UNDER PEER REVIEW

Original Research Article

Separation axioms weaker than $R-\mathcal{I}-R_0$

Abstract

Aims/objectives: The aim of this paper is to study separation axioms weaker than $R-\mathcal{I}-R_0$ axiom in ideal topological space. Also, some properties of $R-\mathcal{I}-R_1$ spaces are discussed.
Study design: Theoretical
Place and Duration of Study: Department of Mathematics, St. Joseph's College (Autonomous) Devagiri, Calicut-673008, India.
Methodology:The application of logical reasoning to deduce new theorems from established principles.
Conclusion: In this paper, we introduced several new separation axioms weaker than the $R-\mathcal{I}-R_0$ axiom namely $R-\mathcal{I}-R_S$, $R-\mathcal{I}-R_D$, $R-\mathcal{I}-R_T$, weakly $R-\mathcal{I}-R_0$, weakly $R-\mathcal{I}-C_0$ spaces. Furthermore, we analyzed the relationships among these spaces and certain characterizations.

Keywords: $R-\mathcal{I}-R_S$*;* $R-\mathcal{I}-R_D$*;* $R-\mathcal{I}-R_T$*; weakly* $R-\mathcal{I}-R_0$*; weakly* $R-\mathcal{I}-C_0$*.*
2010 Mathematics Subject Classification: 54A05

1 INTRODUCTION

The concept of ideal in topological space was first introduced by Kuratowski and Vaidyanathswamy [1]. They have also defined the local function in an ideal topological space. Further, Hamlett and Jankovic in [2] studied the properties of ideal topological spaces. The notion of R_0 topological spaces was introduced by Shanin [3] in 1943. In 1961, Davis [4] studied some properties of the same and also introduced the notion of R_1 topological space. K.K. Dube studied the R_0 and R_1 spaces [5, 6]. Further, investigations of the properties of R_0 topological spaces were carried out by many topologists as in [7, 8, **?**, 10]. In this paper, we discuss some properties of $R-\mathcal{I}-R_1$ spaces.The main purpose is that we use the notion of $R-\mathcal{I}-$open sets to define new separation axioms weaker than $R-\mathcal{I}-R_0$.

2 PRELIMINARIES

By a space (X,τ), we mean a topological space with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X,τ), the system of open neighborhoods of x is denoted by $N(x) = \{U \in \tau : x \in U\}$. For a given subset A of a space (X,τ), $Cl(A)$ and $Int(A)$ are used to denote the closure of A and the interior of A, respectively, with respect to the topology.

A nonempty collection of subsets of a set X is said to be an ideal $\mathcal{I}$ on X, if it satisfies the following two conditions: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal topological space (or ideal space) $(X,\tau,\mathcal{I})$ means a topological space (X,τ) with an ideal $\mathcal{I}$ defined on X. Let (X,τ) be a topological space with an ideal $\mathcal{I}$ defined on X. Then for any subset A of X, $A^*(\mathcal{I},\tau) = \{x \in X/A \cap U \notin \mathcal{I}\ for\ every\ \ U \in N(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ. If there is no ambiguity, we will write $A^*(\mathcal{I})$ or simply A^* for $A^*(\mathcal{I},\tau)$. Also, $Cl^*(A) = A \cup A^*$. It defines a Kuratowski closure operator for the topology $\tau^*(\mathcal{I})$ (or simply τ^*) which is finer than τ [2].

A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be $R-\mathcal{I}-$**open** (resp. regular open) if $Int(Cl^*(A)) = A$ (resp. Int(Cl(A))=A). We call a subset A of $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-$closed if its complement is $R-\mathcal{I}-$open. The intersection of all $R-\mathcal{I}-$closed sets containing A is called the $R-\mathcal{I}-$closure of A and is denoted by $R-\mathcal{I}-$Cl(A). The $R-\mathcal{I}-$interior of A is defined by the union of all $R-\mathcal{I}-$open sets contained in A and is denoted by $R-\mathcal{I}-$Int(A). The family of all $R-\mathcal{I}-$open (resp. $R-\mathcal{I}-$closed) sets of $(X,\tau,\mathcal{I})$ containing a point $x \in X$ is denoted by $R\mathcal{I}O(X,x)$ (resp. $R\mathcal{I}C(X,x)$). A subset N of X is called an $R-\mathcal{I}-$nbd of a point $x \in X$ if there exists an $R-\mathcal{I}-$open set U of $(X,\tau,\mathcal{I})$ such that $x \in U \subset N$.

