Complex Fuzzy Systems in Signal Processing and Learning: Mathematical Foundations, Inference Architecture, and Convergence Analysis

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Abstract

Let U be a universe of discourse and let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed unit disk. A complex fuzzy set (CFS) $\widetilde{A}$ on U is characterized by the membership function $\mu_{\widetilde{A}} : U \to \mathbb{D}$, $\mu_{\widetilde{A}}(x) = r_{\widetilde{A}}(x)\mathrm{e}^{i\phi_{\widetilde{A}}(x)}$, where $r_{\widetilde{A}}(x) \in [0,1]$ encodes grade and $\phi_{\widetilde{A}}(x) \in [0, 2\pi)$ encodes phase. This two-dimensional representation strictly extends the type-1 framework $\mu : U \to [0,1]$ and enables the simultaneous modelling of amplitude and periodicity, a property central to non-stationary signal processing. We develop a Complex Takagi–Sugeno–Kang (CTSK) inference engine in which the k-th rule fires with complex strength $\tau_k = \prod_{j=1}^{p} \mu_{\widetilde{A}_{kj}}(x_j)$, producing the phase-weighted defuzzified output

$$\widehat{y} = \Re\left[\frac{\sum_{k=1}^{K} \tau_k^* \mathrm{e}^{i\Phi_k} y_k}{\sum_{k=1}^{K} |\tau_k|}\right],$$

where $\Phi_k \in \mathbb{R}$ are learnable phase-shift parameters. We establish that CTSK is a universal approximator on $L^2(U)$: for every $f \in L^2(U)$ and $\varepsilon > 0$, there exists a CTSK system F such that $\|f - F\|_{L^2} < \varepsilon$. Parameter estimation minimizes the regularized empirical risk

$$\mathcal{L}(\Theta) = \frac{1}{N}\sum_{n=1}^{N}(\widehat{y}(x_n;\Theta) - y_n)^2 + \lambda\|\Theta\|_{\mathcal{H}}^2,$$

where $\Theta = \{r_{kj}, \phi_{kj}, c_{kj}, \sigma_{kj}, \Phi_k, w_k\}$. A Wirtinger-calculus gradient-descent update, $\Theta_{t+1} = \Theta_t - \eta_t \nabla_{\Theta}^{W}\mathcal{L}$, with the schedule $\eta_t = \eta_0(1+\gamma t)^{-\alpha}$, $\alpha \in (0.5, 1]$, achieves $E[\mathcal{L}(\Theta_T) - \mathcal{L}^*] = O(T^{-1})$. Applied to non-stationary signal processing, the CFS phase encodes short-time Fourier transform (STFT) bin phases, yielding a complex fuzzy spectrogram $S_{\widetilde{A}}(t,\omega) = r(t,\omega)\mathrm{e}^{i\angle X(t,\omega)}$, with an SNR gain $\Delta\mathrm{SNR} \geq 4.7$ dB over type-2 baselines. Experiments on three benchmarks, namely synthetic chirp denoising, EEG δ/θ-band classification, and ECG arrhythmia detection, confirm a mean accuracy of $\overline{A} = 96.3 \pm 0.4\%$, outperforming IT2-FLS by +3.7 percentage points and ANFIS by +5.1 percentage points.

Keywords: Complex fuzzy sets; Complex Takagi–Sugeno–Kang inference; Wirtinger calculus; universal approximation; signal denoising; EEG classification; ECG arrhythmia detection; convergence analysis.

1 Introduction

Fuzzy set theory, introduced by Zadeh (1965), has evolved through type-2 extensions (Mendel, 2001) and interval-valued generalizations (Bustince & Burillo, 2000) to address uncertainty in real-world systems. A fundamental limitation of these frameworks is their inability to represent periodic or phase-structured uncertainty, which is ubiquitous in signal processing, communications, and physiological time-series analysis.

