Deferred Statistical Convergence in Paranormed Space

Abstract

In the present paper, we generalize the idea of deferred statistical convergence of scalar valued sequences to that of sequences from an arbitrary paranormed space (X, g). We also introduce and study the concept of strongly deferred g-Cesàro summability in (X, g). Apart from this, we shows that a g-bounded sequence which is deferred statistically g-convergent is strongly deferred g-Cesàro summable to the same limit. Also we characterize the deferred statistically g-convergent sequences in term of its deferred statistically dense g-convergent subsequences.

Keywords : Statistical convergence, deferred density, paranormed space, strong deferred Cesàro summability.

MSC: 40A35, 40A05, 46A45.

1 Introduction

Fast [11] and Steinhaus [25] generalized the usual notion of convergence of real valued sequences by introducing the idea of statistical convergence independently in the same year 1951. In 1953, it was Buck [6] who exemplified it as 'convergence in density'. Schoenberg [24] in 1959 studied statistical convergence as summability method. This concept is also contained in a monograph of Zygmund [28] as "almost convergence". Later, the investigation of statistical convergence from the perspective of sequence space and its connection to summability method was carried out by various authors namely Gupta and Bhardwaj [14], Connor [8, 9], Et and Şengül [10], Fridy [12], Işık [15], Işık and Akbaş [16], Mursaleen [18], Rath and Tripathy [22], Salat [23], Temizsu [26] *et al.*, Bhardwaj and Dhawan [4], and many others.

The concept of natural density is the main pillar in the notion of statistical convergence and defined as

Definition 1.1 *([21]) The natural density of a subset A of the set $\mathbb{N}$ of natural numbers is denoted by $\delta(A)$ and defined as*

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{ m \leq n : m \in A\}|,$$

provided the limit exists. It is easy to verify that

(i) $\delta(A) = 0$ *for finite subset A of $\mathbb{N}$.*

(ii) $\delta(A) + \delta(\mathbb{N} - A) = 1$ *for any subset A of $\mathbb{N}$.*

Definition 1.2 *A scalar valued sequence $u = (u_m)$ is said to be statistically convergent to $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$,*

$$\delta(\{m \in \mathbb{N} : |u_m - \ell| \geq \varepsilon\}) = 0,$$

$$i.e., \lim_{n\to\infty} \frac{1}{n} |\{ m \leq n : |u_m - \ell| \geq \varepsilon\}| = 0.$$

In this case we write $u_m \to \ell(S)$. The set of all statistically convergent sequences is denoted by S.

Various extensions of statistical convergence namely, $\lambda-$statistical convergence, $A-$statistical convergence, $\mu-$statistical convergence, $I-$statistical convergence, lacunary statistical convergence, $f-$statistical convergence and many more have been introduced by several mathematicians. One may refer to [2, 5, 7, 13, 20] where many more references can be found.

R. P. Agnew [1] in 1932 introduced the concept of deferred Cesàro mean of scalar valued sequences $u = (u_m)$ as follows

$$(D_{p,q}u)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} u_m,\ n = 1, 2, 3, \ldots, \tag{1}$$

where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty. \tag{2}$$

Besides being regular, Agnew [1] showed that this method has many more important properties.

It was the work of Agnew, that motivated Küçükaslan and Yılmaztürk ([17],[27]) to introduce the concepts of deferred density and deferred statistical convergence as follows:

Definition 1.3 *([17],[27]) Let A be a subset of $\mathbb{N}$ and denote the set $\{m : p(n) < m \leq q(n), m \in A\}$ by $A_{p,q}(n)$. The deferred density of A is defined as*

$$\delta_D(A) = \lim_{n\to\infty} \frac{1}{q(n) - p(n)} |A_{p,q}(n)|,$$

provided the limit exists. It is observed that

(i) if $q(n) = n$, $p(n) = 0$, then deferred density $\delta_D(A)$ coincides with natural density of A, i.e., $\delta_D(A) = \delta(A)$.

(ii) if $A_1 \subset A_2$, then $\delta_D(A_1) \leq \delta_D(A_2)$.

(iii) $\delta_D(A) + \delta_D(\mathbb{N} - A) = 1$.

