Closed-Form Expressions of Leonardo-Type Sequences with Homogeneous Counterparts in Balancing and Oresme Numbers

Abstract. In this paper, we establish closed-form solutions for second-order nonhomogeneous linear recurrence relations, identified as generalized Leonardo-type sequences, where the input term is expressed as a polynomial. Our investigation highlights several distinguished cases of generalized Fibonacci numbers, which emerge as the homogeneous analogues of the original nonhomogeneous relations. Specifically, we examine classical families including the balancing numbers, balancing–Lucas numbers, Oresme numbers, and Oresme–Lucas numbers.

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1. Introduction

Recurrence relations have long stood at the foundation of mathematics, extending their influence into physics, engineering, architecture, biology, computer science, and even artistic inquiry. Though their definitions may appear elementary, they conceal profound depth, modeling growth processes, oscillatory behavior, and symbolic structures. Classical second-order families—such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences—remain enduring exemplars of this tradition.

Yet the framework extends far beyond second-order constructions. Higher-order recurrence sequences enrich both theory and application, broadening the classical foundation while unveiling intricate algebraic and analytic structures. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences illustrate this expansion, each governed by characteristic polynomials whose root configurations determine closed-form solutions. Homogeneous recurrences emphasize the relationship between characteristic polynomials and root multiplicities, while nonhomogeneous forms introduce symbolic terms that interact with root structures to generate resonance phenomena. Collectively, these families establish a coherent framework that unites classical recurrence identities with the evolving discipline of symbolic recurrence theory.

This paper seeks to illuminate the architecture of recurrence theory by constructing closed-form solutions for second-order nonhomogeneous linear relations, conceived as generalized Leonardo-type sequences with polynomial inputs. Within this structural edifice, generalized Fibonacci numbers emerge as the harmonious echoes of their nonhomogeneous origins, serving as the homogeneous analogues that mirror the underlying design. Our inquiry threads together venerable families—the the balancing and balancing–Lucas sequences, alongside the Oresme and Oresme–Lucas sequences-each contributing a distinct motif to the symphonic fabric of higher-order recurrence phenomena.

The classical Leonardo sequence arises from the non-homogeneous recurrence relation

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with the initial conditions $l_0 = 1, l_1 = 1$. Its first few terms are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, \ldots$$

Although the precise historical background of the Leonardo sequence is somewhat uncertain, interest in such recurrences can be traced to early volumes of *The Fibonacci Quarterly* (1963). Since that time, numerous studies have investigated Leonardo-type relations under various names and frameworks. A detailed survey of this literature is provided in [21, Section 5].

The Leonardo sequence has remained both a topic of active research and a valuable pedagogical example. Recent contributions have examined its algebraic properties, generating functions, and polynomial extensions (see, for example, [1,2,3,4, 5, 6, 7, 8, 9, 10, 11,12, 13, 14, 15,18, 19, 20]).

Let the second order nonhomogeneous linear recurrence relation be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + p(n) \tag{1.1}$$

with initial conditions $W_0 = u_0, W_1 = u_1$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i$$

and the recurrence coefficients a_1, a_2 are complex scalars or polynomials in $\mathbb{C}[x]$.

A sequence $\{W_n\}_{n\geq 0}$ defined by (1.1) is called a generalized Leonardo-type sequence with input function $p(n)$. For more information on generalized Leonardo-type sequences, see Soykan [22] and [21].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1.$$

Suppose that θ_1 and θ_2 are the roots of the characteristic equation of (1.2)

$$z^2 - a_1 z - a_2 = 0, \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^2 - a_1 z - a_2 = (z-1)^2 = z^2 - 2z + 1 = 0$$

so that $a_1 = 2,\ a_2 = -1$ and (1.2) reduces to

$$V_n = 2V_{n-1} - V_{n-2}. \tag{1.4}$$

In earlier work, particular solutions of second-order nonhomogeneous recurrences (1.1) with polynomial inputs were obtained for $s = 0,1,2,3,4,5,6,7$; see Soykan [23]. The present paper advances those results by deriving unified closed-form solutions through Theorem 1.1. Each recurrence is expressed as the sum of its homogeneous and particular components, with the particular solution determined iteratively. The framework is organized by the multiplicity r of 1 as a root of the characteristic equation (1.3) and the polynomial degree s. Closed-form formulas are obtained for all values $r = 0,1,2$ and $s = 0,1,2,3,4,5,6,7$, illustrating how these parameters jointly shape the solution and how resonance arises when roots coincide. In practice, however, our examples–generalized balancing, and Oresme sequences-require only the non-resonant case $r = 0$. Accordingly, the theorem is presented in detail for $r = 0$, while the cases $r = 1,2$ are acknowledged but omitted as they are not needed here.

Theorem 1.1.