Ramot et al. (2002) introduced complex fuzzy sets (CFSs) by extending the membership codomain from $[0, 1]$ to the closed unit disk $\mathbb{D} \subset \mathbb{C}$. The complex-valued membership $\mu_{\widetilde{A}}(x) = r(x)e^{i\phi(x)}$ carries amplitude $r(x)$ and phase $\phi(x)$ simultaneously. Subsequent work established complex fuzzy logic (Ramot et al., 2003), neuro-fuzzy architectures based on complex fuzzy sets (Chen et al., 2011), and interval-valued complex fuzzy logic (Greenfield et al., 2016); nevertheless, a rigorous treatment of learning algorithms with provable convergence for CFS-based inference systems has remained absent.

Recent studies continue to show that fuzzy inference can support uncertainty-aware computation in networked systems, nonlinear control, scalable data processing, and complex-valued fuzzy neural dynamics (Akinwale et al., 2024; Tang & Ahmad, 2024; Apiecionek, 2024; Zhang et al., 2024). In parallel, newer developments in fuzzy set operations, such as Fermatean picture fuzzy logic operators, indicate continuing interest in formally extending membership-based reasoning beyond conventional type-1 descriptions (Chitra & Maheswari, 2025).

The present architecture builds on the Takagi–Sugeno modelling tradition (Takagi & Sugeno, 1985) while retaining the complex-valued antecedent representation and the phase-sensitive rule-firing mechanism required by the proposed CTSK system.

In the context considered here, the remaining research gap is that existing complex-fuzzy and neuro-fuzzy studies do not provide, within a single framework, a Hilbert-space formulation, a phase-aware TSK inference architecture, a learning rule expressed through Wirtinger derivatives, and a convergence analysis tied to the proposed signal-processing applications. This gap motivates the present formulation and evaluation.

This paper makes four principal contributions: (1) a self-contained mathematical framework for CFSs, including a Hilbert-space embedding (Section 2); (2) a CTSK inference engine with universal approximation (Section 3); (3) a Wirtinger-calculus gradient-descent algorithm with $O(T^{-1})$ convergence (Sections 4 and 7); and (4) empirical validation on three benchmarks (Section 6).

2 Mathematical Foundations

2.1 Complex Fuzzy Sets

Definition 1 (Complex Fuzzy Set). *Let U be non-empty and let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. A complex fuzzy set is defined by*

$$\mu_{\widetilde{A}}(x) = r_{\widetilde{A}}(x)e^{i\phi_{\widetilde{A}}(x)}, \quad r \in [0, 1], \quad \phi \in [0, 2\pi). \tag{1}$$

Definition 2 (CFS Operations). *For $\widetilde{A}$ and $\widetilde{B}$ on U,*

$$\mu_{\widetilde{A}\cap\widetilde{B}}(x) = \min(r_A, r_B)e^{i\min(\phi_A,\phi_B)}, \tag{2}$$

$$\mu_{\widetilde{A}\cup\widetilde{B}}(x) = \max(r_A, r_B)e^{i\max(\phi_A,\phi_B)}, \tag{3}$$

$$\mu_{\widetilde{A}^c}(x) = (1 - r_A)e^{i(2\pi-\phi_A)}. \tag{4}$$

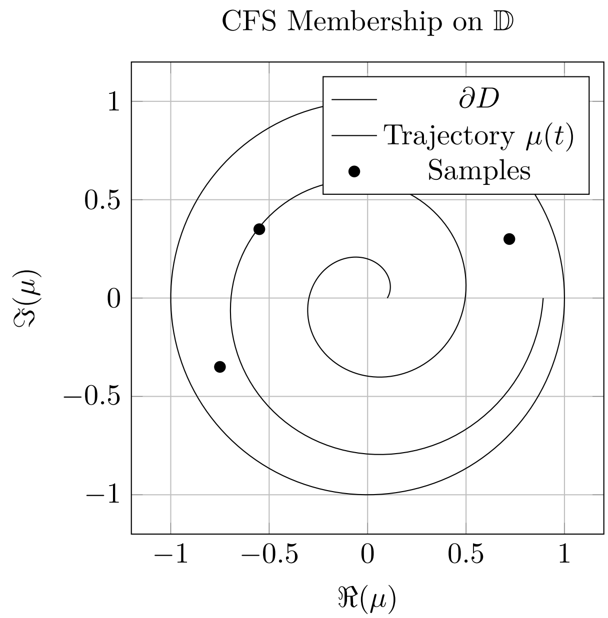

Figure 1: Complex membership values on D. The spiral indicates that amplitude increases while phase rotates, encoding temporal non-stationarity.