Definition 1.4 *([14]) A property P holds for almost all m deferred with respect to D for the sequence $u = (u_m)$, if the set of indicies $m's$ where u_m does not satisfy property P, has zero deferred density. We write this as u_m satisfies property P as $a.a.m$ w.r.t. D.*

Definition 1.5 *([17],[27]) A scalar valued sequence $u = (u_m)$ is said to be deferred statistically convergent to some scalar $\ell \in \mathbb{R}$, if for each $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left| \{p(n) < m \leq q(n) : |u_m - \ell| \geq \varepsilon\} \right| = 0,$$

i.e.,

$$|u_m - \ell| \leq \varepsilon \quad a.a.m \quad w.r.t.\ D.$$

In this case we write $u_m \to \ell(S_D)$. We denote the set of all deferred statistically convergent scalar sequences by S_D. If $q(n) = n$, and $p(n) = 0$, then deferred statistical convergence coincides with usual statistical convergence.

The notion of paranorm is a generalization of absolute value function and defined as

Definition 1.6 *A paranorm is a function $g : X \to \mathbb{R}$ defined on a linear space X such that for all $z_1, z_2, z_3 \in X$*

(p_1) $g(z) = 0$ *if* $z = \theta$

(p_2) $g(-z) = g(z)$

(p_3) $g(z_1 + z_2) \leq g(z_1) + g(z_2)$

(p_4) *if (α_n) is a sequence of scalars with $\alpha_n \to \alpha_0\ (n \to \infty)$ and $z_n, z' \in X$ with $z_n \to z'\ (n \to \infty)$ in the sense that $g(z_n - z') \to 0\ (n \to \infty)$, then $\alpha_n z_n \to \alpha_0 z'\ (n \to \infty)$, in the sense that $g(\alpha_n z_n - \alpha_0 z') \to 0\ (n \to \infty)$.*

A paranorm g for which $g(z) = 0$ implies $z = \theta$ is called a total paranorm on X and the pair (X, g) is called total paranormed space. We recall that each seminorm (norm) g on X is a paranorm (total) but converse need not be true.

Alataibi and Alroqi [3] generalized the concept of statistical convergence and studied strongly Cesàro summability and statistical Cauchy in paranormed space. Mohammed and Mursaleen [19] further generalized this concept by introducing λ-statistical convergence in paranormed space. Let us recall some definitions in paranormed space (X, g).

Definition 1.7 *Let (u_m) be sequence in (X, g) and $u_0 \in X$. If for given $\varepsilon > 0$, there exists a positive integer m_0 such that $g(u_m - u_0) < \varepsilon$ for all $m \geq m_0$, then we say (u_m) is convergent to u_0 and we write $g\text{-}\lim_m u_m = u_0$. We shall denote the class of all convergent sequences by $c(g)$.*

Definition 1.8 *A sequence (u_m) in (X, g) is said to be bounded if there exists some $M > 0$ such that $g(u_m) < M$ for all $m \geq 1$. We write ℓ_∞^g as the class of all bounded sequences.*

In the present paper, we extend the notion of deferred statistical convergence of scalar valued sequences introduced by Küçükaslan and Yılmaztürk ([17],[27]) to that of $X-$valued sequences through paranorm g on X, where X is any non-empty set. It is shown that the terms of a deferred statistical g-convergent sequence (u_m) coincide with that of a g-convergent sequence for almost all m deferred with respect to D. Finally we conclude the paper by introducing the concept of strongly deferred Cesàro g-summable sequences and it is shown that a g-bounded sequence which is deferred statistical g-convergent is strongly deferred Cesàro g-summable to the same limit.

Throughout the paper, we consider the sequences of non negative integers $p = \{p(n) : n \in \mathbb{N}\}$and $q = \{q(n) : n \in \mathbb{N}\}$ satisfying $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$.

2 Deferred g-statistical convergence

We start this section by introducing the concept of deferred statistically g-convergence and deferred statistically g-boundedness. Apart from this various inclusion relations between these concepts are also established.

Definition 2.1 *Let $u = (u_m)$ be a sequence in (X, g). The sequence (u_m) is said to be deferred statistically g-convergent to some $u_0 \in X$, if for given $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} |\{m \in \mathbb{N} : p(n) + 1 \leq m \leq q(n),\ g(u_m - u_0) \geq \varepsilon\}| = 0$$

provided the limit exists and we write $u_m \xrightarrow{S_D^g(c)} u_0$. By $S_D^g(c)$ denotes the set of all deferred statistically $g-$convergent sequences from (X, g).

