Separation axioms weaker than $R - \mathcal{I} - R_0$

Abstract

Aims/objectives: This paper studies separation axioms in ideal topological spaces that are weaker than the $R - \mathcal{I} - R_0$ axiom and examines selected properties of $R - \mathcal{I} - R_1$ spaces. The work is theoretical and is based on logical deduction from established concepts in topology and ideal topological spaces. After recalling the necessary preliminaries on ideals, local functions, $R - \mathcal{I}-$open sets, $R - \mathcal{I}-$closures and $R - \mathcal{I}-$neighbourhoods, the paper considers $R - \mathcal{I}-$regularity and the $R - \mathcal{I} - R_1$ property through the notion of $R - \mathcal{I} - \theta-$closure. Characterisations are then presented to relate $R - \mathcal{I}-$regular spaces and $R - \mathcal{I} - R_1$ spaces to equality conditions involving $R - \mathcal{I}-$closure and $R - \mathcal{I} - \theta-$closure. The paper further introduces and investigates the classes $R - \mathcal{I} - R_S$, $R - \mathcal{I} - R_D$, $R - \mathcal{I} - R_T$, weakly $R - \mathcal{I} - R_0$ and weakly $R - \mathcal{I} - C_0$ spaces. Their interrelationships are established by proving that every $R - \mathcal{I} - R_0$ space is weakly $R - \mathcal{I} - R_0$ and $R - \mathcal{I} - R_T$, and that $R - \mathcal{I} - R_T$ implies both $R - \mathcal{I} - R_D$ and $R - \mathcal{I} - R_S$. Additional results address characterisations of weakly $R - \mathcal{I} - R_0$ and weakly $R - \mathcal{I} - C_0$ spaces, as well as preservation properties under one-one closed maps and product spaces. A finite example demonstrates that the converse implications from weakly $R - \mathcal{I} - R_0$ and $R - \mathcal{I} - R_T$ to $R - \mathcal{I} - R_0$ do not hold in general. The paper also summarizes the resulting implication diagram among the introduced classes. The findings clarify how these weaker separation axioms are positioned relative to $R - \mathcal{I} - R_0$ and related ideal-topological separation properties.

Study design: Theoretical

Place and Duration of Study: Department of Mathematics, St. Joseph's College (Autonomous) Devagiri, Calicut-673008, India.

Methodology:The application of logical reasoning to deduce new theorems from established principles.

Conclusion: In this paper, we introduced several new separation axioms weaker than the $R - \mathcal{I} - R_0$ axiom namely $R - \mathcal{I} - R_S$, $R - \mathcal{I} - R_D$, $R - \mathcal{I} - R_T$, weakly $R - \mathcal{I} - R_0$, weakly $R - \mathcal{I} - C_0$ spaces. Furthermore, we analyzed the relationships among these spaces and certain characterizations.

Keywords: Ideal topological space; $R-\mathcal{I}-$open set; $R-\mathcal{I}-$closure; $R-\mathcal{I}-$kernel; $R-\mathcal{I}-R_0$ space; $R-\mathcal{I}-R_1$ space; $R-\mathcal{I}-R_S$ space; $R-\mathcal{I}-R_D$ space; $R-\mathcal{I}-R_T$ space; weakly $R-\mathcal{I}-R_0$ space; weakly $R-\mathcal{I}-C_0$ space.