(a): *[21, Theorem 7.6. (a)] The case $r = 0$, i.e., all two roots of the characteristic equation of (1.3) is distinct from 1. The solution of (1.1) is given by*

$$\begin{aligned} W_n(W_0, W_1) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = \sum_{i=0}^{s} A_i n^i = A_0 + \sum_{i=1}^{s} A_i n^i$$

is the particular solution of (1.1). For each $0 \le i \le s$, A_i can be calculated with the iteration

$$A_s = -\frac{c_s}{a_1 + a_2 - 1},\ \textit{for } n = s$$

and

$$A_n = -\frac{1}{a_1+a_2-1}(c_n - \sum_{k=n+1}^{s}(-1)^{k-n+1}\binom{k}{n}(a_1+2^{k-n}a_2)A_k), \text{ for } n = s-1, s-2, ..., 2, 1, 0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}) &= V_n(A_0, \sum_{i=0}^{s} A_i) \\ &= \frac{(W_1 - a_1W_0)\sum_{i=0}^{s} A_i - a_2W_0A_0}{W_1^2 - a_2W_0^2 - a_1W_0W_1}V_n(W_0,W_1) + \frac{-a_2(W_0\sum_{i=0}^{s} A_i - A_0W_1)}{W_1^2 - a_2W_0^2 - a_1W_0W_1}V_{n-1}(W_0,W_1) \\ &= -\frac{W_0\sum_{i=0}^{s} A_i - A_0W_1}{W_1^2 - a_2W_0^2 - a_1W_0W_1}V_{n+1}(W_0,W_1) + \frac{W_1\sum_{i=0}^{s} A_i - (W_0a_2 + W_1a_1)A_0}{W_1^2 - a_2W_0^2 - a_1W_0W_1}V_n(W_0,W_1). \end{aligned}$$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1) = \left(-\frac{(W_1 - a_1W_0)\sum_{i=0}^{s} A_i - a_2W_0A_0}{W_1^2 - a_2W_0^2 - a_1W_0W_1} + 1\right)V_n(W_0,W_1) + \frac{a_2(W_0\sum_{i=0}^{s} A_i - A_0W_1)}{W_1^2 - a_2W_0^2 - a_1W_0W_1}V_{n-1}(W_0,W_1) + \sum_{i=0}^{s} A_in^i$$

i.e.,

$$W_n(W_0,W_1) = \frac{W_0\sum_{i=0}^{s} A_i - A_0W_1}{W_1^2 - a_2W_0^2 - a_1W_0W_1}V_{n+1}(W_0,W_1) + \left(-\frac{W_1\sum_{i=0}^{s} A_i - (W_0a_2 + W_1a_1)A_0}{W_1^2 - a_2W_0^2 - a_1W_0W_1} + 1\right)V_n(W_0,W_1) + \sum_{i=0}^{s} A_in^i$$

(b): *The case $r = 1$, i.e., 1 is a simple root of the characteristic equation of (1.3). Its solution can be obtained in direct analogy with case (a).*

(c): *The case $r = 2$, i.e., 1 is a double root of the characteristic equation of (1.3). The construction follows the same method as in case (a), with appropriate multiplicity corrections.*

Proof. The derivation is achieved by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104–105), and Theorem 3.1 (pp. 88–89) for the case $m = 2$, as given in Soykan [21]. The theorem in full generality encompasses all multiplicities $r = 0, 1, 2$: the non-resonant case $r = 0$ arises when all two roots are distinct from 1, whereas $r = 1, 2$ correspond to situations where 1 is a simple or double root of the characteristic equation (1.3). Since the examples treated in this paper involve only $r = 0$, the explicit formulations for $r = 1, 2$ are omitted. They can be derived in analogy with case (a), incorporating multiplicity corrections, but are not required for the present applications.

Remark 1.2. *Clarification on the parameter r. In Theorem 1.1, the parameter r denotes the multiplicity of the root 1 in the characteristic equation, rather than the multiplicity of characteristic roots in general. This distinction is crucial for applications. For instance, in the Oresme case the characteristic polynomial has a repeated root at $\frac{1}{2}$. Since the root 1 does not occur, the Oresme family belongs to the non-resonant setting $r = 0$. Resonance phenomena arise only when the root 1 appears with multiplicity, and repeated roots different* *from* 1 *do not alter the classification. By contrast, in the balancing family the characteristic equation has two distinct roots, neither equal to* 1*, which also places it in the non-resonant case* $r = 0$*. These examples illustrate that the parameter* r *serves specifically to detect resonance at the root* 1*, ensuring that the framework applies consistently across families with distinct roots as well as those with repeated non-*1 *roots.*

2. Examples: Closed-Form Solutions of Second Order nonhomogeneous Linear Recurrence Relations

In this section, we present illustrative applications of Theorem 1.1(a) to selected second-order nonhomogeneous recurrence relations. Throughout, we restrict attention to the non-resonant case $r = 0$, in which the characteristic equation 1.3 possesses two distinct roots different from 1. This assumption suffices for the families under consideration—namely, the generalized balancing and generalized Oresme sequences. For each class, explicit closed-form solutions are derived under polynomial inputs of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$. In this way, the analysis systematically incorporates constant ($s = 0$), linear ($s = 1$), quadratic ($s = 2$), cubic ($s = 3$), quartic ($s = 4$), quintic ($s = 5$), sextic ($s = 6$), and septic ($s = 7$) forcing terms, thereby highlighting both the generality of the method and its relevance across classical recurrence structures.