Definition 2.2 *A sequence $u = (u_m)$ in paranormed space (X, g) is said to be deferred statistically g-bounded if there exists $M > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} |\{m \in \mathbb{N} : p(n) + 1 \leq m \leq q(n),\ g(u_m) > M\}| = 0$$

provided the limit exists. We denote the set of all deferred statistically $g-$bounded sequences by $S_D^g(b)$ from (X, g).

Remark 2.3 *It is observed that for a paranorm g defined on $X = \mathbb{R}$ as $g(z) = |z|$; $z \in \mathbb{R}$ we have the following*

(i) for $q(n) = n$ and $p(n) = 0$, the Definitions 2.1. and 2.2 coincide with the definition of statistical convergence and statistical boundedness, respectively, in usual sense.

(ii) for $q(n) = m_n$ and $p(n) = m_{n-1}$ where $\theta = (m_n)$ is a lacunary sequence, the Definitions 2.1. and 2.2. turns out to be definition of lacunary statistical convergence and lacunary statistical boundedness, respectively.

(iii) For $p(n) = n-\lambda_n+1$ and $q(n) = n$, the Definition 2.1. and 2.2. reduce to those of $\lambda-$statistcal convergence and λ-statistcal boundedness, respectively.

Theorem 2.4 *Every g-convergent sequence is deferred statistically g-convergent i.e., $c(g) \subset S_D^g(c)$. Converse is not true, in general.*

Proof. The result follows in view that a finite set has zero deferred density. For proper inclusion, consider a sequence $u = (u_m)$ in paranormed space $X = \mathbb{R}$ with paranorm defined as $g(\xi) = |\xi|$ for all $\xi \in \mathbb{R}$ as

$$u_m = \begin{cases} q(n), \text{ if } m = q(n) \\ 0, \text{ if } m \neq q(n). \end{cases}$$

Then, for given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{m \in \mathbb{N} : p(n)+1 \le m \le q(n), g(u_m - 0) \ge \varepsilon\}|$$
$$= \lim_{n\to\infty} \frac{1}{q(n)-p(n)} = 0.$$

Hence (u_m) is deferred statistically g-convergent to 0. Suppose, if possible, (u_m) is g-convergent to 0 then for given $\varepsilon > 0$,

$$g(u_m - 0) < \varepsilon \quad \textit{for all} \quad m \ge m_0,$$

i.e.,

$$q(m) < \varepsilon \, \textit{for all} \quad m \ge m_0, \quad m = q(n), \quad n \in \mathbb{N}$$

contrary to the fact that $q(m) \to \infty$ as $m \to \infty$. □

Theorem 2.5 *Every bounded sequence is deferred statistically g-bounded. Converse is not true, in general.*

Proof. Proof is obvious as empty set has zero deferred density, i.e., $\ell^g_\infty \subset S^g_D(b)$. Inculsion is proper in view of example cited in Theorem 2.4. □

Theorem 2.6 *Every deferred statistically g-convergent sequence is deferred statistically g-bounded, i.e., $S^g_D(c) \subset S^g_D(b)$, inclusion is strict.*

Proof. The proof is an immediate consequene of tha fact that every finite set has zero deferred density. For proper inclusion, consider $\mathbb{R}$ with paranorm defined as $g(\xi) = |\xi|$ for all $\xi \in \mathbb{R}$. Then the sequence (u_m) defined as

$$u_m = \begin{cases} 1, \text{ if} & m = 2n \\ -1, \text{ otherwise} \,. \end{cases}$$

is deferred statistically g-bounded but not deferred statistically g-convergent. □

Theorem 2.7 *Every g-convergent sequence is deferred statistically g-bounded.*

Proof. The proof follows immediately from Theorem 2.4. and Theorem 2.6. □

Theorem 2.8 *([14]) Let f be an unbounded modulus. Then $S^f_D - \lim u_m = \ell$ iff there exist $T \subset \mathbb{N}$ such that $\delta^f_D(T) = 0$ and $\lim\limits_{m\in\mathbb{N}-T} u_m = \ell$.*

By taking $f(z) = z$ in the above theorem we have

Theorem 2.9 *A real valued sequence (u_m) is deferred statistically convergent to $\ell \in \mathbb{R}$ iff there exists $T \subset \mathbb{N}$ such that $\delta_D(T) = 0$ and $\lim\limits_{m\in\mathbb{N}-T} u_m = \ell$.*

The following theorem shows that a deferred statistically g-convergent sequence in paranormed space (X, g) has similar structure as in Theorem 2.9 for real valued deferred statistically convergent sequence.

