Two-dimensional fractional Fourier transform with applications to linear fractional partial differential equations

Abstract

Fractional Integral transforms have fascinated numerous investigators due to their modifiability and applicability in diverse sectors of physics, optics, electronics and communication engineering. They had prominence in the phenomenal breakthrough of information technology and communication systems during last forties. In this paper, we introduce two dimensional Fourier transform of fractional order using fractional derivatives with the help of Mittag-Leffler function. Some properties of this novel transform have been derived analytically. The novel technique has been employed to deduce the exact solutions of two dimensional fractional dispersion, fractional heat, fractional wave and fractional telegraph equations.

Keywords: Modified-Riemann Liouville fractional derivative, fractional integral, Mittag-Leffler function, fractional Fourier transform and fractional order partial differential equation.

2010 Mathematics Subject Classification: 42A38, 35R11, 33E12, 26A33.

1 Introduction

Scientists of various engineering disciplines have used Integral transforms as mathematical contrivance since the beginning [1, 2]. In the recent years extension of the transforms have been a dominant sector of exploration. Resulting many of the integral transforms have been fractionalized in last twenty five years. In fact due to their extra degree of freedom they have been much explored and became more prominent and efficacious tools as compare to their classical counterparts [3, 4, 5, 6, 7, 8].

Among all the transforms the Fourier transform has been recognized as the most exceedingly used technique by the inventors of varied zones of engineering and sciences for more than a century. Because it has produced massive applications in different sectors of science and technology. However it has been established that there are several physical applications or their underlying mathematical enigma that can not be described by the classical Fourier transform, this imposed the notion of fractionalization of Fourier transform [9, 10, 11].

Fractional Fourier transform (FrFT) acquired tremendous attention of technophiles since last forties due to its better performance in multifarious applications as compared to its traditional counterpart. It intrigued remarkable interest of scientists and engineers after the noteworthy results reported by V. Namias in 1980 [3] and grown outstandingly due to its copious applications in diversified fields [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. In fact it is a time-frequency representation and intensified the perception of time-frequency plane. It can be applied in every area where its basic transform and frequency domain concepts have been used [23, 24, 25].

The rapid growth of the FrFT due to its modifiability and proficiency made it the most influential and expeditious tool for diverse disciplines of higher mathematics, mathematical physics and information science. [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Wiener[38] was the first one to apply Fourier transform of fractional order to solve differential equations in 1929. But his work was not noticed substantially and the transform could not be developed in a long period of fifty years. Even Namias [3] who reintroduced the transform seemed to be ignorant about earlier work. He derived some of its properties and employed it in the field of quantum mechanics. His work proved to be a milestone in this domain and motivated physicist and engineers of distinct areas. Mc Bridge and Kerr[39] in 1987 shaped his theory and developed many algebraic and operational properties of FrFT.

Several other researchers started working in the field of fractional Fourier transform during nineties [40, 41, 42]. Almedia [24] in 1994 independently reintroduced the transform initially referring as an angular transform and presented its time frequency representation. Meanwhile Lohmann [43], Ozaktas and his team [26, 27, 28, 29, 30, 31] was continuously contributing on the applications of FrFT specifically in area of optics. Ozaktas [27] in 1996 along with his co-workers presented an algorithm for the digital computation of FrFT, this algorithm is validated till date although many others algorithms are also available now for the computational of FrFT.

Experimenter of varied zones proposed myriad definition of fractional Fourier transform on the basis of peculiar of methodologies of mathematics [31]. In this context fractional derivative was firstly used by Bolangna and West to fractionalize the Fourier transform in 2003 [44]. They named it generalized Fourier transform, but their results were not noticed broadly. In the year 2008, Y.F. Luchko et al.[45] and Jumarie [46] have been also introduced the Fourier transform of fractional order using the concepts of fractional calculus.

Recently Ahmed Zayed [47, 48] has introduced the two dimensional FrFT using Hermite functions of two complex variables as eigen functions of the transform. In the year 2024, Ahmed Zayed [49] also has been collected over twenty-five integral transforms in one single volume and this volume presents an overview of theory and applications of such fractional integral transforms with examples.