In Theorem 1.1, the parameter r is defined as the multiplicity of the root 1 in the characteristic equation. This definition is narrower than the general notion of root multiplicity: it does not account for repeated roots other than 1. The distinction is important in applications. For example, in the Oresme case the characteristic equation admits a repeated root at $\frac{1}{2}$. Since the root 1 does not occur, the Oresme family remains in the non-resonant setting $r = 0$. Thus, resonance phenomena arise only when the root 1 appears with multiplicity, and the presence of a repeated root different from 1 does not alter the classification. This ensures that Theorem 1.1 applies directly to the Oresme sequences without modification.

By contrast, in the balancing family the characteristic equation has two distinct roots, neither equal to 1. Here again we are in the non-resonant case $r = 0$, but for a different reason: the roots are distinct and avoid 1. Together, these examples illustrate the scope of the framework. The parameter r serves to detect resonance specifically at the root 1, while other repeated roots (such as $\frac{1}{2}$ in the Oresme case) do not trigger resonance corrections. This clarification highlights why both balancing and Oresme families fall under the non-resonant classification, despite their different root structures.

To demonstrate the scope of the framework, recall from the introduction that the general second-order nonhomogeneous recurrence relation is given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + p(n), \quad n \geq 2,$$

with initial conditions $W_0 = k_0, W_1 = k_1$, where the input polynomial takes the form

$$p(n) = \sum_{i=0}^{s} c_i n^i, \quad c_i \in \mathbb{C} \text{ or } c_i \in \mathbb{C}[x].$$

Here, the coefficients a_1, a_2 are arbitrary complex scalars (or polynomials in $\mathbb{C}[x]$), and the choice of (k_0, k_1) specifies the initial configuration of the sequence. In the examples that follow, we focus on two distinguished parameter sets:

$$(a_1, a_2) = (6, -1), \quad (a_1, a_2) = \left(1, -\tfrac{1}{4}\right),$$

corresponding respectively to the generalized balancing and generalized Oresme families. For each case, closed-form solutions are obtained under polynomial inputs of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$, illustrating how the general framework adapts to different initial values and forcing terms while unifying the treatment of these classical sequences.

2.1. Generalized balancing Numbers: The case $a_1 = 6$ and $a_2 = -1$. In this subsection, we consider the case $a_1 = 6, a_2 = -1$. A generalized balancing sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$V_n = 6V_{n-1} - V_{n-2} \tag{2.1}$$

with the initial values $V_0 = v_0, V_1 = v_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = 6V_{-(n-1)} - V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n. For more information on generalized balancing numbers, see Soykan [16].

The Binet formula of generalized balancing numbers can be written as

$$V_n = \frac{V_1 - \beta V_0}{\alpha - \beta}\alpha^n - \frac{V_1 - \alpha V_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation $z^2 - 6z + 1 = 0$. Moreover

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2} \\ \beta &= 3 - 2\sqrt{2} \end{aligned}$$

So

$$V_n = \frac{V_1 - (3 - 2\sqrt{2})V_0}{4\sqrt{2}}(3 + 2\sqrt{2})^n - \frac{V_1 - (3 + 2\sqrt{2})V_0}{4\sqrt{2}}(3 - 2\sqrt{2})^n. \tag{2.2}$$

Now we define three special cases of the sequence $\{V_n\}$. balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$\begin{aligned} B_n &= 6B_{n-1} - B_{n-2}, \qquad B_0 = 0, B_1 = 1, && (2.3) \\ H_n &= 6H_{n-1} - H_{n-2}, \qquad H_0 = 2, H_1 = 6, && (2.4) \\ C_n &= 6C_{n-1} - C_{n-2}, \qquad C_0 = 1, C_1 = 3. && (2.5) \end{aligned}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (2.3)-(2.5) hold for all integer n.

For all integers n, balancing, modified Lucas-balancing and Lucas-balancing numbers can be expressed using Binet's formulas as

$$\begin{aligned} B_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \\ C_n &= \frac{\alpha^n + \beta^n}{2}, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 6, \\ a_2 &= -1, \\ W_0 &= 0, \\ W_1 &= 1, \end{aligned}$$

so that we have the case $V_n = B_n$ (balancing numbers).