2010 Mathematics Subject Classification: 54A05

1 INTRODUCTION

The concept of an ideal in a topological space was first introduced by Kuratowski and Vaidyanathaswamy (Kuratowski, 1966; Vaidyanathaswamy, 1960). They also defined the local function in an ideal topological space. Further, Jankovic and Hamlett (1990) studied the properties of ideal topological spaces. The notion of R_0 topological spaces was introduced by Shanin (1943). In 1961, Davis (1961) studied some properties of these spaces and also introduced the notion of an R_1 topological space. Dube studied R_0 and R_1 spaces (Dube, 1974, 1982). Many topologists carried out further investigations of the properties of the R_0 topological spaces (Di Maio, 1985; Mishra & Dube, 1973; Naimpally, 1967; Tong, 1983). Related studies have used generalized open sets, such as αIg-open sets, to formulate separation axioms in ideal topological spaces (Maragathavalli & Vinodhini, 2018). Recent work has also considered algebraic associations of topological spaces with rings or modules (Tognon, 2026), topologies induced by ideals, primals, filters and grills (Matejdes, 2024), and idealized expansions of open sets, including $hI-$open sets in ideal topological spaces (Acikgoz & Noiri, 2025). We Sangeetha, M. V., & Chacko, B. (2019), Sangeetha, M. V., & Chacko, B. (2020a), Sangeetha, M. V., & Chacko, B. (2020b) studied ideal topological spaces through $R-\mathcal{I}-$oprn sets which includes supra spaces, separation axioms and weak separation axioms in ideal topological spaces.

Although several forms of separation axioms have been studied in topological and ideal topological settings, the position of separation axioms that are strictly weaker than $R-\mathcal{I}-R_0$ through $R-\mathcal{I}-$open sets requires further clarification. In particular, the relationships among the new weak forms, their characterisations and their behaviour under suitable mappings and products remain insufficiently organized within the present framework.

In this paper, we discuss some properties of $R-\mathcal{I}-R_1$ spaces. The main objective is to use the notion of $R-\mathcal{I}-$open sets to define new separation axioms weaker than $R-\mathcal{I}-R_0$ and to establish selected relationships among them.

2 PRELIMINARIES

By a space (X,τ), we mean a topological space with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X,τ), the system of open neighborhoods of x is denoted by $N(x) = \{U \in \tau : x \in U\}$. For a given subset A of a space (X,τ), $Cl(A)$ and $Int(A)$ are used to denote the closure of A and the interior of A, respectively, with respect to the topology.

A nonempty collection of subsets of a set X is said to be an ideal $\mathcal{I}$ on X, if it satisfies the following two conditions: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal topological space (or ideal space) $(X,\tau,\mathcal{I})$ means a topological space (X,τ) with an ideal $\mathcal{I}$ defined on X. Let (X,τ) be a topological space with an ideal $\mathcal{I}$ defined on X. Then for any subset A of X, $A^*(\mathcal{I},\tau) = \{x \in X/A \cap U \notin \mathcal{I}\ for\ every\ \ U \in N(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ. If there is no ambiguity, we will write $A^*(\mathcal{I})$ or simply A^* for $A^*(\mathcal{I},\tau)$. Also, $Cl^*(A) = A \cup A^*$. It defines a Kuratowski closure operator for the topology $\tau^*(\mathcal{I})$ (or simply τ^*) which is finer than τ. Jankovic, D., & Hamlett, T. R. (1990).

A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be $R-\mathcal{I}-$**open** (resp. regular open) if $Int(Cl^*(A)) = A$ (resp. Int(Cl(A))=A). We call a subset A of $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-$closed if its complement is $R-\mathcal{I}-$open. The intersection of all $R-\mathcal{I}-$closed sets containing A is called the $R-\mathcal{I}-$closure of A and is denoted by $R-\mathcal{I}-$Cl(A). The $R-\mathcal{I}-$interior of A is defined by the union of all $R-\mathcal{I}-$open sets contained in A and is denoted by $R-\mathcal{I}-$Int(A). The family of all $R-\mathcal{I}-$open (resp. $R-\mathcal{I}-$closed) sets of $(X,\tau,\mathcal{I})$ containing a point $x \in X$ is denoted by $R\mathcal{I}O(X,x)$ (resp. $R\mathcal{I}C(X,x)$). A subset N of X is called an $R-\mathcal{I}-$nbd of a point $x \in X$ if there exists an $R-\mathcal{I}-$open set U of $(X,\tau,\mathcal{I})$ such that $x \in U \subset N$.