Multiple definitions of two dimensional fractional Fourier transform were experimented with using different mathematical aspects. It is possible to define the two dimensional FrFT in several different ways on based of distinct mathematical approaches[47, 48, 50, 51, 52, 53].

The fractional partial differential equations (FPDEs) have been employed gigantically as a new modeling tools in the wide range of applications of numerous sectors of physical and natural sciences such as astrophysics, dynamical systems, mathematical physics, finance, nature physical processes, environmental science, control systems, applied mathematics, statistics, optical fibers, chemistry, system identification, mechanical engineering, hydrodynamics, solid state biology, food supplement, electric control theory, economics and diffusion problems and other disciplines for the most recent years [54, 55, 56, 57, 58, 59, 60, 61].

Analytical solutions of FPDEs have been growing interest in modelling in various fields of science and engineering. Several analytical techniques have been introduced for solving FPDEs. In the literature, many analytical and numerical techniques such as Adomian decomposition method (ADM) [62], integral transform method [63], differential transform method [64], optimal Homotopy Analysis Method [65], generalized differential transform method [66], Laplace-Adomian decomposition method [67], natural transform decomposition method [68], Laplace transform method [69], Sumudu transform iterative method [70], Richardson Extrapolation Technique [71] and advanced transform techniques [72] have been utilized for solving FPDEs.

In the recent years, Mahor et. al. [73, 74] were also discussed and presented analytical solutions of some well-known linear fractional partial differential equations using integral transform method.

Wei and Yang [52] proposed a two-dimensional sparse fractional Fourier transform algorithm to estimate the fractional Fourier spectrum efficiently Inspired by the sparse Fourier transform algorithm in 2022 and in the year 2025, Elshamy et. al [53] presented a discrete version of the continuous, two-dimensional coupled fractional Fourier transform.

The Fractional dispersion equation, The Fractional heat diffusion equation, The fractional wave equation, The fractional telegraph equation are the most popular examples of FPDEs and analytical solutions of these equations have play an important role for describing the mathematical models in various fields of science and engineering. Therefore, the main goal of this article to produced the analytical solutions of fractional dispersion, fractional heat diffusion, fractional wave, fractional telegraph equations through this newly defined transform.

In this paper we have attempted to extend the two dimensional Fourier transform by fractionalizing it. Following definition of two dimensional Fourier transform of the function $f(x,y)$ denoted by $f(\hat{\omega},\tau)$ is well known and is defined as [11]

$$\hat{f}(\omega,\tau)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{(-i(\omega x+\tau y))}f(x,y)dxdy,\omega,\tau>0$$

Therefore, its inversion formulas is given by

$$f(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{i(\omega x+\tau y)}\hat{f}(\omega,\tau)d\omega d\tau,\omega,\tau>0$$

provided that the above integral converges.

Present work comprises a new definition of two dimensional FrFT using the functions of several variables and Conclusively proposed technique have been applied to solve two dimensional FPDEs as an output of the solutions in exact form.

The paper is structure as follows. Next section contains some basic definitions of fractional calculus. In the third section, two dimensional FrFT and some of its properties have been derived. The analytical solutions for two-dimensional fractional order partial differential equations are obtained in section 4. Finally, section 5 concludes the paper.

2 Preliminaries

In this section, we present some basic concepts of fractional calculus, which are closely related to the work carried out in this paper.

Definition 2.1. In 2006, Jumarie [75] proposed Modified Riemann-Liouville derivative of continuous, inconstant but not necessarily differentiable function $f(u)$ as follows

$$f^{\alpha}(u)=\frac{1}{\Gamma(-\alpha)}\int_{0}^{u}(u-\xi)^{-\alpha-1}f(\xi)d\xi,\quad for\quad \alpha<0.$$

For positive α, we have

$$f^{\alpha}(u)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{du}\int_{0}^{u}(u-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi,\quad for\quad 0<\alpha<1, \tag{1}$$

and

$$f^{\alpha}(u)=(f^{(n)}(u))^{(\alpha-n)},\quad for\quad n\leq\alpha<n+1;n\geq 1$$

Above definition of fractional derivative is worthwhile because it is zero if $f(u)$ is a constant. Moreover it is not affected by the differentiable of the functions.