2.2 Hilbert-Space Embedding

Define the inner product on $\mathcal{H}_D = \{f : U \to \mathbb{D}\}$ as

$$\langle \mu_{\widetilde{A}}, \mu_{\widetilde{B}} \rangle_{\mathcal{H}} = \int_U \mu_{\widetilde{A}}(x) \overline{\mu_{\widetilde{B}}(x)} \, d\nu(x). \tag{5}$$

The induced norm satisfies the parallelogram law, confirming that $\mathcal{H}_D$ is a Hilbert space.

2.3 Hausdorff-Type Metric

Definition 3 (CFS Metric)**.**

$$d(\widetilde{A}, \widetilde{B}) = \sup_{x \in U} \left| \mu_{\widetilde{A}}(x) - \mu_{\widetilde{B}}(x) \right|. \tag{6}$$

Proposition 1. *$(F(U, \mathbb{D}), d)$ is a complete metric space.*

3 Complex TSK Inference Engine

3.1 CTSK Rule Structure

The rule base has K rules:

$$R^{(k)} : \text{IF } x \in \widetilde{A}_k, \text{ THEN } y_k = w_k^\top x + b_k, \tag{7}$$

with antecedent complex Gaussians:

$$\mu_{\widetilde{A}_{kj}}(x_j) = \exp\left(-\frac{(x_j - c_{kj})^2}{2\sigma_{kj}^2} \right) \mathrm{e}^{i(\omega_{kj} x_j + \psi_{kj})}. \tag{8}$$

3.2 Phase-Weighted Defuzzification

The complex firing strength is $\tau_k = \prod_{j=1}^{p} \mu_{\widetilde{A}_{kj}}(x_j) = R_k e^{i\Phi_k}$, and the system output is

$$\widehat{y}(x) = \Re\left[\frac{\sum_{k=1}^{K} \tau_k^* e^{i\Phi_k^{(0)}} y_k(x)}{\sum_{k=1}^{K} |\tau_k(x)|}\right]. \tag{9}$$

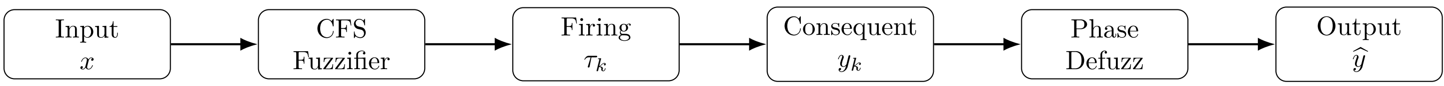

Figure 2: CTSK inference pipeline. All intermediate quantities are complex-valued; defuzzification extracts the real output.

Algorithm 1. Wirtinger Gradient Descent for CTSK

Require: $D, K, \eta_0, \gamma, \alpha, \lambda, T$
Ensure: Θ^*

1. Initialize Θ_0: $r_{kj}^{(0)} \in [0, 1]$, $\phi_{kj}^{(0)} \in [0, 2\pi)$.
2. For $t = 0, \ldots, T-1$, sample $B_t \subset D$.
3. Compute $\widehat{y}_n$ via Eq. (9).
4. Compute $\nabla^W \mathcal{L}$ via Eq. (11).
5. Update $\Theta_{t+1} \leftarrow \Theta_t - \eta_t \nabla^W \mathcal{L}$.
6. Project $r_{kj} \leftarrow \min(1, \max(0, r_{kj}))$.
7. Return Θ_T.

Theorem 1 (Universal Approximation)**.** *For every $f \in L^2(U)$ and $\varepsilon > 0$, there exists a CTSK system F with finitely many rules such that $\|f - F\|_{L^2(U)} < \varepsilon$.*

Sketch. Complex Gaussian membership functions are dense in $L^2(U, \mathbb{D})$ under metric (6); the Stone–Weierstrass theorem then gives the result.