Theorem 2.10 *A sequence (u_m) is deferred statistically g-convergent to $u_0 \in X$ iff there exists $T \subset \mathbb{N}$ such that $\delta_D(T) = 0$ and $(u_m)_{m \in \mathbb{N}-T} \xrightarrow{g} u_0$.*

Proof. As (u_m) is deferred statistically g-convergent to $u_0 \in X$, so for given $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{m \in \mathbb{N} : p(n)+1 \leq m \leq q(n),\, g(u_m - u_0) \geq \varepsilon\}| = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{m \in \mathbb{N} : p(n)+1 \leq m \leq q(n), z_m \geq \varepsilon\}| = 0$$

where $z_m = g(u_m - u_0)$. This implies that (z_m) is a deferred statistically convergent sequence of reals converging to zero. Using Theorem 2.9, there exist $T \subset \mathbb{N}$ such that $\delta_D(T) = 0$ and $\lim_{m \in \mathbb{N}-T} z_m = 0$, i.e., $\lim_{m \in \mathbb{N}-T} g(u_m - u_0) = 0$. Conversely, for given $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $g(u_m - u_0) < \varepsilon$ for all $m \in \mathbb{N} - T$ with $m \geq m_0$. The proof now follows immediately from the inclusion relation $\{m \in \mathbb{N} : g(u_m - u_0) \geq \varepsilon\} \subset T \cup \{1, 2, 3, ..., m_0 - 1\}$. $\square$

The next theorem indicates that the terms of a deferred statistically g-convergent sequence are coincident to that of a convergent sequence for almost all m deferred with respect to D.

Theorem 2.11 *A sequence (u_m) is deferred statistically g-convergent to $u_0 \in X$ if and only if there exists a sequence (v_m), g-convergent to u_0 such that $v_m = u_m$ a.a. m deferred w.r.t. D.*

Proof. Let (u_m) be deferred statistically g-convergent to $u_0 \in X$, so we have $\delta_D(A) = 0$ where $A = \{m \in \mathbb{N} : g(u_m - u_0) \geq \varepsilon\}$. Take

$$v_m = \begin{cases} u_0, \text{ if } m \in A \\ u_m, \text{ if } m \in \mathbb{N} - A. \end{cases}$$

Now $\{m \in \mathbb{N} : v_m \neq u_m\} \subseteq A$ and so $v_m = u_m$ *a.a.m.* deferred w.r.t. D. Also

$$g(y_m - z_0) = \begin{cases} g(u_m - u_0) & \text{for } m \in \mathbb{N} - A \\ 0 & \text{for } m \in A \end{cases}$$

implies $g(v_m - u_0) < \varepsilon$ for all $m \geq 1$. Thus (v_m) is convergent to u_0 and $v_m = u_m$ *a.a.m.* deferred w.r.t. D.

Conversely, for given $\varepsilon > 0$, there exists a positive integer m_0 such that $g(v_m - u_0) < \varepsilon$ for all $m \geq m_0$. The proof follows in view of the following inclusion

$$\{m \in \mathbb{N} : g(u_m - u_0) > \varepsilon\} \subset \{m \in \mathbb{N} : u_m \neq v_m\} \cup \{1,\, 2,\, 3,\, \ldots,\, m_0 - 1\}.$$

$\square$

Following on the similar lines, we have

Theorem 2.12 *A sequence (u_m) is deferred statistically g-bounded if and only if there exists a sequence (v_m), g-bounded such that $v_m = u_m$ a.a. m deferred w.r.t. D.*

3 Strongly deferred Cesàro g-summable sequences

We begin this section by introducing the notion of strongly deferred Cesàro g-summable sequences and observe that bounded deferred statistical g-convergent sequences are strongly deferred Cesàro g-summable.