Example 2.1. *In this example, we consider the case $a_1 = 6, a_2 = -1, W_0 = 0, W_1 = 1$ in Theorem 1.1 (a). So $V_n(0,1) = B_n$ represents nth balancing number i.e., $V_n = B_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_0$$

is given by

$$W_n(0,1) = (\frac{1}{4}c_0 + 1)B_n - \frac{1}{4}c_0 B_{n-1} - \frac{1}{4}c_0,$$

i.e.,

$$W_n(0,1) = \frac{1}{4}c_0 B_{n+1} + (1 - \frac{5}{4}c_0)B_n - \frac{1}{4}c_0.$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_1 n + c_0$$

is given by

$$W_n(0,1) = \frac{1}{4}(c_0 + 2c_1 + 4)B_n - \frac{1}{4}(c_0 + c_1)B_{n-1} - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1),$$

i.e.,

$$W_n(0,1) = \frac{1}{4}(c_0 + c_1)B_{n+1} + \frac{1}{4}(4 - 4c_1 - 5c_0)B_n - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1).$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2)B_{n-1} + A_2 n^2 + A_1 n + A_0,$$

i.e.,

$$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 - 8)B_n + A_2 n^2 + A_1 n + A_0,$$

where

$$\begin{aligned} A_2 &= -\frac{1}{4}c_2, \\ A_1 &= -\frac{1}{4}(c_1 + 2c_2), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 22c_3 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3)B_{n-1} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

i.e.,

$$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 + 8c_3 - 8)B_n + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{4}(c_2 + 3c_3), \\ A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3). \end{aligned}$$

(e): *The case $s=4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 22c_3 + 60c_4 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4)B_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

i.e.,

$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 + 8c_3 + 12c_4 - 8)B_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

where

$$\begin{aligned}
A_4 &= -\frac{1}{4}c_4, \\
A_3 &= -\frac{1}{4}(c_3 + 4c_4), \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4), \\
A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3 + 20c_4), \\
A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4).
\end{aligned}$$

(f): *The case $s=5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 22c_3 + 60c_4 + 184c_5 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4 + 32c_5)B_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4 + 32c_5)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 + 8c_3 + 12c_4 + 8c_5 - 8)B_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$
\begin{aligned}
A_5 &= -\frac{1}{4}c_5, \\
A_4 &= -\frac{1}{4}(c_4+5c_5), \\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5), \\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5), \\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5), \\
A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4+32c_5).
\end{aligned}
$$

(g): *The case $s=6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{16}(4c_0+8c_1+18c_2+44c_3+120c_4+368c_5+1293c_6+16)B_n - \frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6)B_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(0,1) = \frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6)B_{n+1} - \frac{1}{16}(20c_0+16c_1+18c_2+16c_3+24c_4+16c_5+93c_6-16)B_n + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$
\begin{aligned}
A_6 &= -\frac{1}{4}c_6, \\
A_5 &= -\frac{1}{4}(c_5+6c_6), \\
A_4 &= -\frac{1}{8}(2c_4+10c_5+45c_6), \\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5+50c_6), \\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5+90c_6), \\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5+192c_6), \\
A_0 &= -\frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6).
\end{aligned}
$$

(h): *The case $s=7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1) = \frac{1}{16}(4c_0+8c_1+18c_2+44c_3+120c_4+368c_5+1293c_6+5174c_7+16)B_n - \frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6+865c_7)B_{n-1}+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

i.e.,

$W_n(0,1) = \frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6+865c_7)B_{n+1} - \frac{1}{16}(20c_0+16c_1+18c_2+16c_3+24c_4+16c_5+93c_6+16c_7-16)B_n+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

where

$$\begin{aligned}
A_7 &= -\frac{1}{4}c_7, \\
A_6 &= -\frac{1}{4}(c_6+7c_7), \\
A_5 &= -\frac{1}{8}(2c_5+12c_6+63c_7), \\
A_4 &= -\frac{1}{8}(2c_4+10c_5+45c_6+175c_7), \\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5+50c_6+210c_7), \\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5+90c_6+336c_7), \\
A_1 &= -\frac{1}{16}(4c_1+8c_2+18c_3+40c_4+120c_5+384c_6+1617c_7), \\
A_0 &= -\frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6+865c_7).
\end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned}
a_1 &= 6, \\
a_2 &= -1, \\
W_0 &= 2, \\
W_1 &= 6,
\end{aligned}$$

so that we have the case $V_n = H_n$ (modified Lucas-balancing numbers).