3 On $R-\mathcal{I}-R_1$ spaces

Definition 3.1. [11] An ideal topological space $(X,\tau,\mathcal{I})$ is said to be $R-\mathcal{I}-$regular at a point $x \in X$ if for each pair consisting of a point $x \in x$ and a $R-\mathcal{I}-$closed set B disjoint from x, there exist disjoint $R-\mathcal{I}-$open sets containing x and B respectively.

Definition 3.2. [12] An ideal topological space $(X,\tau,\mathcal{I})$ is called an $R-\mathcal{I}-R_1$ space if for $x,y \in X$ with $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ there exist disjoint $R-\mathcal{I}-$open sets U,V such that $R-\mathcal{I}-Cl(\{x\}) \subset U$ and $R-\mathcal{I}-Cl(\{y\}) \subset V$.

Result 3.1. *[12] An ideal topological space is $R-\mathcal{I}-R_1$ if and only if $x \in X - R-\mathcal{I}-Cl(\{y\})$ implies that x and y have disjoint $R-\mathcal{I}-$open neighbourhoods.*

Definition 3.3. A point x of X is called a $R-\mathcal{I}-$cluster point of A if $R-\mathcal{I}-Cl(U) \cap A \neq \phi$ for every $U \in R\mathcal{I}O(X,x)$. The set of all $R-\mathcal{I}-$cluster points of A is called the $R-\mathcal{I}-\theta-$closure of A and is denoted by $R-\mathcal{I}-\theta Cl(A)$.
A subset A is said to be $R-\mathcal{I}-\theta-$closed if $A = R-\mathcal{I}-\theta Cl(A)$. The complement of a $R-\mathcal{I}-\theta-$closed set is said to be $R-\mathcal{I}-\theta-$open.

Theorem 3.1. *For any subset A of an ideal topological space $(X,\tau,\mathcal{I})$, $R-\mathcal{I}-Cl(A) \subset R-\mathcal{I}-\theta Cl(A)$.*

Theorem 3.2. *For an ideal topological space $(X,\tau,\mathcal{I})$, the following properties are equivalent.*

i $(X, \tau, \mathcal{I})$ *is a* $R - \mathcal{I} -$*regular.*

ii $R - \mathcal{I} - \theta Cl(A)$ = $R - \mathcal{I} - Cl(A)$ *for every subset* A *of* X*.*

iii $R - \mathcal{I} - \theta Cl(A)$ = A *for every* $R - \mathcal{I} -$*closed subset* A *of* X*.*

Proof. (i) $\Rightarrow$ (ii)
It is enough to show that $R - \mathcal{I} - \theta Cl(A) \subset R - \mathcal{I} - Cl(A)$.
Assume $x \in R - \mathcal{I} - Cl(A)$. Then by the $R - \mathcal{I} -$regularity of $(X, \tau, \mathcal{I})$, there exist disjoint $R - \mathcal{I} -$open sets U and V such that $x \in U$ and $R - \mathcal{I} - Cl(A) \subset V$. Hence $A \cap R - \mathcal{I} - Cl(U) \subset R - \mathcal{I} - Cl(A) \cap R - \mathcal{I} - Cl(U) \subset V \cap R - \mathcal{I} - Cl(U) = \phi$. Hence, $x \notin R - \mathcal{I} - \theta Cl(A)$ and so (ii).
(ii) $\Rightarrow$ (iii)
Let A be a $R - \mathcal{I} -$closed subset of X. Then $R - \mathcal{I} - Cl(A) = A$ and by (ii) $R - \mathcal{I} - \theta Cl(A) = A$.
(iii) $\Rightarrow$ (i)
Let U be a $R - \mathcal{I} -$open set in X and let $x \in U$. Then $X - U$ is a $R - \mathcal{I} -$closed set that does not contain u. By (iii) $R - \mathcal{I} - \theta Cl(X - U) = X - U$.Then, there exists a $R - \mathcal{I} -$open set V containing x such that $R - \mathcal{I} - Cl(V) \cap (X - U) = \phi$. Thus $(X - U) \subset X - R - \mathcal{I} - Cl(V)$. Hence, (i). ☐