3 On $R-\mathcal{I}-R_1$ spaces

Definition 3.1. Sangeetha, M. V., & Chacko, B. (2020a). An ideal topological space $(X,\tau,\mathcal{I})$ is said to be $R-\mathcal{I}-$regular at a point $x \in X$ if for each pair consisting of a point $x \in x$ and a $R-\mathcal{I}-$closed set B disjoint from x, there exist disjoint $R-\mathcal{I}-$open sets containing x and B respectively.

Definition 3.2. Sangeetha, M. V., & Chacko, B. (2020b). An ideal topological space $(X,\tau,\mathcal{I})$ is called an $R-\mathcal{I}-R_1$ space if for $x, y \in X$ with $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ there exist disjoint $R-\mathcal{I}-$open sets U, V such that $R-\mathcal{I}-Cl(\{x\}) \subset U$ and $R-\mathcal{I}-Cl(\{y\}) \subset V$.

Result 3.1. *Sangeetha, M. V., & Chacko, B. (2020b). An ideal topological space is $R-\mathcal{I}-R_1$ if and only if $x \in X-R-\mathcal{I}-Cl(\{y\})$ implies that x and y have disjoint $R-\mathcal{I}-$open neighbourhoods.*

Definition 3.3. A point x of X is called a $R-\mathcal{I}-$cluster point of A if $R-\mathcal{I}-Cl(U)\cap A \neq \phi$ for every $U \in R\mathcal{I}O(X,x)$. The set of all $R-\mathcal{I}-$cluster points of A is called the $R-\mathcal{I}-\theta-$closure of A and is denoted by $R-\mathcal{I}-\theta Cl(A)$.
A subset A is said to be $R-\mathcal{I}-\theta-$closed if $A = R-\mathcal{I}-\theta Cl(A)$. The complement of a $R-\mathcal{I}-\theta-$closed set is said to be $R-\mathcal{I}-\theta-$open.

Theorem 3.1. *For any subset A of an ideal topological space $(X,\tau,\mathcal{I})$, $R-\mathcal{I}-Cl(A) \subset R-\mathcal{I}-\theta Cl(A)$.*

Theorem 3.2. *For an ideal topological space $(X,\tau,\mathcal{I})$, the following properties are equivalent.*

i $(X,\tau,\mathcal{I})$ *is a $R-\mathcal{I}-$regular.*

ii $R-\mathcal{I}-\theta Cl(A) = R-\mathcal{I}-Cl(A)$ *for every subset A of X.*

iii $R-\mathcal{I}-\theta Cl(A) = A$ *for every $R-\mathcal{I}-$closed subset A of X.*

Proof. (i) ⇒ (ii)
It is enough to show that $R-\mathcal{I}-\theta Cl(A) \subset R-\mathcal{I}-Cl(A)$.
Assume $x \in R-\mathcal{I}-Cl(A)$. Then by the $R-\mathcal{I}-$regularity of $(X,\tau,\mathcal{I})$, there exist disjoint $R-\mathcal{I}-$open sets U and V such that $x \in U$ and $R-\mathcal{I}-Cl(A) \subset V$. Hence $A \cap R-\mathcal{I}-Cl(U) \subset R-\mathcal{I}-Cl(A)\cap R-\mathcal{I}-Cl(U) \subset V \cap R-\mathcal{I}-Cl(U) = \phi$. Hence, $x \notin R-\mathcal{I}-\theta Cl(A)$ and so (ii).
(ii) ⇒ (iii)
Let A be a $R-\mathcal{I}-$closed subset of X. Then $R-\mathcal{I}-Cl(A) = A$ and by (ii) $R-\mathcal{I}-\theta Cl(A) = A$.
(iii) ⇒ (i)
Let U be a $R-\mathcal{I}-$open set in X and let $x \in U$. Then $X-U$ is a $R-\mathcal{I}-$closed set that does not contain u. By (iii) $R-\mathcal{I}-\theta Cl(X-U) = X-U$.Then, there exists a $R-\mathcal{I}-$open set V containing x such that $R-\mathcal{I}-Cl(V)\cap(X-U) = \phi$. Thus $(X-U) \subset X-R-\mathcal{I}-Cl(V)$. Hence, (i). ☐