Example 2.2. *In this example, we consider the case $a_1 = 6, a_2 = -1, W_0 = 2, W_1 = 6$ in Theorem 1.1 (a). So $V_n(2,6) = H_n$ represents nth modified Lucas-balancing number i.e., $V_n = H_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_0$$

is given by

$$W_n(2,6) = \frac{1}{32}(c_0+32)H_n + \frac{1}{32}c_0H_{n-1} - \frac{1}{4}c_0,$$

i.e.,

$$W_n(2,6) = -\frac{1}{32}c_0H_{n+1} + \frac{1}{32}(7c_0+32)H_n - \frac{1}{4}c_0.$$

(b): *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_1 n + c_0$$

is given by

$$W_n(2,6) = \frac{1}{64}(2c_0 + 5c_1 + 64)H_n + \frac{1}{64}(2c_0 + c_1)H_{n-1} - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1),$$

i.e.,

$$W_n(2,6) = -\frac{1}{64}(2c_0 + c_1)H_{n+1} + \frac{1}{64}(14c_0 + 11c_1 + 64)H_n - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1).$$

(c): *The case $s=2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(2,6) = \frac{1}{64}(2c_0 + 5c_1 + 12c_2 + 64)H_n + \frac{1}{64}(2c_0 + c_1)H_{n-1} + A_2 n^2 + A_1 n + A_0$$

i.e.,

$$W_n(2,6) = -\frac{1}{64}(2c_0 + c_1)H_{n+1} + \frac{1}{64}(14c_0 + 11c_1 + 12c_2 + 64)H_n + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{4}c_2, \\ A_1 &= -\frac{1}{4}(c_1 + 2c_2), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(2,6) = \frac{1}{128}(4c_0 + 10c_1 + 24c_2 + 61c_3 + 128)H_n + \frac{1}{128}(4c_0 + 2c_1 - 7c_3)H_{n-1} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

i.e.,

$$W_n(2,6) = -\frac{1}{128}(4c_0 + 2c_1 - 7c_3)H_{n+1} + \frac{1}{128}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 128)H_n + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{4}(c_2+3c_3), \\ A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3), \\ A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3). \end{aligned}$$

(e): *The case $s=4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,6) = \frac{1}{128}(4c_0+10c_1+24c_2+61c_3+168c_4+128)H_n + \frac{1}{128}(4c_0+2c_1-7c_3-24c_4)H_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(2,6) = -\frac{1}{128}(4c_0+2c_1-7c_3-24c_4)H_{n+1} + \frac{1}{128}(28c_0+22c_1+24c_2+19c_3+24c_4+128)H_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_4 &= -\frac{1}{4}c_4, \\ A_3 &= -\frac{1}{4}(c_3+4c_4), \\ A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4), \\ A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4), \\ A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4). \end{aligned}$$

(f): *The case $s=5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,6) = \frac{1}{128}(4c_0+10c_1+24c_2+61c_3+168c_4+520c_5+128)H_n + \frac{1}{128}(4c_0+2c_1-7c_3-24c_4-88c_5)H_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(2,6) = -\frac{1}{128}(4c_0+2c_1-7c_3-24c_4-88c_5)H_{n+1} + \frac{1}{128}(28c_0+22c_1+24c_2+19c_3+24c_4-8c_5+128)H_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_5 &= -\frac{1}{4}c_5, \\ A_4 &= -\frac{1}{4}(c_4+5c_5), \\ A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5), \\ A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5), \\ A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5), \\ A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4+32c_5). \end{aligned}$$

(g): *The case $s=6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,6) = \frac{1}{128}(4c_0+10c_1+24c_2+61c_3+168c_4+520c_5+1824c_6+128)H_n + \frac{1}{128}(4c_0+2c_1-7c_3-24c_4-88c_5-300c_6)H_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(2,6) = -\frac{1}{128}(4c_0+2c_1-7c_3-24c_4-88c_5-300c_6)H_{n+1} + \frac{1}{128}(28c_0+22c_1+24c_2+19c_3+24c_4-8c_5+24c_6+128)H_n + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_6 &= -\frac{1}{4}c_6, \\ A_5 &= -\frac{1}{4}(c_5+6c_6), \\ A_4 &= -\frac{1}{8}(2c_4+10c_5+45c_6), \\ A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5+50c_6), \\ A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5+90c_6), \\ A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5+192c_6), \\ A_0 &= -\frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6). \end{aligned}$$

(h): *The case $s=7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(2,6) = \frac{1}{256}(8c_0+20c_1+48c_2+122c_3+336c_4+1040c_5+3648c_6+14\,657c_7+256)H_n+\frac{1}{256}(8c_0+4c_1-14c_3-48c_4-176c_5-600c_6-2579c_7)H_{n-1}+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

i.e.,

$W_n(2,6) = -\frac{1}{256}(8c_0+4c_1-14c_3-48c_4-176c_5-600c_6-2579c_7)H_{n+1}+\frac{1}{256}(56c_0+44c_1+48c_2+38c_3+48c_4-16c_5+48c_6-817c_7+256)H_n+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

where

$$\begin{aligned}
A_7 &= -\frac{1}{4}c_7, \\
A_6 &= -\frac{1}{4}(c_6+7c_7), \\
A_5 &= -\frac{1}{8}(2c_5+12c_6+63c_7), \\
A_4 &= -\frac{1}{8}(2c_4+10c_5+45c_6+175c_7), \\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5+50c_6+210c_7), \\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5+90c_6+336c_7), \\
A_1 &= -\frac{1}{16}(4c_1+8c_2+18c_3+40c_4+120c_5+384c_6+1617c_7), \\
A_0 &= -\frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6+865c_7).
\end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned}
a_1 &= 6, \\
a_2 &= -1, \\
W_0 &= 1, \\
W_1 &= 3,
\end{aligned}$$

so that we have the case $V_n = C_n$ (Lucas-balancing numbers).