Theorem 3.3. *An ideal topological space* $(X, \tau, \mathcal{I})$ *is* $R - \mathcal{I} - R_1$ *if and only if* $R - \mathcal{I} - \theta Cl(\{x\}) = R - \mathcal{I} - Cl(\{x\})$ *for each* $x \in X$*.*

Proof. Let $(X, \tau, \mathcal{I})$ be $R - \mathcal{I} - R_1$ and $x \in X$. Suppose $y \notin R - \mathcal{I} - Cl(\{x\})$. Then by result 1, x and y belong to disjoint $R - \mathcal{I} -$open sets U and V respectively. So $\{x\} \cap R - \mathcal{I} - Cl(V) \subset U \cap R - \mathcal{I} - Cl(V) = \phi$ and $y \notin R - \mathcal{I} - \theta Cl(\{x\})$. Hence $R - \mathcal{I} - \theta Cl(\{x\}) \subset R - \mathcal{I} - Cl(\{x\})$ and so $R - \mathcal{I} - \theta Cl(\{x\}) = R - \mathcal{I} - Cl(\{x\})$.
Conversely, let $x \neq y$ and $y \notin R - \mathcal{I} - Cl(\{x\})$. Then $y \notin R - \mathcal{I} - \theta Cl(\{x\})$. So, there exists a $R - \mathcal{I} -$open set V containing y and $R - \mathcal{I} - Cl(V) \cap \{x\} = \phi$. Thus $x \in X - R - \mathcal{I} - Cl(V) = U(say)$. Thus, U and V are disjoint $R - \mathcal{I} -$open sets containing x and y respectively. Hence $(X, \tau, \mathcal{I})$ is $R - \mathcal{I} - R_1$. ☐

Remark 3.1. Every $R - \mathcal{I} -$regular space is $R - \mathcal{I} - R_1$.

4 Weaker than $R - \mathcal{I} - R_0$ spaces

Definition 4.1. [11] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. The $R - \mathcal{I} -$kernal of A is denoted by $\mathcal{I}_R Ker(A)$ and is defined as the set
$\mathcal{I}_R Ker(A) = \cap \{G \in R\mathcal{I}O(X) : A \subset G\}$.

Definition 4.2. [11] An ideal topological space $(X, \tau, \mathcal{I})$ is called a $R - \mathcal{I} - R_0$ space if every $R - \mathcal{I} -$open set contains the $R - \mathcal{I} -$closure of each of its singletons.

Definition 4.3. An ideal topological space $(X, \tau, \mathcal{I})$ is $R - \mathcal{I} - R_S$ if and only if for any points $x, y \in X$, $R - \mathcal{I} - Cl(\{x\}) \neq R - \mathcal{I} - Cl(\{y\})$ implies $R - \mathcal{I} - Cl(\{x\}) \cap R - \mathcal{I} - Cl(\{y\}) = \phi$ or $\{x\}$ or $\{y\}$.

Definition 4.4. An ideal topological space $(X, \tau, \mathcal{I})$ is $R - \mathcal{I} - R_D$ if and only if for $x \in X$, $R - \mathcal{I} - Cl(\{x\}) \cap R - \mathcal{I} - Ker(\{x\}) = \{x\}$ implies that $(R - \mathcal{I} - Cl(\{x\}) \setminus \{x\})$ is $R - \mathcal{I} -$closed.

Definition 4.5. An ideal topological space $(X, \tau, \mathcal{I})$ is $R - \mathcal{I} - R_T$ if and only if for each $x \in X$, both $R - \mathcal{I} - Ker(\{x\}) \setminus R - \mathcal{I} - Cl(\{x\})$ and $R - \mathcal{I} - Cl(\{x\}) \setminus R - \mathcal{I} - Ker(\{x\})$ are degenerate.

