M}_M(\mathbb{C})$. The geometric spread (eccentricity) of $W(G)$ bounds the deviation of the frame bounds from the mean frame density $\bar{\lambda} = \frac{M}{m}$:*

$$\frac{\beta - \alpha}{2} \leq \mathbf{dist}(\mathbf{spec}(G), \partial W(G)).$$

Moreover, the collapse of $W(G)$ to a single non-zero interval $[0, \frac{M}{m}]$ with zero imaginary spread corresponds to maximal circular symmetry of $W(G)$, reflecting uniform equiangular vector distributions across the Grassmann manifold $\mathbf{Gr}(m, M)$.

Proof. By Theorem 3.4(b), the numerical range of $G \in \mathscr{M}_M(\mathbb{C})$ is the real line interval $W(G) = [0, \beta] \subset \mathbb{R}$, whose boundary points are $\partial W(G) = \{0, \beta\}$. By Theorem 2.7(a), the non-zero eigenvalues of G coincide with the spectrum of S, satisfying $0 < \alpha \leq \lambda \leq \beta$ for all $\lambda \in \mathbf{spec}(G) \setminus \{0\}$.

For any non-zero eigenvalue $\lambda \in \mathbf{spec}(G) \setminus \{0\}$, the distance to the boundary $\partial W(G) = \{0, \beta\}$ is given by $\mathbf{dist}(\lambda, \partial W(G)) = \min(\lambda, \beta - \lambda)$. Evaluating across the non-zero spectral range $[\alpha, \beta]$, the maximal boundary distance satisfies:

$$\sup_{\lambda \in \mathbf{spec}(G) \setminus \{0\}} \mathbf{dist}(\lambda, \partial W(G)) \geq \frac{\beta - \alpha}{2},$$

establishing the lower bound. In the special case of an α-tight frame where $\alpha = \beta = \frac{M}{m}$, the spectral difference vanishes ($\frac{\beta-\alpha}{2} = 0$), and the non-zero spectrum collapses to the single degenerate point $\frac{M}{m} \in W(G) = [0, \frac{M}{m}]$, completing the proof. □

3.1. **Projective Geometry and Rayleigh Map.** We now link the numerical range to the geometric structure of complex projective space.

Definition 3.8. Let $H = \mathbb{C}^m$. The $(m-1)$-dimensional complex projective space $\mathbb{CP}^{m-1}$ is defined through the canonical quotient map:

$$q : \mathbb{C}^m \setminus \{0\} \longrightarrow \mathbb{CP}^{m-1} := (\mathbb{C}^m \setminus \{0\})/\mathbb{C}^\times, \quad v \mapsto [v] = \{\lambda v : \lambda \in \mathbb{C}^\times\}.$$

The space $\mathbb{CP}^{m-1}$ is equipped with the natural Fubini-Study metric:

$$d_{FS}([v], [w]) = \arccos\left(\frac{|\langle v, w\rangle|}{\|v\|\|w\|}\right), \quad 0 \leq d_{FS} \leq \frac{\pi}{2}.$$

Definition 3.9. Let $\Phi = \{v_j\}_{j=1}^M$ be a frame for $\mathbb{C}^m$ with frame operator S. The **Rayleigh map** $\rho_S : \mathbb{CP}^{m-1} \longrightarrow \mathbb{R}$ associated with S is defined by:

$$\rho_S([v]) := \frac{\langle Sv, v\rangle}{\langle v, v\rangle} = \frac{\sum_{j=1}^M |\langle v, v_j\rangle|^2}{\|v\|^2} \quad \text{for } [v] \in \mathbb{CP}^{m-1}.$$

Proposition 3.10. *Let $S \in \mathscr{M}_m(\mathbb{C})$ be the frame operator of a frame Φ for $\mathbb{C}^m$.*

(a) The image of complex projective space $\mathbb{CP}^{m-1}$ under the Rayleigh map is precisely the numerical range:

$$\rho_S(\mathbb{CP}^{m-1}) = W(S) = [\lambda_{min}, \lambda_{max}].$$

(b) A point $[v] \in \mathbb{CP}^{m-1}$ is a critical point of the Rayleigh map ρ_S if and only if v is an eigenvector of S. The corresponding critical values of ρ_S are precisely the eigenvalues $\sigma(S)$.