EXAMPLE 2.3. *In this example, we consider the case $a_1 = 6, a_2 = -1, W_0 = 1, W_1 = 3$ in Theorem 1.1 (a). So $V_n(1,3) = C_n$ represents nth Lucas-balancing number i.e., $V_n = C_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_0$$

is given by

$$W_n(1,3) = \frac{1}{16}(c_0+16)C_n + \frac{1}{16}c_0C_{n-1} - \frac{1}{4}c_0,$$

i.e.,

$$W_n(1,3) = -\frac{1}{16}c_0C_{n+1} + \frac{1}{16}(7c_0+16)C_n - \frac{1}{4}c_0.$$

(b): *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_1 n + c_0$$

is given by

$$W_n(1,3) = \frac{1}{32}(2c_0 + 5c_1 + 32)C_n + \frac{1}{32}(2c_0 + c_1)C_{n-1} - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1),$$

i.e.,

$$W_n(1,3) = -\frac{1}{32}(2c_0 + c_1)C_{n+1} + \frac{1}{32}(14c_0 + 11c_1 + 32)C_n - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1).$$

(c): *The case $s=2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(1,3) = \frac{1}{32}(2c_0 + 5c_1 + 12c_2 + 32)C_n + \frac{1}{32}(2c_0 + c_1)C_{n-1} + A_2 n^2 + A_1 n + A_0$$

i.e.,

$$W_n(1,3) = -\frac{1}{32}(2c_0 + c_1)C_{n+1} + \frac{1}{32}(14c_0 + 11c_1 + 12c_2 + 32)C_n + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{4}c_2, \\ A_1 &= -\frac{1}{4}(c_1 + 2c_2), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(1,3) = \frac{1}{64}(4c_0 + 10c_1 + 24c_2 + 61c_3 + 64)C_n + \frac{1}{64}(4c_0 + 2c_1 - 7c_3)C_{n-1} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

i.e.,

$$W_n(1,3) = -\frac{1}{64}(4c_0 + 2c_1 - 7c_3)C_{n+1} + \frac{1}{64}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 64)C_n + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

where

$$\begin{aligned}
A_3 &= -\frac{1}{4}c_3, \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3), \\
A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3), \\
A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3).
\end{aligned}$$

(e): *The case $s = 4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(1,3) = \frac{1}{64}(4c_0 + 10c_1 + 24c_2 + 61c_3 + 168c_4 + 64)C_n + \frac{1}{64}(4c_0 + 2c_1 - 7c_3 - 24c_4)C_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(1,3) = -\frac{1}{64}(4c_0 + 2c_1 - 7c_3 - 24c_4)C_{n+1} + \frac{1}{64}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 24c_4 + 64)C_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned}
A_4 &= -\frac{1}{4}c_4, \\
A_3 &= -\frac{1}{4}(c_3 + 4c_4), \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4), \\
A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3 + 20c_4), \\
A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4).
\end{aligned}$$

(f): *The case $s = 5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(1,3) = \frac{1}{64}(4c_0 + 10c_1 + 24c_2 + 61c_3 + 168c_4 + 520c_5 + 64)C_n + \frac{1}{64}(4c_0 + 2c_1 - 7c_3 - 24c_4 - 88c_5)C_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

i.e.,

$W_n(1,3) = -\frac{1}{64}(4c_0 + 2c_1 - 7c_3 - 24c_4 - 88c_5)C_{n+1} + \frac{1}{64}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 24c_4 - 8c_5 + 64)C_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

where

$$\begin{aligned}
A_5 &= -\frac{1}{4}c_5, \\
A_4 &= -\frac{1}{4}(c_4+5c_5), \\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5), \\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5), \\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5), \\
A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4+32c_5).
\end{aligned}$$

(g): *The case $s=6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(1,3) = \frac{1}{64}(4c_0+10c_1+24c_2+61c_3+168c_4+520c_5+1824c_6+64)C_n + \frac{1}{64}(4c_0+2c_1-7c_3-24c_4-88c_5-300c_6)C_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

i.e.,

$W_n(1,3) = -\frac{1}{64}(4c_0+2c_1-7c_3-24c_4-88c_5-300c_6)C_{n+1} + \frac{1}{64}(28c_0+22c_1+24c_2+19c_3+24c_4-8c_5+24c_6+64)C_n + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned}
A_6 &= -\frac{1}{4}c_6, \\
A_5 &= -\frac{1}{4}(c_5+6c_6), \\
A_4 &= -\frac{1}{8}(2c_4+10c_5+45c_6), \\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5+50c_6), \\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5+90c_6), \\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5+192c_6), \\
A_0 &= -\frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6).
\end{aligned}$$