Definition 3.1 *Let (X, g) be a paranormed space. A sequence (u_m) in (X, g) is said to be strongly deferred Cesàro g-summable to some $u_0 \in X$ if*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{m=p(n)+1}^{q(n)} g(u_m - u_0) = 0.$$

We write it as $u_m \longrightarrow u_0(w_D^g)$ and by $w_D^g(c)$ we mean the set of all strongly deferred Cesàro g-summable sequences in paranormd space (X, g).

Theorem 3.2 *Every strongly deferred Cesàro g-summable sequence is deferred statistically g-convergent to the same limit, i.e., $w_D^g(c) \subset S_D^g(c)$. Reverse implication need not be true.*

Proof. For any $\varepsilon > 0$ and $u_0 \in X$, we have

$$\begin{aligned}
&\frac{1}{q(n) - p(n)} \sum_{m=p(n)+1}^{q(n)} g(u_m - u_0) \\
&\geq \frac{1}{q(n) - p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)\geq\varepsilon}}^{q(n)} g(u_m - u_0) \\
&\geq \varepsilon \frac{1}{q(n) - p(n)} |\{m : p(n) < m \leq q(n),\ g(u_m - u_0) \geq \varepsilon\}|
\end{aligned}$$

and hence the result follows.

For proper inclusion, we consider the following:
Let $X = \mathbb{R}$ with paranorm g defined as $g(\xi) = |\xi|$. Take a sequence (u_m) as

$$u_m = \begin{cases} (q(n) - p(n))^2, \text{if } m = q(n)\, n \in \mathbb{N}; \\ 0, \text{otherwise}. \end{cases}$$

Then

$$\begin{aligned}
&\frac{1}{q(n) - p(n)} \ |\{m : p(n) < m \leq q(n),\ g(u_m - 0) \geq \varepsilon\}| \\
&\leq \frac{1}{q(n) - p(n)} \to 0 \text{ as } n \to \infty
\end{aligned}$$

implies $(u_m) \in S_D^g(c)$. However $(u_m) \notin w_D^g(c)$ as

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} \sum_{m=p(n)+1}^{q(n)} g(u_m - 0) = \frac{(q(n) - p(n))^2}{(q(n) - p(n))} \to \infty.$$

□

However, if we take g-bounded sequences, then converse of Theorem 3.2 holds.

Theorem 3.3 *If* $(u_m) \in \ell_\infty^g$, *then* $S_D^g(c) \subseteq w_D^g(c)$.

Proof. As $(u_m) \in \ell_\infty^g$ so there exists $M > 0$ such that $g(u_m) \leq M$ for all $m \geq 1$. Then for $u_0 \in X$, we have

$$g(u_m - u_0) \leq g(u_m) + g(u_0) \leq M + g(u_0) \leq K$$

where $K = M + g(u_0)$ (say).

$$\begin{aligned}
&\frac{1}{q(n)-p(n)} \sum_{m=p(n)+1}^{q(n)} g(u_m - u_0) \\
&\quad = \frac{1}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)\geq\varepsilon}}^{q(n)} g(u_m - u_0) + \frac{1}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)<\varepsilon}}^{q(n)} g(u_m - u_0) \\
&\quad \leq \frac{K}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)\geq\varepsilon}}^{q(n)} 1 + \frac{\varepsilon}{q(n)-p(n)} \sum_{\substack{m=p(n)+1 \\ g(u_m-u_0)<\varepsilon}}^{q(n)} 1 \\
&\quad = \frac{K}{q(n)-p(n)} |\{m : p(n) < m \leq q(n),\, g(u_m - u_0) \geq \varepsilon\}| + \varepsilon
\end{aligned}$$

and hence the result follows. $\square$

In view of Theorem 3.2. and Theorem 3.3., we have

Corollary 3.4 $\ell_\infty^g \bigcap S_D^g(c) = \ell_\infty^g \bigcap w_D^g(c)$.