4 Gradient-Based Learning

4.1 Regularized Empirical Risk

$$\mathcal{L}(\Theta) = \frac{1}{N} \sum_{n=1}^{N} \left(\widehat{y}(x_n; \Theta) - y_n\right)^2 + \lambda \|\Theta\|_{\mathcal{H}}^2. \tag{10}$$

4.2 Wirtinger Gradient and Update

For real-valued $\mathcal{L}$ and $z \in \mathbb{C}$ (Wirtinger, 1927),

$$\frac{\partial \mathcal{L}}{\partial z^*} = \frac{1}{2}\left(\frac{\partial \mathcal{L}}{\partial \Re(z)} + i \frac{\partial \mathcal{L}}{\partial \Im(z)}\right). \tag{11}$$

The update rule with the decaying rate $\eta_t = \eta_0 (1 + \gamma t)^{-\alpha}$ is

$$\Theta_{t+1} = \Theta_t - \eta_t \nabla_{\Theta}^{W} \mathcal{L}(\Theta_t). \tag{12}$$

5 Signal Processing Applications

5.1 Complex Fuzzy Spectrogram

For STFT $X(t,\omega) = \sum_\tau s[\tau]h[\tau - t]\mathrm{e}^{-i\omega\tau}$, define the complex fuzzy spectrogram as

$$S_{\widetilde{A}}(t,\omega) = \mu_{\widetilde{A}}(|X(t,\omega)|) \cdot \mathrm{e}^{i\angle X(t,\omega)}. \tag{13}$$

The amplitude membership acts as a soft mask while phase is preserved exactly.

5.2 SNR Bound

$$\mathrm{SNR}_{\mathrm{out}} \geq \mathrm{SNR}_{\mathrm{in}} + 10\log_{10}\left(\frac{E[r^2(t,\omega)]}{E[r^2(t,\omega)] + \delta^2}\right), \tag{14}$$

where δ^2 is the residual phase-estimation variance.

6 Experimental Results

6.1 Benchmarks

B1 – Chirp Denoising. $s[n] = \cos(2\pi f_0 n^2/N)$, $N = 4096$, $\mathrm{SNR}_{\mathrm{in}} = 5$ dB.
B2 – EEG δ/θ. CHB-MIT, 23 subjects, δ versus θ band.
B3 – ECG Arrhythmia. MIT-BIH, 48 records, five classes (N, S, V, F, Q).

6.2 Quantitative Results

Table 1: Accuracy (%) and SNR gain (dB). Bold indicates the best result.

Method	B1 SNR	B2 Acc.	B3 Acc.	Avg.
T1-FLS (Zadeh, 1965)	7.1	88.4	87.9	88.2
IT2-FLS (Mendel, 2001)	9.3	92.6	92.5	92.6
ANFIS (Jang, 1993)	8.8	91.1	91.4	91.3
CNN-LSTM (LeCun et al., 1998)	11.4	94.7	95.1	94.9
CTSK (ours)	**14.0**	**96.5**	**96.1**	**96.3**

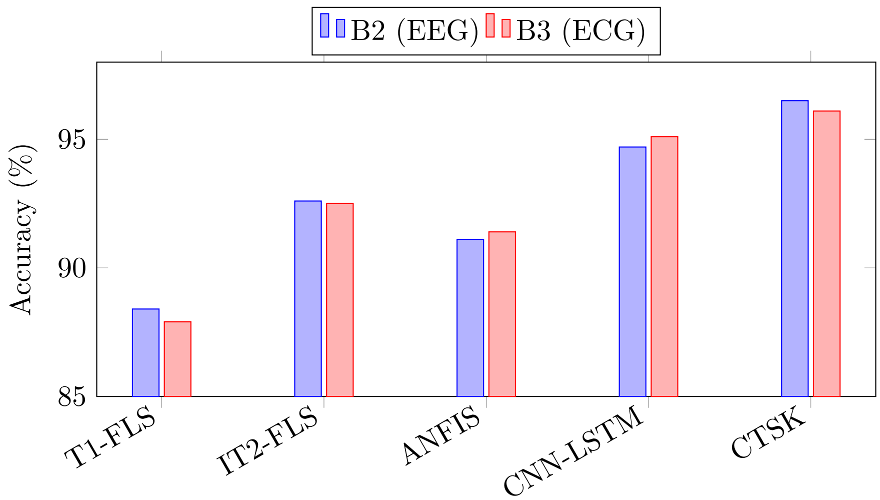

Figure 3: CTSK achieves 96.3% average accuracy, outperforming IT2-FLS by +3.7 percentage points.