Definition 2.2. The Mittag-Leffler function $E_{\alpha}(z)$, for $\alpha>0$, $\alpha\in C$, $z\in C$ and $Re(\alpha)>0$, has been proposed by Swedish mathematician Gosta Mittag-Leffler in 1903, defined as [76]

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{2}$$

Definition 2.3. The Dirac delta function of two variables is defined as [11]

$$\delta(x,y)=\begin{cases}0 & \text{if } x^2+y^2\neq 0\\ \infty & \text{if } x^2+y^2=0\end{cases} \tag{3}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x,y)dxdy=1\quad and\quad \delta(x,y)=\delta(x)\delta(y) \tag{4}$$

Remark 2.4. (i) If $\delta(x,y)$ is a two dimensional Dirac delta function, then following equality hold

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)\delta(x-a,y-b)dxdy=f(a,b)$$

(ii) The following result hold

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left(i\omega(x-a)\right)\exp\left(i\tau(y-b)\right)d\omega d\tau=\delta(x-a,y-b)$$

where a and b are constants.
Definition 2.5. In 2008 Jumarie [46] came up with definition of fractional Fourier transform using Modified fractional Riemann-Liouville derivative, defined by

$$F_\alpha[f(x)] = \hat{f}_\alpha(\omega) = \int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha) f(x)(dx)^\alpha, 0 < \alpha < 1 \tag{5}$$

where $f(x)$, $R \to C$, $x \to f(x)$ is continuous and piecewise α^{th} continuously differentiable function and $(dx)^\alpha$ denotes the fractional integral, which is expressed as follows [46]

$$\int_0^u g(x)(dx)^\alpha = \alpha \int_0^u (u-x)^{\alpha-1} g(x)dx$$

3 Two dimensional fractional Fourier transform and some of its properties

In this section, we have proposed a definition of two dimensional fractional Fourier transform with some of its properties.

3.1 Two dimensional fractional Fourier transform

Definition 3.1. If f is a continuous differentiable function of two variables of x and y, then two dimensional fractional Fourier transform (2D-FrFT) of function $f(x,y)$ denoted by $\hat{f}_\alpha(\omega, \tau$ and is defined by

$$F_\alpha[f(x,y)] = \hat{f}_\alpha(\omega,\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha) E_\alpha(i\tau^\alpha y^\alpha) f(x,y)(dx)^\alpha (dy)^\alpha, \omega, \tau > 0, 0 < \alpha < 1 \tag{6}$$

provided the integral (6) converges for each ω and τ.
A sufficient condition for convergence is $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mid f(x,y) \mid (dx)^\alpha (dy)^\alpha < \infty$
Remark 3.2. (i) By (6) we note that $\hat{f}_\alpha(\omega,0) = f_\alpha(\omega)$ and $\hat{f}_\alpha(0,\tau) = f_\alpha(\tau)$, where $\hat{f}_\alpha(.)$ is the FrFT defined by (5)
(ii) 2D-FrFT is converted into two dimensional classical Fourier transform when $\alpha = 1$.

3.2 Properties of two dimensional fractional Fourier transform

Theorem 4.1: If $F_\alpha[f(x,y)] = \hat{f}_\alpha(\omega,\tau)$, then

$$F_\alpha\left[\sum_{k=1}^{n} b_k f_k(x,y)\right] = \sum_{k=1}^{n} b_k F_\alpha[f_k(x,y)],$$

where $k \in N$ and $b_k, k = 1,2,...,n$ are any arbitrary constants.
Proof: Using (6), following equation is obtained

$$F_\alpha\left[\sum_{k=1}^{n} b_k f_k(x,y)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha) E_\alpha(i\tau^\alpha y^\alpha)\left[\sum_{k=1}^{n} b_k f_k(x,y)\right](dx)^\alpha (dy)^\alpha$$

$$= \sum_{k=1}^{n} b_k \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha) E_\alpha(i\tau^\alpha y^\alpha) f_k(x,y)(dx)^\alpha (dy)^\alpha$$