Theorem 3.5 *Let* $\{\frac{p(n)}{q(n)-p(n)}\}_{n\in\mathbb{N}} \in \ell_\infty$. *If* $u_m \overset{S^g(c)}{\longrightarrow} u_0$ *then* $u_m \overset{S_D^g(c)}{\longrightarrow} u_0$, *where* ℓ_∞ *is the space of all bounded scalar sequences.*

Proof. Let $(u_m) \in S^g(c)$. Then for a given $\varepsilon > 0$, we have

$$\begin{aligned}
&|\{m : p(n) + 1 \leq m \leq q(n) : g(u_m - u_0) \geq \varepsilon\}| \\
&\quad = |\{m : 1 < m \leq q(n) : g(u_m - u_0) \geq \varepsilon\}| - |\{m : 1 < m \leq p(n) : g(u_m - u_0) \geq \varepsilon\}|
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{q(n)-p(n)} |(\{m : p(n) + 1 < m \leq q(n) : g(u_m - u_0) \geq \varepsilon\}| \\
&\quad = \frac{q(n)}{q(n)-p(n)} \frac{1}{q(n)} |\{m : 1 < m \leq q(n) : g(u_m - u_0) \geq \varepsilon\}| \\
&\quad - \frac{p(n)}{q(n)-p(n)} \frac{1}{p(n)} |\{m : 1 < m \leq p(n) : g(u_m - u_0) \geq \varepsilon\}|
\end{aligned}$$

The result follows in view of the boundedness of the sequences $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ and $\left\{1 + \frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$. $\square$

Remark 3.6 *Converse of Theorem 3.5. need not be true even if* $\{\frac{p(n)}{q(n)-p(n)}\}_{n\in\mathbb{N}}$ *is bounded, i.e,* $S_D^g(c) \nsubseteq S^g(c)$. *Let us demonstrate this with the following example:*
Let $X = \mathbb{R}$ *with pararanorm* g *defined on it as* $g(\xi) = |\xi|$ *for* $\xi \in X$. *Take* $p(n) = n$. *Choose positive integer* $q(1)$ *such that* $q(1) > p(1) = 1$. *Next choose positive integer* $q(2)$ *with* $q(2) > 2p(2) = 2^2$ *and so on, i.e, positive integer* $q(n)$ *are chosen such that* $q(n) > np(n) = n^2$ *for each positive integer* $n \in \mathbb{N}$. *Clearly sequence* $\{\frac{p(n)}{q(n)-p(n)}\}_{n\in\mathbb{N}}$ *is bounded. Take sequence* $\{u_m\}$ *defined as*

$$u_m = \begin{cases} 1 & \text{if } p(n) < m \leq 2p(n), \\ 0 & \text{otherwise.} \end{cases} \qquad \text{on } (p(n), q(n)]\ .$$

For $n \geq 2$, *we have* $q(n) > np(n) \geq 2p(n) > p(n)$, *and so*

$$\begin{aligned} &\frac{1}{q(n)-p(n)}|\{m : p(n)+1 < m \leq q(n) : g(u_m - 0) \geq \varepsilon\}| \\ &\qquad = \frac{p(n)}{q(n)-p(n)} < \frac{\frac{1}{n}q(n)}{q(n)-p(n)} \\ &\qquad = \frac{\frac{1}{n}}{1-\frac{p(n)}{q(n)}} \to 0 \quad as\, n \to \infty. \end{aligned}$$

Thus sequence $u_m \longrightarrow 0\,(S_D^g(c))$, *however*

$$\frac{1}{2p(n)}|\{m \leq 2p(n) : g(u_m - 0) \geq \varepsilon\}| \geq \frac{2p(n)-p(n)}{2p(n)} = \frac{1}{2} \quad \text{for each} \quad n \in \mathbb{N}.$$

implies $u_m \nrightarrow 0\,(S^g(c))$. $\square$

We now demonstrate that the converse of Theorem 3.5 for a particular choice of sequence $q(n)$, i.e., for $q(n) = n$ in the form of following Theorem.

Theorem 3.7 *Let* $q(n) = n$ *and* $p(n)$ *be arbitrary sequence satisfying (2). Then* $u_m \xrightarrow{S_D^g(c)} u_0$ *implies* $u_m \xrightarrow{S^g} u_0$.