Definition 4.6. An ideal topological space $(X, \tau, \mathcal{I})$ is weakly-$R - \mathcal{I} - R_0$ if $\bigcap_{x \in X} R - \mathcal{I} - Cl(\{x\}) = \phi$.

Definition 4.7. An ideal topological space $(X,\tau,\mathcal{I})$ is weakly-$R-\mathcal{I}-C_0$ if $\bigcap_{x\in X} R-\mathcal{I}-Ker(\{x\}) = \phi$.

Remark 4.1. Every $R-\mathcal{I}-R_0$ space is weakly-$R-\mathcal{I}-R_0$.

Remark 4.2. Every $R-\mathcal{I}-R_0$ space is $R-\mathcal{I}-R_T$.

The converses of the above statements are not true in general.

Example 4.1. *Let $X=\{a,b,c\},\tau=\{\phi,X,\{b\},\{c\},\{b,c\},\{a,b\}\},\mathcal{I}=\{\phi,\{c\}\}$. The $R-\mathcal{I}-$open sets are $\{b\},\{c\},\{a,b\},X$. Then $(X,\tau,\mathcal{I})$ is not $R-\mathcal{I}-R_0$ but weakly $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_T$.*

Theorem 4.2. *If $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_T$, then it is $R-\mathcal{I}-R_D$.*

Proof. Let X be $R-\mathcal{I}-R_T$ and denote $(x) = R-\mathcal{I}-Cl(\{x\})\cap R-\mathcal{I}-Ker(\{x\})$. Then $R-\mathcal{I}-Cl(\{x\}) = (x)\cup D$ and $R-\mathcal{I}-Ker(\{x\}) = (x)\cup E$, where D and E be degenerate sets such that D is not a subset of $R-\mathcal{I}-Ker(\{x\})$ and E is not a subset of $R-\mathcal{I}-Cl(\{x\})$. If $(x)=\{x\}$, then $R-\mathcal{I}-Cl(\{x\})=\{x\}\cup D$ and $R-\mathcal{I}-Ker(\{x\})=\{x\}\cup E$.
Let U be a $R-\mathcal{I}-$open set containing $R-\mathcal{I}-Ker(\{x\})$. Then $X-U$ is a $R-\mathcal{I}-$closed set and $(X-U)\cap R-\mathcal{I}-Cl(\{x\})=\phi$ or D. If $(X-U)\cap R-\mathcal{I}-Cl(\{x\})=D$, then D is a $R-\mathcal{I}-$closed set, being the intersection of two $R-\mathcal{I}-$closed sets. If $(X-U)\cap R-\mathcal{I}-Cl(\{x\})=\phi$, then $R-\mathcal{I}-Cl(\{x\})\subseteq U, D\subset U$. Since D is not a subset of $R-\mathcal{I}-Ker(\{x\}$, there exists a $R-\mathcal{I}-$open set V such that $x\in V$ and D is not a subset of V. Then $R-\mathcal{I}-Cl(\{x\})\cap(X-V)=D$ is a $R-\mathcal{I}-$closed set. Hence $(R-\mathcal{I}-Cl(\{x\})-\{x\})$ is $R-\mathcal{I}-$closed whenever $(x)=\{x\}$. Thus, $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_D$. ☐

Theorem 4.3. *If $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_T$, then it is $R-\mathcal{I}-R_S$.*