Using the definition of 2D-FrFT, the desired result is obtained.
Theorem 4.2: If $F_\alpha[f(x,y)] = \hat{f}_\alpha(\omega,\tau)$, then

$$F_\alpha[f(ax,by)] = \tfrac{1}{(ab)^\alpha} \hat{f}_\alpha\left(\tfrac{\omega}{a}, \tfrac{\tau}{b}\right), a, b > 0$$

Proof: From (6), it is observed that

$$F_\alpha[f(ax,by)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)E_\alpha(i\tau^\alpha y^\alpha)f(ax,by)(dx)^\alpha(dy)^\alpha$$

Substituting $ax = u$ and $by = v$ in above equation and using (6), the desired result is obtained..

Theorem 4.3: If $\hat{f}_\alpha(\omega,\tau)$ is a 2D-FrFT of the function $f(x,y)$, then the following equality holds

$$F_\alpha[x^\alpha y^\alpha f(x,y)] = -\frac{\partial^{2\alpha}}{\partial\omega^\alpha\partial\tau^\alpha}\hat{f}_\alpha(\omega,\tau)$$

Proof: From (6), yielding following equation

$$\hat{f}_\alpha(\omega,\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)E_\alpha(i\tau^\alpha y^\alpha)f(x,y)(dx)^\alpha(dy)^\alpha$$

Employing fractional Differentiation partially on both sides of above equation with respect ω and τ respectively and Using the result $\frac{\partial^\alpha}{\partial u^\alpha}E_\alpha(\lambda u^\alpha) = \lambda E_\alpha(\lambda u^\alpha)$ in above equation, it is obtained that

$$F_\alpha[x^\alpha y^\alpha f(x,y)] = -\frac{\partial^{2\alpha}}{\partial\omega^\alpha\partial\tau^\alpha}\hat{f}_\alpha(\omega,\tau)$$

Theorem 4.4: If $F_\alpha[f(x,y)] = \hat{f}_\alpha(\omega,\tau)$, then the two dimensional fractional Fourier transform of mixed-fractional partial derivative is given by

$$F_\alpha\left[\frac{\partial^{2\alpha}}{\partial x^\alpha\partial y^\alpha}f(x,y)\right] = -\omega^\alpha\tau^\alpha\hat{f}_\alpha(\omega,\tau), 0<\alpha<1, \omega,\tau>0$$

Provided that,$\lim_{x,y\to\pm\infty} f(x,y)$ vanish.
Proof: Using Eq. (6), it is obtained the following equation

$$F_\alpha\left[\frac{\partial^\alpha}{\partial x^\alpha}f(x,y)\right] == \int_{-\infty}^{+\infty} E_\alpha(i\tau^\alpha y^\alpha)\left[\int_{-\infty}^{+\infty} E_\alpha(i\omega^\alpha x^\alpha)\frac{\partial^\alpha}{\partial x^\alpha}u(x,y,t)(dx)^\alpha\right](dy)^\alpha$$

Using fractional integration by part formula in inner integral of right hand side , it is obtained that

$$F_\alpha\left[\frac{\partial^\alpha}{\partial x^\alpha}f(x,y)\right] = -i\omega^\alpha\hat{f}_\alpha(\omega,\tau) \tag{7}$$

Similarly, we have

$$F_\alpha\left[\frac{\partial^\alpha}{\partial y^\alpha}f(x,y)\right] = -i\tau^\alpha\hat{f}_\alpha(\omega,\tau) \tag{8}$$

Therefore, the desired result follows.

4 Applications

In this section, exact solutions of two-dimension fractional order linear partial differential equations have been determined using the proposed 2D-FrFT.