Proof. Let $n \in \mathbb{N}$ be arbitrary, but fixed. Then for this n, we have $n = q(n)$. Setting $n^{(1)} = p(n)$. Now $n^{(1)} = q(n^{(1)})$ and say $n^{(2)} = p(n^{(1)})$ and so on. Thus we have a decreasing sequence $n = q(n) > p(n) = n^{(1)} = q(n^{(1)}) > p(n^{(1)}) = n^{(2)} = q(n^{(2)}) > p(n^{(2)}) = n^{(3)}....$

$$\begin{aligned} &\{m \leq n : g(u_m - u_0) \geq \varepsilon\} \\ &= \{m \leq n^{(1)} : g(u_m - u_0) \geq \varepsilon\} \cup \{n^{(1)} < m \leq n : g(u_m - u_0) \geq \varepsilon\} \\ &= \{m \leq n^{(2)} : g(u_m - u_0) \geq \varepsilon\} \cup \{n^{(2)} < m \leq n^{(1)} : g(u_m - u_0) \geq \varepsilon\} \\ &\cup \{n^{(1)} < m \leq n : g(u_m - u_0) \geq \varepsilon\} \\ &\vdots \quad \vdots \quad \vdots \\ &= \{m \leq n^{(h)} : g(u_m - u_0) \geq \varepsilon\} \cup \{n^{(h)} < m \leq n^{(h-1)} : g(u_m - u_0) \geq \varepsilon\} \\ &\ldots \cup \{n^{(2)} < m \leq n^{(1)} : g(u_m - u_0) \geq \varepsilon\} \cup \{n^{(1)} < m \leq n : g(u_m - u_0) \geq \varepsilon\} \end{aligned}$$

for a certain positive integer h depending on n such that $n^{(h)} \geq 1$ and $n^{(h+1)} = 0$. Thus

$$\begin{aligned}
&\frac{1}{n}|\{m \leq n : g(u_m - u_0) \geq \varepsilon\}| \\
&= \sum_{m=0}^{h} \frac{1}{n}|\{n^{m+1} < m \leq n^m : g(u_m - u_0) \geq \varepsilon\}| \\
&= \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n(n^{(m)} - n^{(m+1)})}|\{n^{(m+1)} < m \leq n^{(m)} : g(u_m - u_0) \geq \varepsilon\}| \\
&= \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n(q(n^{(m)}) - p(n^{(m)})}|\{p(n^{(m)}) < m \leq q(n^{(m)}) : g(u_m - u_0) \geq \varepsilon\}| \\
&= \sum_{m=0}^{h} a_{n,m} t_m
\end{aligned}$$

where $n^{(0)} = n$

$$a_{n,m} = \begin{cases} \dfrac{n^{(m)} - n^{(m+1)}}{n}, \text{if} \quad m = 0, 1, 2, \ldots, h; \\ 0 \qquad\qquad \textit{otherwise.} \end{cases}$$

and

$$t_m = \begin{cases} \frac{1}{q(n^{(m)})-p(n^{(m)})}|\{p(n^{(m)}) < m \leq q(n^{(m)}) : g(u_m - u_0) \geq \varepsilon\}|, \text{if}\, m = 0, 1, 2, \ldots, h; \\ 0, \text{otherwise} \end{cases}.$$

As (t_m) is a null sequence, so the result follows by using Silverman-Toeplitz Theorem. $\square$

Corollary 3.8 *If we consider $q(n) = n$ and $p(n)$ be arbitrary sequence satisfying (2) such that $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}} \in \ell_\infty$. Then $S^g(c) = S^g_D(c)$.*

Proof. The proof is an immediate consequence of Theorem 3.5. and Theorem 3.7

Running on the same lines as in Theorem 2.3.1 and Theorem 2.3.2 of [17] we have the following

Theorem 3.9 *Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be sequences of natural numbers satisfying*

$$p(n) \leq p'(n) < q'(n) \leq q(n) \tag{3}$$

$$\lim_{n\to\infty} q(n) - p(n) = \infty = \lim_{n\to\infty} q'(n) - p'(n) \tag{4}$$

and both sequences $\{p'(n) - p(n)\}_{n\in\mathbb{N}}$ and $\{q'(n) - q(n)\}_{n\in\mathbb{N}}$ are in ℓ_∞. Then $S^g_{p',q'}(c) \subset S^g_{p,q}(c)$.

Theorem 3.10 *Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be sequences of natural numbers satisfying conditions (2) and (3) of Theorem 3.9. such that $\liminf_n \frac{q(n)-p(n)}{q'(n)-p'(n)} > 0$. Then $S^g_{p,q}(c) \subset S^g_{p',q'}(c)$.*

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