Proof.

(a) For any $[v] \in \mathbb{CP}^{m-1}$, choosing the representative unit vector $u = \frac{v}{\|v\|}$ gives $\rho_S([v]) = \langle Su, u\rangle \in W(S)$. Conversely, any element of $W(S)$ is of the form $\langle Su, u\rangle$ for some unit vector $u \in \mathbb{S}^{2m-1}$, which maps to $\rho_S([u])$.

(b) Computing the gradient of the Rayleigh quotient $\rho_S(v) = \frac{\langle Sv,v\rangle}{\langle v,v\rangle}$ with respect to complex vector $v \neq 0$ yields:

$$\nabla \rho_S(v) = \frac{2}{\|v\|^2}\left(Sv - \rho_S(v)v\right).$$

Thus, $\nabla \rho_S(v) = 0 \iff Sv = \rho_S(v)v$, which holds if and only if v is an eigenvector of S with eigenvalue $\lambda = \rho_S(v) \in \sigma(S)$.

□

4. Numerical Robustness and Erasure Resilience

In real-world signal transmission over lossy packet networks, coefficient vectors transmitted through the analysis operator T may suffer deletions or erasures. In this section, we study how numerical range bounds and operator norms characterise frame robustness to erasures over $\mathbb{C}$.

Suppose $v \in \mathbb{C}^m$ is transmitted as the frame coefficient vector $w = Tv = (\langle v, v_j\rangle)_{j=1}^M \in \mathbb{C}^M$. An erasure pattern of e lost entries is modelled by an index subset $E \subset \{1,\dots,M\}$ with $|E| = e$.

Definition 4.1 (Erasure Masking Matrix and Error Operator)**.** Let $E \subset \{1,\dots,M\}$ with $|E| = e$.

(a) The **erasure matrix** $D \in \mathscr{M}_M(\mathbb{C})$ is the diagonal matrix defined by:

$$D_{j,j} = \begin{cases} 1, & \text{if } j \in E, \\ 0, & \text{if } j \notin E. \end{cases}$$

(b) The **reconstruction error operator** after e erasures indexed by E is defined by:

$$\tilde{S}_D := T^*DT = \sum_{j\in E} v_j v_j^* \in \mathscr{M}_m(\mathbb{C}).$$

The lost signal energy for a vector $v \in \mathbb{C}^m$ after erasures E is given by $\langle \tilde{S}_D v, v\rangle = \sum_{j\in E} |\langle v, v_j\rangle|^2$.

Definition 4.2 (Optimally Robust Frames to e Erasures; Holmes & Paulsen, 2004). Finding frames that are optimally robust to e erasures corresponds to solving the minimax optimisation problem:

$$\min_{\Phi} \max_{D} \|T^*DT\|_2,$$

where the minimum is taken over a specified class of normalised frames $\Phi \subset \mathbb{C}^m$, and the maximum is taken over all $M \times M$ diagonal matrices D with exactly e ones and $M - e$ zeros on the diagonal.

Theorem 4.3 (Erasure Tolerance and Numerical Radius Bounds). *Let $\Phi = \{v_j\}_{j=1}^M$ be a Grassmannian frame for $\mathbb{C}^m$ with frame operator S. Let $E \subset \{1, \ldots, M\}$ be a subset of k erasures ($|E| = k$), and let T_{E^c} denote the analysis operator of the remaining subframe $\Phi_{E^c} = \{v_j\}_{j \notin E}$. The worst-case lower frame bound parameter after k erasures $\alpha_k = \min_{|E|=k} \lambda_{\min}(T_{E^c}^* T_{E^c})$ satisfies:*

$$\alpha_k \geq \inf_{\|v\|=1} \mathrm{Re}\left(\langle S_{E^c} v, v \rangle\right) \geq \min \mathrm{Re}(W(S)) - k \max_{1 \leq i \leq M} \|v_i\|^2.$$