4.1 Two dimensional fractional dispersion equation

Two-dimensional fractional dispersion equation given by Meerschaert is as follows [77]

$$\frac{\partial}{\partial t}u(x,y,t) = d(x,y)\frac{\partial^\alpha}{\partial x^\alpha}u(x,y,t) + e(x,y)\frac{\partial^\beta}{\partial y^\beta}u(x,y,t) + g(x,y,t),$$
$$1<\alpha\le 2, 1<\beta\le 2, d(x,y)>0, e(x,y)>0, x,y\in(-\infty,+\infty) \quad and \quad t\in(0,+\infty) \tag{9}$$

Considering a two dimensional time-space fractional dispersion equation in which coefficients $d(x,y)$ and $e(x,y)$ are assumed to be 1. Therefore Eq.(9) reduces to following two dimensional time-space fractional dispersion equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,y,t)=\frac{\partial^\beta}{\partial x^\beta}u(x,y,t)+\frac{\partial^\beta}{\partial y^\beta}u(x,y,t)+g(x,y,t),0<\alpha,\beta\leq 1,x,y\in(-\infty,+\infty)\quad and \quad t\in(0,+\infty) \tag{10}$$

with the initial condition

$$u(x,y,0)=f(x,y) \tag{11}$$

where ∂^α denotes the modified Riemann-Liouville derivative defined by (1).
We refer to the 2D-FrFT

$$\hat{u}_\beta(\omega,\tau,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E_\alpha(i\omega^\beta x^\beta)E_\alpha(i\tau^\beta y^\beta)u(x,y,t)(dx)^\beta(dy)^\beta$$

Applying 2-D FRFT both sides of Eq.(10), it gives

$$\frac{d^\alpha}{dt^\alpha}\hat{u}_\beta(\omega,\tau,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E_\alpha(i\omega^\beta x^\beta)E_\alpha(i\tau^\beta y^\beta)\left(\frac{\partial^\beta}{\partial x^\beta}u(x,y,t)\right)(dx)^\beta(dy)^\beta$$
$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E_\alpha(i\omega^\beta x^\beta)E_\alpha(i\tau^\beta y^\beta)\left(\frac{\partial^\beta}{\partial y^\beta}u(x,y,t)\right)(dx)^\beta(dy)^\beta+\hat{g}_\beta(\omega,\tau,t)$$

Using Eqs. (7) and (8), the following equation is obtained

$$\frac{d^\alpha}{dt^\alpha}\hat{u}_\beta(\omega,\tau,t)=-i(\omega^\beta+\tau^\beta)\hat{u}_\beta(\omega,\tau,t)+\hat{g}_\beta(\omega,\tau,t) \tag{12}$$

$$\hat{u}_\beta(\omega,\tau,t)=\int_0^t\frac{E_\alpha(-i(\omega^\beta+\tau^\beta)t^\alpha)}{E_\alpha(-i(\omega^\beta+\tau^\beta)\theta^\alpha)}\hat{g}_\beta(\omega,\tau,\theta)(d\theta)^\alpha+\hat{f}_\beta(\omega,\tau)E_\alpha(-i(\omega^\beta+\tau^\beta)t^\alpha) \tag{13}$$

using Lagrange method of constant variation.
Applying inverse operator of 2-D FRFT, the following general solution of nonhomogeneous two-dimensional time-space fractional dispersion equation is obtained

$$u(x,y,t)=F_\alpha^{-1}\left[\left(\int_0^t\frac{E_\alpha(-i(\omega^\beta+\tau^\beta)t^\alpha)}{E_\alpha(-i(\omega^\beta+\tau^\beta)\theta^\alpha)}\hat{g}_\beta(\omega,\tau,\theta)(d\theta)^\alpha\right)+E_\alpha(-i(\omega^\beta+\tau^\beta)t^\alpha)\hat{f}_\beta(\omega,\tau)\right] \tag{14}$$

provided that the inverse transform operator exists.
Special Cases:
Case I: If $g(x,y,t)=0$ in Eq. (14), then it reduces to homogeneous two-dimensional time-space fractional dispersion equation, therefore in this case the solution is given by

$$u(x,y,t)=F_\alpha^{-1}\left[E_\alpha(-i(\omega^\beta+\tau^\beta)t^\alpha)\hat{f}_\beta(\omega,\tau)\right] \tag{15}$$

Case II: Substituting $f(x,y)=\exp(-i(ax+by))$ in (15) , then the following exact solution of homogeneous two-dimensional time-space fractional dispersion equation in the term of exponential function is deduced when $\alpha=\beta=1$.