Theorem 3.3. *An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_1$ if and only if $R-\mathcal{I}-\theta Cl(\{x\}) = R-\mathcal{I}-Cl(\{x\})$ for each $x \in X$.*

Proof. Let $(X,\tau,\mathcal{I})$ be $R-\mathcal{I}-R_1$ and $x \in X$. Suppose $y \notin R-\mathcal{I}-Cl(\{x\})$. Then by result 1, x and y belong to disjoint $R-\mathcal{I}-$open sets U and V respectively. So $\{x\} \cap R-\mathcal{I}-Cl(V) \subset U \cap R-\mathcal{I}-Cl(V) = \phi$ and $y \notin R-\mathcal{I}-\theta Cl(\{x\})$. Hence $R-\mathcal{I}-\theta Cl(\{x\}) \subset R-\mathcal{I}-Cl(\{x\})$ and so $R-\mathcal{I}-\theta Cl(\{x\}) = R-\mathcal{I}-Cl(\{x\})$.
Conversely, let $x \neq y$ and $y \notin R-\mathcal{I}-Cl(\{x\})$. Then $y \notin R-\mathcal{I}-\theta Cl(\{x\})$. So, there exists a $R-\mathcal{I}-$open set V containing y and $R-\mathcal{I}-Cl(V) \cap \{x\} = \phi$. Thus $x \in X-R-\mathcal{I}-Cl(V) = U(say)$. Thus, U and V are disjoint $R-\mathcal{I}-$open sets containing x and y respectively. Hence $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_1$. □

Remark 3.1. Every $R-\mathcal{I}-$regular space is $R-\mathcal{I}-R_1$.

4 Weaker than $R-\mathcal{I}-R_0$ spaces

Definition 4.1. Sangeetha, M. V., & Chacko, B. (2020a). Let $(X,\tau,\mathcal{I})$ be an ideal topological space and $A \subset X$. The $R-\mathcal{I}-$kernal of A is denoted by $\mathcal{I}_R Ker(A)$ and is defined as the set
$\mathcal{I}_R Ker(A) = \cap\{G \in R\mathcal{I}O(X) : A \subset G\}$.

Definition 4.2. Sangeetha, M. V., & Chacko, B. (2020a). An ideal topological space $(X,\tau,\mathcal{I})$ is called a $R-\mathcal{I}-R_0$ space if every $R-\mathcal{I}-$open set contains the $R-\mathcal{I}-$closure of each of its singletons.

Definition 4.3. An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_S$ if and only if for any points $x, y \in X,\quad R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ implies $R-\mathcal{I}-Cl(\{x\}) \cap R-\mathcal{I}-Cl(\{y\}) = \phi$ or $\{x\}$ or $\{y\}$.

Definition 4.4. An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_D$ if and only if for $x \in X,\quad R-\mathcal{I}-Cl(\{x\}) \cap R-\mathcal{I}-Ker(\{x\}) = \{x\}$ implies that $(R-\mathcal{I}-Cl(\{x\}) \setminus \{x\})$ is $R-\mathcal{I}-$closed.

Definition 4.5. An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_T$ if and only if for each $x \in X$, both $R-\mathcal{I}-Ker(\{x\}) \setminus R-\mathcal{I}-Cl(\{x\})$ and $R-\mathcal{I}-Cl(\{x\}) \setminus R-\mathcal{I}-Ker(\{x\})$ are degenerate.

Definition 4.6. An ideal topological space $(X,\tau,\mathcal{I})$ is weakly-$R-\mathcal{I}-R_0$ if $\bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\}) = \phi$.