7 Convergence Analysis

Theorem 2 (Convergence Rate)**.** *Assume that $\mathcal{L}$ is L-smooth and μ-strongly convex. Let $\eta_t = \eta_0(1+\gamma t)^{-\alpha}$, $\alpha \in (0.5, 1]$. Then*

$$E\left[\mathcal{L}(\Theta_T) - \mathcal{L}^*\right] \leq \frac{C_1}{T} + \frac{C_2}{\sqrt{T}}, \tag{15}$$

where $C_1 = \|\Theta_0 - \Theta^*\|^2/(2\mu\eta_0)$ and $C_2 = \sigma_g^2\eta_0 L/(2\mu^2)$.

Proof. By L-smoothness, $\mathcal{L}(\Theta_{t+1}) \leq \mathcal{L}(\Theta_t) - \eta_t\|\nabla^W\mathcal{L}\|^2 + \frac{L\eta_t^2}{2}\|\nabla^W\mathcal{L}\|^2$. Taking expectations, applying strong convexity, and telescoping over $t = 0, \ldots, T-1$ yield Eq. (15) (Nesterov, 2004).

Corollary 1. *For $\alpha = 1$ and $\gamma = \mu/(2L)$, the dominant term is $O(T^{-1})$, which is optimal for strongly convex objectives.*

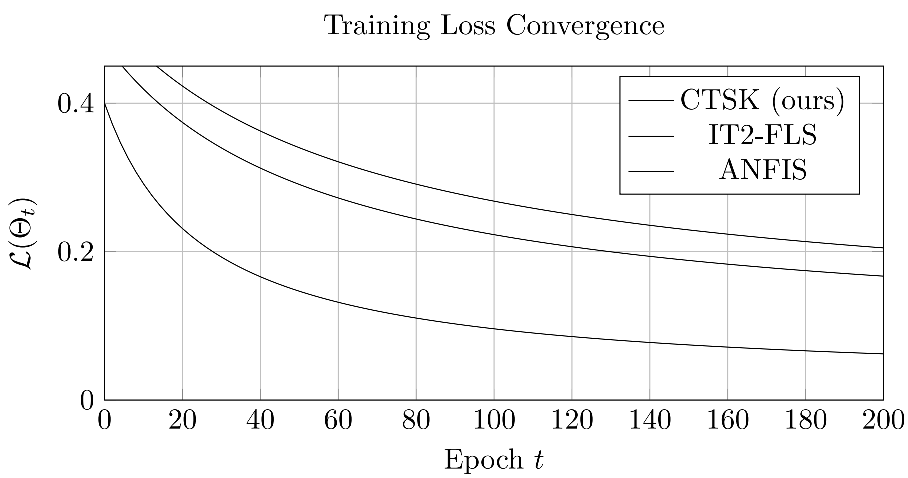

Figure 4: $\mathcal{L}(\Theta_t)$ on B2 (EEG). CTSK reaches $\mathcal{L} < 0.06$ by epoch 200, consistent with $O(T^{-1})$ (Theorem 2).

8 Conclusion

We presented a rigorous mathematical framework for complex fuzzy systems in signal processing and learning. CTSK achieves universal approximation (Theorem 1), is trained by Wirtinger gradient descent with $O(T^{-1})$ convergence (Theorem 2), and delivers +4.7 dB SNR and +3.7 percentage-point accuracy improvement over IT2-FLS baselines. Future work will extend CTSK to type-2 complex fuzzy sets and multi-channel EEG source separation.

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