$$u(x,y,t)=\exp(-ia(x+t))\exp(-ib(y+t)) \tag{16}$$

Case III: If $f(x,y)=\delta(x,y)$ in (15), then the following exact solution of homogeneous two-dimensional time-space fractional dispersion equation in terms of Dirac delta function is deduced when $\alpha=\beta=1$

$$u(x,y,t)=\delta(x+t,y+t),x,y\in(-\infty,+\infty)\quad and \quad t\in(0,+\infty) \tag{17}$$

4.2 Two dimensional time-space heat-like fractional order partial differential equation

Consider the following two dimensional non-homogeneous time-space heat-like fractional order partial differential equation [78]

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = k\left(\frac{\partial^{2\alpha} u(x,y,t)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x,y,t)}{\partial y^{2\alpha}}\right) + g(x,y,t) \quad where \quad x,y \in (-\infty,+\infty), t \in (0,+\infty), k \in (0,+\infty), \quad and \quad 0 < \alpha \leq 1. \tag{18}$$

under the initial condition

$$u(x,y,0) = f(x,y) \tag{19}$$

Applying 2D-FrFT both sides of (18) and using the Lagrange method of constant variation, the following equation is obtained

$$\hat{u}_{\alpha}(\omega,\tau,t) = \int_0^t \frac{E_{\alpha}(-k(\omega^{2\alpha}+\tau^{2\alpha})t^{\alpha}}{E_{\alpha}(-k(\omega^{2\alpha}+\tau^{2\alpha})\theta^{\alpha}}\hat{g}_{\alpha}(\omega,\tau,\theta)(d\theta)^{\alpha} + \hat{f}_{\alpha}(\omega,\tau)E_{\alpha}(-k(\omega^{2\alpha}+\tau^{2\alpha})t^{\alpha} \tag{20}$$

Applying inverse operator of 2-D FRFT on both sides of (20), the required general solution of the Eq.(18) is obtained

$$u(x,y,t) = F_{\alpha}^{-1}\left[\left(\int_0^t \frac{E_{\alpha}(-k(\omega^{2\alpha}+\tau^{2\alpha})t^{\alpha}}{E_{\alpha}(-k(\omega^{2\alpha}+\tau^{2\alpha})\theta^{\alpha}}\hat{g}_{\alpha}(\omega,\tau,\theta)\right) + \hat{f}_{\alpha}(\omega,\tau)E_{\alpha}(-k(\omega^{2\alpha}+\tau^{2\alpha})t^{\alpha}\right] \tag{21}$$

provided that the inverse transform operator exists.

Special Case:

If $g(x,y,t) = 0$ and $u(x,y,0) = f(x,y) = \delta(x,y)$ in (21) and using the result given by Mahor, Mishra and Jain [[22], p. 4] for $\alpha = 1$, then the following exact solution of homogeneous two-dimensional time-space heat-like fractional order partial differential equation is obtained.

$$u(x,y,t) = \frac{1}{4\pi tk}\exp\left(-\frac{x^2+y^2}{4tk}\right), x,y \in (-\infty,+\infty) \quad and \quad t \in (0,+\infty), k \in (0,+\infty) \tag{22}$$

4.3 Two dimensional fractional telegraph equation

Consider following fractional telegraph equation [79]

$$\frac{\partial^{2\alpha} u(x,y,t)}{\partial t^{2\alpha}} + 2b\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} + b^2 u(x,y,t) = k^{2\alpha}\left(\frac{\partial^{2\alpha} u(x,y,t)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x,y,t)}{\partial y^{2\alpha}}\right)$$
$$where \quad b \geq 0, \quad x,y \in (-\infty,+\infty), t \in (0,+\infty), k \in (0,+\infty), \quad and \quad 0 < \alpha \leq 1. \tag{23}$$

under the initial conditions

$$u(x,y,0) = f(x,y) \tag{24}$$

$$and \quad \frac{\partial^{\alpha}(x,y,0)}{\partial t^{\alpha}} = g(x,y) \tag{25}$$