(h): *The case $s=7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(1,3) = \frac{1}{128}(8c_0+20c_1+48c_2+122c_3+336c_4+1040c_5+3648c_6+14657c_7+128)C_n+\frac{1}{128}(8c_0+4c_1-14c_3-48c_4-176c_5-600c_6-2579c_7)C_{n-1}+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

i.e.,

$W_n(1,3) = -\frac{1}{128}(8c_0+4c_1-14c_3-48c_4-176c_5-600c_6-2579c_7)C_{n+1}+\frac{1}{128}(56c_0+44c_1+48c_2+38c_3+48c_4-16c_5+48c_6-817c_7+128)C_n+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

where

$$\begin{aligned}
A_7 &= -\frac{1}{4}c_7, \\
A_6 &= -\frac{1}{4}(c_6+7c_7), \\
A_5 &= -\frac{1}{8}(2c_5+12c_6+63c_7), \\
A_4 &= -\frac{1}{8}(2c_4+10c_5+45c_6+175c_7), \\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5+50c_6+210c_7), \\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5+90c_6+336c_7), \\
A_1 &= -\frac{1}{16}(4c_1+8c_2+18c_3+40c_4+120c_5+384c_6+1617c_7), \\
A_0 &= -\frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6+865c_7).
\end{aligned}$$

2.2. Generalized Oresme Numbers: The case $a_1 = 1$ and $a_2 = -\frac{1}{4}$. Now, we consider the case $a_1 = 1, a_2 = -\frac{1}{4}$. A generalized Oresme sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$V_n = V_{n-1} - \frac{1}{4}V_{n-2} \tag{2.6}$$

with the initial values $V_0 = v_0, V_1 = v_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = 4V_{-(n-1)} - 4V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.6) holds for all integer n. For more information on generalized Oresme numbers, see Soykan [17].

Binet formula of generalized Oresme numbers can be given as

$$V_n = (D_1 + D_2 n)\alpha^n \tag{2.7}$$

where

$$\begin{aligned}
D_1 &= V_0, \\
D_2 &= \frac{1}{\alpha}(V_1 - \alpha V_0).
\end{aligned}$$

i.e.,

$$V_n = (V_0 + \frac{1}{\alpha}(V_1 - \alpha V_0)\, n)\alpha^n$$

Here, $\alpha = \beta = \frac{1}{2}$ are the roots of the quadratic equation

$$z^2 - z + \frac{1}{4} = 0. \tag{2.8}$$

i.e. the roots of characteristic equation (2.8) are equal. Note that

$$V_n = (V_0 + 2\left(V_1 - \frac{1}{2}V_0\right) n) \times \frac{1}{2^n}.$$

Now we define three special cases of the sequence $\{V_n\}$. Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence

$$\begin{aligned}
G_{n+2} &= G_{n+1} - \frac{1}{4}G_n, \qquad G_0 = 0, G_1 = 1, && (2.9)\\
H_{n+2} &= H_{n+1} - \frac{1}{4}H_n, \qquad H_0 = 2, H_1 = 1, && (2.10)\\
O_{n+2} &= O_{n+1} - \frac{1}{4}O_n, \qquad O_0 = 0, O_1 = \frac{1}{2}. && (2.11)
\end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
G_{-n} &= 4G_{-(n-1)} - 4G_{-(n-2)},\\
H_{-n} &= 4H_{-(n-1)} - 4H_{-(n-2)},\\
O_{-n} &= 4O_{-(n-1)} - 4O_{-(n-2)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.9)-(2.11) hold for all integer n.

For all integers n, modified Oresme, Oresme-Lucas and Oresme numbers can be expressed using Binet's formulas as

$$\begin{aligned}
G_n &= n\alpha^{n-1} = \frac{n}{2^{n-1}},\\
H_n &= 2\alpha^n = \frac{1}{2^{n-1}},\\
O_n &= n\alpha^n = \frac{n}{2^n},
\end{aligned}$$

respectively.

It is important to note that in Theorem 1.1, the parameter r refers specifically to the multiplicity of the root 1. This distinction avoids confusion in the Oresme case, where the characteristic equation possesses a repeated root at $\frac{1}{2}$. Since the root 1 does not appear, the Oresme family remains in the non-resonant case $r = 0$, and the general framework applies directly without resonance corrections.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1, \\ a_2 &= -\frac{1}{4}, \\ W_0 &= 0, \\ W_1 &= 1, \end{aligned}$$

so that we have the case $V_n = G_n$ (modified Oresme numbers).