Definition 4.7. An ideal topological space $(X,\tau,\mathcal{I})$ is weakly-$R-\mathcal{I}-C_0$ if $\bigcap_{x\in X} R-\mathcal{I}-Ker(\{x\}) = \phi$.

Remark 4.1. Every $R-\mathcal{I}-R_0$ space is weakly-$R-\mathcal{I}-R_0$.

Remark 4.2. Every $R-\mathcal{I}-R_0$ space is $R-\mathcal{I}-R_T$.

The converses of the above statements are not true in general.

Example 4.1. *Let $X = \{a,b,c\}, \tau = \{\phi, X, \{b\}, \{c\}, \{b,c\}, \{a,b\}\}, \mathcal{I} = \{\phi, \{c\}\}$. The $R-\mathcal{I}-$open sets are $\{b\}, \{c\}, \{a,b\}, X$. Then $(X,\tau,\mathcal{I})$ is not $R-\mathcal{I}-R_0$ but weakly $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_T$.*

Theorem 4.2. *If $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_T$, then it is $R-\mathcal{I}-R_D$.*

Proof. Let X be $R-\mathcal{I}-R_T$ and denote $(x) = R-\mathcal{I}-Cl(\{x\}) \cap R-\mathcal{I}-Ker(\{x\})$. Then $R-\mathcal{I}-Cl(\{x\}) = (x) \cup D$ and $R-\mathcal{I}-Ker(\{x\}) = (x) \cup E$, where D and E be degenerate sets such that D is not a subset of $R-\mathcal{I}-Ker(\{x\})$ and E is not a subset of $R-\mathcal{I}-Cl(\{x\})$. If $(x) = \{x\}$, then $R-\mathcal{I}-Cl(\{x\}) = \{x\} \cup D$ and $R-\mathcal{I}-Ker(\{x\}) = \{x\} \cup E$.
Let U be a $R-\mathcal{I}-$open set containing $R-\mathcal{I}-Ker(\{x\})$. Then $X-U$ is a $R-\mathcal{I}-$closed set and $(X-U) \cap R-\mathcal{I}-Cl(\{x\}) = \phi$ or D. If $(X-U) \cap R-\mathcal{I}-Cl(\{x\}) = D$, then D is a $R-\mathcal{I}-$closed set, being the intersection of two $R-\mathcal{I}-$closed sets. If $(X-U) \cap R-\mathcal{I}-Cl(\{x\}) = \phi$, then $R-\mathcal{I}-Cl(\{x\}) \subseteq U, D \subset U$. Since D is not a subset of $R-\mathcal{I}-Ker(\{x\}$, there exists a $R-\mathcal{I}-$open set V such that $x \in V$ and D is not a subset of V. Then $R-\mathcal{I}-Cl(\{x\}) \cap (X-V) = D$ is a $R-\mathcal{I}-$closed set. Hence $(R-\mathcal{I}-Cl(\{x\}) - \{x\})$ is $R-\mathcal{I}-$closed whenever $(x) = \{x\}$. Thus, $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-R_D$. □

Theorem 4.3. *If $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-R_T$, then it is $R-\mathcal{I}-R_S$.*