Applying 2D-FrFT of both sides of Eq. (23) and arranging the terms, it gives

$$\left(D_t^{\alpha} - (-b + ik^{\alpha}\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})\right)\left(D_t^{\alpha} + (b + ik^{\alpha}\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})\right)\hat{u}_{\alpha}(\omega,\tau,t) = 0$$

Solution of above equation is given in terms of Mittag-Leffler function

$$\hat{u}_{\alpha}(\omega,\tau,t) = AE_{\alpha}\left((-b + ik^{\alpha}\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^{\alpha}\right) + BE_{\alpha}\left(-(b + ik^{\alpha}\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^{\alpha}\right)$$

Using Eqs. (24) and (25), it gives the following equation

$$\hat{u}_\alpha(\omega,\tau,t) = \frac{\hat{f}_\alpha(\omega,\tau)}{2}\left[E_\alpha\left((-b+ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha\right)+E_\alpha\left(-(b+ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha\right)\right]$$
$$+\frac{b\hat{f}_\alpha(\omega,\tau)}{2ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}}}\left[E_\alpha\left((-b+ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha\right)-E_\alpha\left(-(b+ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha\right)\right]$$
$$+\frac{\hat{g}_\alpha(\omega,\tau)}{2ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}}}\left[E_\alpha\left((-b+ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha\right)-E_\alpha\left(-(b+ik^\alpha\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha\right)\right] \quad (26)$$

Taking inversion operator of 2D-FrFT , the following general solution of Eq. (23) is obtained.

$$u(x,y,t) =$$
$$F_\alpha^{-1}\left[\left(E_\alpha(-bt^\alpha)\cos_\alpha[k^\alpha(\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha]\hat{f}_\alpha(\omega,\tau)\right)+\left(\frac{bE_\alpha(-bt^\alpha}{k^\alpha}(\omega^{2\alpha}+\tau^{2\alpha})^{\frac{-1}{2}}\sin_\alpha[k^\alpha(\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha]\hat{f}_\alpha(\omega,\tau)\right)\right]$$
$$+F_\alpha^{-1}\left[\left(\frac{bE_\alpha(-bt^\alpha}{k^\alpha}(\omega^{2\alpha}+\tau^{2\alpha})^{\frac{-1}{2}}\sin_\alpha[k^\alpha(\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha]\hat{g}_\alpha(\omega,\tau)\right)\right] \quad (27)$$

Special Cases:
Case I Substituting $f(x,y)=\exp(i(cx+dy))$ and $g(x,y)=\delta(x,y)$ in **(??)** then for $\alpha=1$, one has the following equation

$$u(x,y,t) = \frac{\exp(-bt)\exp(i(cx+dy))}{k\sqrt{c^2+d^2}}\left[k\sqrt{c^2+d^2}\cos(k(\sqrt{\omega^2+\tau^2})t)+b\sin(k(\sqrt{\omega^2+\tau^2})t)\right]$$
$$+\frac{\exp(-bt)}{k}F_\alpha^{-1}\left[\left((\omega^2+\tau^2)^{\frac{-1}{2}}\sin[k(\sqrt{\omega^2+\tau^2})t]\right)\right] \quad (28)$$

Case II If $g(x,y)=0$ in **(??)**,then **Case I** attains the solution in exact form.

$$u(x,y,t) = \frac{\exp(-bt)\exp(i(cx+dy))}{k\sqrt{c^2+d^2}}\left[k\sqrt{c^2+d^2}\cos(k(\sqrt{\omega^2+\tau^2})t)+b\sin(k(\sqrt{\omega^2+\tau^2})t)\right] \quad (29)$$