Proof. Let X be $R-\mathcal{I}-R_T$ and let $x,y\in X$. If $R-\mathcal{I}-Cl(\{x\})\neq R-\mathcal{I}-Cl(\{y\})$ and there is an element $s\in X$ such that $s\neq x,\ s\neq y$ but $s\in R-\mathcal{I}-Cl(\{x\})\cap R-\mathcal{I}-Cl(\{y\})$, then $s\in R-\mathcal{I}-Cl(\{x\})$ and $s\in R-\mathcal{I}-Cl(\{y\})$. Hence $x\in R-\mathcal{I}-Ker(\{s\}$ and $y\in R-\mathcal{I}-Ker(\{s\}$. But $R-\mathcal{I}-Ker(\{s\}=(s)\cup E$ where E is a degenerate set and E is not a subset of $R-\mathcal{I}-Cl(\{s\})$. Now four cases are possible as follows:
(i) $x\in(s)$ and $y\in(s)$.
Then $x\in R-\mathcal{I}-Cl(\{s\})$ and $y\in R-\mathcal{I}-Cl(\{s\})$. But $s\in R-\mathcal{I}-Cl(\{x\})$ and $s\in R-\mathcal{I}-Cl(\{y\})$. Hence $R-\mathcal{I}-Cl(\{x\})=R-\mathcal{I}-Cl(\{y\})=R-\mathcal{I}-Cl(\{s\})$, which is impossible.
(ii) $\{x\}=E, y\in(s)$.
Then $x\notin R-\mathcal{I}-Cl(\{s\})$ and $y\in R-\mathcal{I}-Cl(\{s\})$. Since $s\in R-\mathcal{I}-Cl(\{y\})$, $R-\mathcal{I}-Cl(\{y\}) = R-\mathcal{I}-Cl(\{s\})$. Now, either $y\in R-\mathcal{I}-Cl(\{x\})$ or $y\notin R-\mathcal{I}-Cl(\{x\})$. Let $y\in R-\mathcal{I}-Cl(\{x\})$. Since $x\notin R-\mathcal{I}-Cl(\{s\}), x\in X-(R-\mathcal{I}-Cl(\{s\}))$, which is $R-\mathcal{I}-$open, $R-\mathcal{I}-Ker(\{x\})\subseteq X-(R-\mathcal{I}-Cl(\{s\}))$. Hence $R-\mathcal{I}-Cl(\{s\})\subseteq R-\mathcal{I}-Cl(\{x\})-R-\mathcal{I}-Ker(\{x\})$. But $\{s,y\}\subseteq R-\mathcal{I}-Cl(\{s\})$. Hence $R-\mathcal{I}-Cl(\{x\})-R-\mathcal{I}-Ker(\{x\})$ is not a degenerate set, a contradiction to the fact that X is $R-\mathcal{I}-R_T$. Now let $y\notin R-\mathcal{I}-Cl(\{x\})$. Since $y\in R-\mathcal{I}-Cl(\{s\})$ and $s\in R-\mathcal{I}-Cl(\{x\})$, we have $y\in R-\mathcal{I}-Cl(\{x\})$, a contradiction.
(iii) $x\in(s)$ and $\{y\}=E$.
The proof is similar to that of the above case.
(iv) $\{x\}=\{y\}=E$.
Then $R-\mathcal{I}-Cl(\{x\})=R-\mathcal{I}-Cl(\{y\})$, which is not possible.
Thus if If $R-\mathcal{I}-Cl(\{x\})\neq R-\mathcal{I}-Cl(\{y\})$, then $R-\mathcal{I}-Cl(\{x\})\cap R-\mathcal{I}-Cl(\{y\})$ is either ϕ or $\{x\}$ or $\{y\}$. Hence X is $R-\mathcal{I}-R_S$. ☐

Theorem 4.4. *An ideal topological space $(X, \tau, \mathcal{I})$ is weakly-$R-\mathcal{I}-R_0$ if and only if for each $x \in X,\ \ R-\mathcal{I}-Ker(\{x\}) \neq X$.*

Proof. Assume X is weakly-$R-\mathcal{I}-R_0$. If there exists an element u that does not belong to any proper $R-\mathcal{I}-$open set, then u belongs to $R-\mathcal{I}-Cl(\{x\})$ for all $x \in X$. Hence $u \in \bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\})$, a contradiction.
Conversely, assume $R-\mathcal{I}-Ker(\{x\}) \neq X$ for each $x \in X$, which means that for each $x \in X$, there exists a proper $R-\mathcal{I}-$open set U such that $x \in U$. If X is not weakly-$R-\mathcal{I}-R_0$, then there exists at least an element $u \in \bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\})$. Then X is the only $R-\mathcal{I}-$open set containing u. This contradiction proves that X is weakly-$R-\mathcal{I}-R_0$. □