Proof. Let X be $R-\mathcal{I}-R_T$ and let $x, y \in X$. If $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ and there is an element $s \in X$ such that $s \neq x,\ s \neq y$ but $s \in R-\mathcal{I}-Cl(\{x\}) \cap R-\mathcal{I}-Cl(\{y\})$, then $s \in R-\mathcal{I}-Cl(\{x\})$ and $s \in R-\mathcal{I}-Cl(\{y\})$. Hence $x \in R-\mathcal{I}-Ker(\{s\}$ and $y \in R-\mathcal{I}-Ker(\{s\}$. But $R-\mathcal{I}-Ker(\{s\} = (s) \cup E$ where E is a degenerate set and E is not a subset of $R-\mathcal{I}-Cl(\{s\})$.
Now four cases are possible as follows:
(i) $x \in (s)$ and $y \in (s)$.
Then $x \in R-\mathcal{I}-Cl(\{s\})$ and $y \in R-\mathcal{I}-Cl(\{s\})$. But $s \in R-\mathcal{I}-Cl(\{x\})$ and $s \in R-\mathcal{I}-Cl(\{y\})$. Hence $R-\mathcal{I}-Cl(\{x\}) = R-\mathcal{I}-Cl(\{y\}) = R-\mathcal{I}-Cl(\{s\})$, which is impossible.
(ii) $\{x\} = E, y \in (s)$.
Then $x \notin R-\mathcal{I}-Cl(\{s\})$ and $y \in R-\mathcal{I}-Cl(\{s\})$. Since $s \in R-\mathcal{I}-Cl(\{y\})$, $R-\mathcal{I}-Cl(\{y\}) = R-\mathcal{I}-Cl(\{s\})$. Now, either $y \in R-\mathcal{I}-Cl(\{x\})$ or $y \notin R-\mathcal{I}-Cl(\{x\})$. Let $y \in R-\mathcal{I}-Cl(\{x\})$. Since $x \notin R-\mathcal{I}-Cl(\{s\}), x \in X-(R-\mathcal{I}-Cl(\{s\}))$, which is $R-\mathcal{I}-$open, $R-\mathcal{I}-Ker(\{x\}) \subseteq X-(R-\mathcal{I}-Cl(\{s\}))$. Hence $R-\mathcal{I}-Cl(\{s\}) \subseteq R-\mathcal{I}-Cl(\{x\}) - R-\mathcal{I}-Ker(\{x\})$. But $\{s, y\} \subseteq R-\mathcal{I}-Cl(\{s\})$. Hence $R-\mathcal{I}-Cl(\{x\}) - R-\mathcal{I}-Ker(\{x\})$ is not a degenerate set, a contradiction to the fact that X is $R-\mathcal{I}-R_T$. Now let $y \notin R-\mathcal{I}-Cl(\{x\})$. Since $y \in R-\mathcal{I}-Cl(\{s\})$ and $s \in R-\mathcal{I}-Cl(\{x\})$, we have $y \in R-\mathcal{I}-Cl(\{x\})$, a contradiction.
(iii) $x \in (s)$ and $\{y\} = E$.
The proof is similar to that of the above case.
(iv) $\{x\} = \{y\} = E$.
Then $R-\mathcal{I}-Cl(\{x\}) = R-\mathcal{I}-Cl(\{y\})$, which is not possible.
Thus if If $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$, then $R-\mathcal{I}-Cl(\{x\}) \cap R-\mathcal{I}-Cl(\{y\})$ is either ϕ or $\{x\}$ or $\{y\}$. Hence X is $R-\mathcal{I}-R_S$. □

Theorem 4.4. *An ideal topological space $(X, \tau, \mathcal{I})$ is weakly-$R-\mathcal{I}-R_0$ if and only if for each $x \in X,\ \ R-\mathcal{I}-Ker(\{x\}) \neq X$.*

Proof. Assume X is weakly-$R-\mathcal{I}-R_0$. If there exists an element u that does not belong to any proper $R-\mathcal{I}-$open set, then u belongs to $R-\mathcal{I}-Cl(\{x\})$ for all $x \in X$. Hence $u \in \bigcap_{x \in X} R-\mathcal{I}-Cl(\{x\})$, a contradiction.
Conversely, assume $R-\mathcal{I}-Ker(\{x\}) \neq X$ for each $x \in X$, which means that for each $x \in X$, there exists a proper $R-\mathcal{I}-$open set U such that $x \in U$. If X is not weakly-$R-\mathcal{I}-R_0$, then there exists at least an element $u \in \bigcap_{x \in X} R-\mathcal{I}-Cl(\{x\})$. Then X is the only $R-\mathcal{I}-$open set containing u. This contradiction proves that X is weakly-$R-\mathcal{I}-R_0$. □