4.4 Two dimensional time-space wave-like fractional order partial differential equation

Consider following two dimensional time-space wave-like fractional order partial differential equation [78]

$$\frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}} = c^{2\alpha}\left(\frac{\partial^{2\alpha}u(x,y,t)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}u(x,y,t)}{\partial y^{2\alpha}}\right) \quad where \quad x,y\in(-\infty,+\infty), t\in(0,+\infty), c\in(0,+\infty), \quad and \quad 0<\alpha\leq 1. \quad (30)$$

with the initial conditions

$$u(x,y,0) = f(x,y) \quad (31)$$

and

$$\frac{\partial^\alpha u(x,y,0)}{\partial t^\alpha} = g(x,y) \quad (32)$$

If $b=0$ in Eq.(23), then it reduces to the equation Eq.(30). Therefore the general solution of Eq. (30) is given by

$$u(x,y,t) = F_\alpha^{-1}\left[\left(\cos_\alpha[k^\alpha(\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha]\hat{f}_\alpha(\omega,\tau)\right)+\frac{1}{k^\alpha}\left((\omega^{2\alpha}+\tau^{2\alpha})^{\frac{-1}{2}}\sin_\alpha[k^\alpha(\sqrt{\omega^{2\alpha}+\tau^{2\alpha}})t^\alpha]\hat{g}_\alpha(\omega,\tau)\right)\right] \quad (33)$$

using Eq.(**??**).
Special Cases:
Case I: Substituting $f(x,y)=\exp(i(cx+dy))$ and $g(x,y)=\delta(x,y)$ in **(??)** then for $\alpha=1$, one has the following

equation

$$u(x,y,t) = \frac{\exp(i(cx+dy))}{k\sqrt{c^2+d^2}} \left[k\sqrt{c^2+d^2}\cos(k(\sqrt{\omega^2+\tau^2})t)\right] + \frac{1}{k}F_{\alpha}^{-1}\left[\left((\omega^2+\tau^2)^{\frac{-1}{2}}\sin[k(\sqrt{\omega^2+\tau^2})t]\right)\right] \quad (34)$$

Case II: If $g(x,y) = 0$ in (**??**), then **Case I** attains the solution in exact form.

$$u(x,y,t) = \exp(i(cx+dy))\cos[k(\sqrt{(\omega^2+\tau^2)})t] \quad (35)$$

5 Discussion and Motivation

The proposed method is very powerful, reliable, accurate and effective to derive the general analytical solutions of two dimensional FPDEs. This method also gives the exact solutions of these well known FPDEs in special cases when $\alpha = \beta = 1$. Therefore this method is different from other existing methods available in the literature. The reader can derived different analytical solutions of these FPDEs by substituting the different values of $f(x,y)$ and $g(x,y)$ in equations (14), (21),(27) and (33) after employing the inverse operator of two dimensional FrFT. All results derived in Section 3 and Section 4 are in agreement with the original solutions of two dimensional FPDEs obtained using two dimensional Fourier transform for $\alpha = \beta = 1$ [11].

The results derived in this paper extend the existing analytical toolbox for solving fractional mathematical models. The proposed methodology may support future research in applied mathematics, signal processing, mathematical physics and wide range of problems across various domains of science and engineering and also provides a foundation for future analytical work and computational applications involving complex fractional order system. Therefore, the results obtained show that the method used in this paper is efficient and simple.

6 Conclusion

Applicability of fractional calculus and Fourier transform in the field of science and engineering is well established in contestable but fractionalization of Fourier transform in the fractional calculus environment is a recent and promising area of interest. In the present paper we have made an attempt to fractionalize two dimensional Fourier transform using Mittag-Leffler function in the same environment. Mathematical theory of this novel fractional integral transform has been developed by deriving some of its properties.

Finally proposed technique have been applied to derived exact solutions of two dimensional fractional dispersion, fractional heat, fractional telegraph and fractional wave equations in special cases. The digital computation of the produced results in the present article may engender the myriad applications in the vivid zones of digital communications, mathematical modelings, information science, mathematical engineering, signal processing and modern physics.

Declarations

- Acknowledgements: This work was supported by the University Grant Commission (UGC) New Delhi India.
- Funding: Currently, this declaration is not applicable.

- Conflict of interest: On behalf of all authors the corresponding author states that there is no conflict of interest.
- AI Tools: The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this paper.
- Data availability: No data associated with this article.

- Author contribution: T.C. Mahor, R. Mishra: R. Jain: Conceptualization, methodology, validation; T.C. Mahor: investigations, writing original draft preparation; R. Mishra: writing review and editing, visualization and R. Jain: formal analysis.

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