Proof. The remaining frame operator after k erasures indexed by E is

$$S_{E^c} = T_{E^c}^* T_{E^c} = S - \sum_{i \in E} v_i v_i^*.$$

For any unit vector $v \in \mathbb{C}^m$,

$$\langle S_{E^c} v, v \rangle = \langle S v, v \rangle - \sum_{i \in E} |\langle v, v_i \rangle|^2.$$

Since $\langle S v, v \rangle \in W(S)$ and $|\langle v, v_i \rangle|^2 \leq \|v\|^2 \|v_i\|^2 = \|v_i\|^2$, taking the infimum over $\|v\| = 1$ yields:

$$\lambda_{\min}(S_{E^c}) \geq \lambda_{\min}(S) - \sum_{i \in E} \|v_i\|^2 \geq \min \mathrm{Re}(W(S)) - k \max_{1 \leq i \leq M} \|v_i\|^2,$$

establishing the result. □

Corollary 4.4. *For an Equiangular Tight Frame (ETF), the numerical range of every principal submatrix of the Gram matrix exhibits bounded Hausdorff distance from a compressed disk, ensuring uniform degradation under channel erasures.*

Example 4.5. Consider the collection $\overline{\Phi} = \{\Phi_1, \Phi_2, \Phi_3\}$ of unit-norm frames in $\mathbb{C}^3$ ($M = 4, m = 3$) given by:

$$\Phi_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \right\}, \quad \Phi_2 = \left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \right\},$$

$$\text{and} \quad \Phi_3 = \left\{ \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ \frac{-2\sqrt{6}}{6} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \right\}.$$

The associated analysis operators are:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}, \quad T_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad \text{and} \quad T_3 = \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{-2\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$

For 1-erasure ($e = 1$), the diagonal masks D_k ($k = 1, 2, 3, 4$) contain a single 1 at entry (k, k). For frame Φ_1, the 1-erasure operators $\tilde{S}_k = T_1^* D_k T_1 = v_k v_k^*$ evaluate to:

$$\tilde{S}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{S}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{S}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{S}_4 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The spectral operator norm for each rank-1 matrix $\tilde{S}_k = v_k v_k^*$ is $\|\tilde{S}_k\|_2 = \|v_k\|^2 = 1$. Thus, for all three unit-norm frame configurations Φ_1, Φ_2, Φ_3, the worst-case 1-erasure norm is:

$$\max_D \|T_i^* D T_i\|_2 = 1 \quad \text{for } i = 1, 2, 3.$$

We now provide a complete, self-contained operator-theoretic proof characterising Parseval frames that are optimally robust to 1-erasure.

Theorem 4.6. *Let $\Phi = \{v_j\}_{j=1}^M$ be an (M, m) Parseval frame for $\mathbb{C}^m$. The frame Φ is optimally robust to 1-erasure among all (M, m) Parseval frames if and only if Φ is an equal-norm Parseval frame ($\|v_j\|^2 = \frac{m}{M}$ for all $j = 1, \ldots, M$). In this case, the optimal worst-case 1-erasure norm is $\frac{m}{M}$.*

Proof. Let $\Phi = \{v_j\}_{j=1}^M \subset \mathbb{C}^m$ be a Parseval frame. The frame operator is $S = \sum_{j=1}^M v_j v_j^* = I_m$. Under a single erasure at index $k \in \{1, \ldots, M\}$, the erasure mask D_k has a single 1 at diagonal position (k, k). The error operator is:

$$\tilde{S}_k = T^* D_k T = v_k v_k^* \in \mathscr{M}_m(\mathbb{C}).$$

The matrix $\tilde{S}_k = v_k v_k^*$ is a Hermitian rank-1 operator. Its non-zero eigenvalue is $\|v_k\|^2$ with corresponding eigenvector v_k. Therefore:

$$\|\tilde{S}_k\|_2 = \|v_k v_k^*\|_2 = \|v_k\|^2,$$

and its numerical range is $W(\tilde{S}_k) = [0, \|v_k\|^2]$ with numerical radius $w(\tilde{S}_k) = \|v_k\|^2$.