Theorem 4.5. *An ideal topological space $(X, \tau, \mathcal{I})$ is weakly-$R-\mathcal{I}-C_0$ if and only if for each $x \in X,\ \ R-\mathcal{I}-Cl(\{x\}) \neq X$.*

Proof. The proof is similar to that of the above theorem. □

Theorem 4.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, J)$ be a one-one closed map. Then if X is weakly-$R-\mathcal{I}-R_0$, so is Y.*

Proof. $\bigcap_{x\in X} R-\mathcal{I}-Cl(\{f(x)\}) = \phi$ since $\bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\}) = \phi$. Since $\bigcap_{y\in Y} R-\mathcal{I}-Cl(\{y\}) \subset \bigcap_{x\in X} R-\mathcal{I}-Cl(\{f(x)\})$, Y is also weakly-$R-\mathcal{I}-R_0$. □

5 Conclusion

This study defined and examined separation axioms weaker than the $R-\mathcal{I}-R_0$ axiom in ideal topological spaces. The classes$R-\mathcal{I}-R_S$, $R-\mathcal{I}-R_D$, $R-\mathcal{I}-R_T$, weakly $R-\mathcal{I}-R_0$ and weakly $R-\mathcal{I}-C_0$ were introduced and positioned within the framework generated by $R-\mathcal{I}-$open sets, $R-\mathcal{I}-$closures and $R-\mathcal{I}-$kernels. The results show that $R-\mathcal{I}-$regularity leads to the $R-\mathcal{I}-R_1$ condition through the equality of $R-\mathcal{I}-\theta-$closure and $R-\mathcal{I}-$closure for singletons. The paper also establishes that every $R-\mathcal{I}-R_0$ space is weakly $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_T$, while $R-\mathcal{I}-R_T$ implies both $R-\mathcal{I}-R_D$ and $R-\mathcal{I}-R_S$. Characterisations of weakly $R-\mathcal{I}-R_0$ and weakly $R-\mathcal{I}-C_0$ spaces are also obtained through conditions on $R-\mathcal{I}-$kernels and$R-\mathcal{I}-$closures of singletons. The finite example included in the manuscript supports the distinction between $R-\mathcal{I}-R_0$, weakly $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_T$. These relationships clarify the hierarchy of the proposed weak separation axioms and provide a basis for further investigations in ideal topological spaces, including possible converse statements, independence questions and the construction of additional finite separating examples.

$$\begin{array}{ccccc} R-\mathcal{I}-R_1 & \Rightarrow & R-\mathcal{I}-R_0 & \Rightarrow & \text{weakly-}R-\mathcal{I}-R_0 \\ & & \Downarrow & & \\ R-\mathcal{I}-R_D & \Leftarrow & R-\mathcal{I}-R_T & \Rightarrow & R-\mathcal{I}-R_S \end{array}$$

6 Limitations

This paper is theoretical and does not include applications beyond one finite ideal topological space. The introduced notions require further comparison with existing separation axioms. Several results depend on properties of $R-\mathcal{I}-$closure, $R-\mathcal{I}-$kernel and product spaces that may be stated more explicitly in future work. Additional examples, independence results and preservation theorems under suitable mappings would further clarify the scope of the proposed classes.

7 Declaration of AI Use

This manuscript was prepared through the combined contributions of all authors, including contributions to study design, data, content development, results, interpretation, and related scholarly work. The author(s) acknowledge the use of Grammarly and ChatGPT to assist with grammar checking, language refinement, and reference formatting. These AI-assisted tools were not used by the authors and did not replace the intellectual contributions or the scholarly judgment of the author(s). All AI-assisted outputs, including content, references, and interpretations, were carefully reviewed, revised, verified, and approved by the author(s). The author(s) accept full responsibility for the accuracy, integrity, and final content of the manuscript.

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