Theorem 4.5. *An ideal topological space $(X, \tau, \mathcal{I})$ is weakly-$R-\mathcal{I}-C_0$ if and only if for each $x \in X,\ \ R-\mathcal{I}-Cl(\{x\}) \neq X$.*

Proof. The proof is similar to that of the above theorem. □

Theorem 4.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, J)$ be a one-one closed map. Then if X is weakly-$R-\mathcal{I}-R_0$, so is Y.*

Proof. $\bigcap_{x\in X} R-\mathcal{I}-Cl(\{f(x)\}) = \phi$ since $\bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\}) = \phi$. Since $\bigcap_{y\in Y} R-\mathcal{I}-Cl(\{y\}) \subset \bigcap_{x\in X} R-\mathcal{I}-Cl(\{f(x)\})$, Y is also weakly-$R-\mathcal{I}-R_0$. □

Theorem 4.7. *If an ideal topological space $(X, \tau, \mathcal{I})$ is weakly-$R-\mathcal{I}-R_0$, then for every ideal topological space Y, the product space $X \times Y$ is also weakly-$R-\mathcal{I}-R_0$.*

Proof. $\bigcap_{(x,y)\in X\times Y} R-\mathcal{I}-Cl(\{(x,y)\}) = \bigcap_{(x,y)\in X\times Y} R-\mathcal{I}-Cl(\{x\}) \times R-\mathcal{I}-Cl(\{y\}) \subset \phi \times Y = \phi$. □

Theorem 4.8. *If $X \times Y$ is weakly-$R-\mathcal{I}-R_0$, then either X or Y is weakly-$R-\mathcal{I}-R_0$.*

Proof. Suppose that neither X nor Y is weakly-$R-\mathcal{I}-R_0$. Then there exist elements $u \in X$ and $v \in Y$ such that the only $R-\mathcal{I}-$open sets containing u and v are X and Y respectively. Thus, $(u, v) \in X \times Y$ does not belong to any proper $R-\mathcal{I}-$open set, which is a contradiction. Hence, either X or Y is weakly-$R-\mathcal{I}-R_0$. □

5 Conclusion

The study successfully defined and investigated separation axioms weaker than the $R-\mathcal{I}-R_0$ axiom, introducing $R-\mathcal{I}-R_S$, $R-\mathcal{I}-R_D$, $R-\mathcal{I}-R_T$, weakly $R-\mathcal{I}-R_0$, weakly $R-\mathcal{I}-C_0$ spaces. In addition, we analyze certain characterizations, and the relationships between these spaces are found as follows:

$$\begin{array}{ccccc} R-\mathcal{I}-R_1 & \Rightarrow & R-\mathcal{I}-R_0 & \Rightarrow & \text{weakly-}R-\mathcal{I}-R_0 \\ & & \Downarrow & & \\ R-\mathcal{I}-R_D & \Leftarrow & R-\mathcal{I}-R_T & \Rightarrow & R-\mathcal{I}-R_S \end{array}$$

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Limitations

The paper is theoretical and does not include applications or examples beyond a small finite ideal topological space. Some introduced notions require clearer motivation and comparison with existing separation axioms. Several proofs depend on properties of (R-I)-closure, (R-I)-kernel, and product spaces that should be stated explicitly. The manuscript also does not discuss whether the new axioms are independent or provide enough counterexamples to separate all introduced classes. Further work may include additional examples, corrected implication diagrams, independence results, and clearer preservation theorems under suitable mappings.

Declaration of AI Use

This manuscript was prepared through the combined contributions of all author(s), including contributions to the study design, data, content development, results, interpretation, and related scholarly work. The author(s) acknowledge the use of Grammarly and ChatGPT to assist with grammar checking, language refinement, reference formatting. These AI-assisted tools were not used as authors and did not replace the intellectual contributions or scholarly judgment of the author(s). All AI-assisted outputs, including content, references, and interpretations, were carefully reviewed, revised, verified, and approved by the author(s). The author(s) accept full responsibility for the accuracy, integrity, and final content of the manuscript.