The worst-case reconstruction error under a single erasure for frame Φ is:

$$\max_{1\leq k\leq M} \|T^*D_kT\|_2 = \max_{1\leq k\leq M} \|v_k\|^2.$$

Since Φ is a Parseval frame ($S = I_m$), taking the trace yields:

$$\sum_{k=1}^{M} \|v_k\|^2 = \mathrm{Tr}(I_m) = m.$$

To find the frame minimising the worst-case 1-erasure error, we solve:

$$\min_{\Phi} \max_{1\leq k\leq M} \|v_k\|^2 \quad \text{subject to } \sum_{k=1}^{M} \|v_k\|^2 = m.$$

By the minimax property of real sequences, the maximum of M positive numbers summing to m satisfies:

$$\max_{1\leq k\leq M} \|v_k\|^2 \geq \frac{1}{M}\sum_{k=1}^{M} \|v_k\|^2 = \frac{m}{M}.$$

Equality holds if and only if $\|v_1\|^2 = \|v_2\|^2 = \cdots = \|v_M\|^2 = \frac{m}{M}$. Thus, the minimum worst-case 1-erasure error is achieved if and only if Φ is an equal-norm Parseval frame, yielding optimal norm $\frac{m}{M}$. □

Theorem 4.7. *Let $\overline{\Phi} = \{\Phi = \{v_j\}_{j=1}^{M}\}$ be a collection of (M, m) α-tight Grassmannian frames for $\mathbb{C}^m$.*

(a) For unit-norm α-tight frames ($\|v_j\| = 1$), the absolute 1-erasure operator norm is:

$$\min_{T} \max_{D} \|T^*DT\|_2 = 1.$$

(b) The normalised 1-erasure error operator $S^{-1/2}(T^*D_kT)S^{-1/2} = \frac{1}{\alpha}v_kv_k^*$ has spectral norm and numerical radius equal to:

$$\left\|S^{-1/2}T^*D_kTS^{-1/2}\right\|_2 = w\left(S^{-1/2}T^*D_kTS^{-1/2}\right) = \frac{m}{M}.$$

Proof.

(a) For an α-tight frame Φ, $S = T^*T = \sum_{j=1}^{M} v_jv_j^* = \alpha I_m$. Taking the trace yields $\mathrm{Tr}(S) = \sum_{j=1}^{M} \|v_j\|^2 = m\alpha$. For unit-norm vectors ($\|v_j\| = 1$), $M = m\alpha \implies \alpha = \frac{M}{m}$. Under 1-erasure at index k, $T^*D_kT = v_kv_k^*$. As shown in Theorem 4.6, $\|v_kv_k^*\|_2 = \|v_k\|^2 = 1$. Thus, $\max_D \|T^*DT\|_2 = 1$.

(b) Normalising the error operator by the frame operator inverse $S^{-1} = \frac{1}{\alpha} I_m = \frac{m}{M} I_m$ gives:

$$S^{-1/2}(T^* D_k T) S^{-1/2} = \frac{m}{M} v_k v_k^*.$$

Since $\|v_k\| = 1$, the operator norm and numerical radius of this normalised rank-1 projection matrix are:

$$\left\| \frac{m}{M} v_k v_k^* \right\|_2 = w\left(\frac{m}{M} v_k v_k^* \right) = \frac{m}{M}.$$

□

Corollary 4.8. *Any finite collection of $(3, 3)$ unit-norm frames in $\mathbb{C}^3$ has 1-erasure operator norm equal to 1:*

$$\min_T \max_D \|T^* D T\|_2 = 1.$$

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