EXAMPLE 2.4. *In this example, we consider the case* $a_1 = 1, a_2 = -\frac{1}{4}, W_0 = 0, W_1 = 1$ *in Theorem 1.1 (a). So* $V_n(0,1) = V_n$ *represents nth modified Oresme number, i.e.,* $V_n = G_n$.

- **(a):** *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_0$$

is given by

$$W_n(0,1) = -\left(4c_0 - 1\right)G_n + c_0 G_{n-1} + 4c_0$$

i.e.,

$$W_n(0,1) = -4c_0 G_{n+1} + G_n + 4c_0.$$

- **(b):** *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_1 n + c_0$$

is given by

$$W_n(0,1) = -(4c_0 - 4c_1 - 1)G_n + (c_0 - 2c_1)G_{n-1} + A_1 n + A_0$$

i.e.,

$$W_n(0,1) = -4(c_0 - 2c_1)G_{n+1} - (4c_1 - 1)G_n + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= 4c_1, \\ A_0 &= 4(c_0 - 2c_1). \end{aligned}$$

- **(c):** *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_2 n^2 + c_1 n + c_0$$

is given by

$$W_n(0,1) = -(4c_0 - 4c_1 + 20c_2 - 1)G_n + (c_0 - 2c_1 + 8c_2)G_{n-1} + A_2n^2 + A_1n + A_0$$

i.e.,

$$W_n(0,1) = -4(c_0 - 2c_1 + 8c_2)G_{n+1} - (4c_1 - 12c_2 - 1)G_n + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_2 &= 4c_2, \\ A_1 &= 4(c_1 - 4c_2), \\ A_0 &= 4(c_0 - 2c_1 + 8c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(0,1) = -(4c_0 - 4c_1 + 20c_2 - 100c_3 - 1)G_n + (c_0 - 2c_1 + 8c_2 - 44c_3)G_{n-1} + A_3n^3 + A_2n^2 + A_1n + A_0$$

i.e.,

$$W_n(0,1) = -4(c_0 - 2c_1 + 8c_2 - 44c_3)G_{n+1} - (4c_1 - 12c_2 + 76c_3 - 1)G_n + A_3n^3 + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_3 &= 4c_3, \\ A_2 &= 4(c_2 - 6c_3), \\ A_1 &= 4(c_1 - 4c_2 + 24c_3), \\ A_0 &= 4(c_0 - 2c_1 + 8c_2 - 44c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1, \\ a_2 &= -\frac{1}{4}, \\ W_0 &= 0, \\ W_1 &= \frac{1}{2}, \end{aligned}$$

so that we have the case $V_n = O_n$ (Oresme numbers).

Example 2.5. *In this example, we consider the case $a_1 = 1, a_2 = -\frac{1}{4}, W_0 = 0, W_1 = \frac{1}{2}$ in Theorem 1.1 (a). So $V_n(0, \frac{1}{2}) = V_n$ represents nth Oresme number, i.e., $V_n = O_n$.*

(a): *The case $s=0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_0$$

is given by

$$W_n(0,\frac{1}{2}) = -(8c_0-1)O_n + 2c_0O_{n-1} + 4c_0$$

i.e.,

$$W_n(0,\frac{1}{2}) = -8c_0O_{n+1} + O_n + 4c_0.$$

(b): *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_1n + c_0$$

is given by

$$W_n(0,\frac{1}{2}) = -(8c_0-8c_1-1)O_n + 2(c_0-2c_1)O_{n-1} + A_1n + A_0$$

i.e.,

$$W_n(0,\frac{1}{2}) = -8(c_0-2c_1)O_{n+1} - (8c_1-1)O_n + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= 4c_1, \\ A_0 &= 4(c_0-2c_1). \end{aligned}$$

(c): *The case $s=2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(0,\frac{1}{2}) = -(8c_0-8c_1+40c_2-1)O_n + 2(c_0-2c_1+8c_2)O_{n-1} + A_2n^2 + A_1n + A_0$$

i.e.,

$$W_n(0,\frac{1}{2}) = -8(c_0-2c_1+8c_2)O_{n+1} - (8c_1-24c_2-1)O_n + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_2 &= 4c_2, \\ A_1 &= 4(c_1-4c_2), \\ A_0 &= 4(c_0-2c_1+8c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(0,\frac{1}{2}) = -(8c_0 - 8c_1 + 40c_2 - 200c_3 - 1)O_n + 2(c_0 - 2c_1 + 8c_2 - 44c_3)O_{n-1} + A_3n^3 + A_2n^2 + A_1n + A_0$$

i.e.,

$$W_n(0,\frac{1}{2}) = -8(c_0 - 2c_1 + 8c_2 - 44c_3)O_{n+1} - (8c_1 - 24c_2 + 152c_3 - 1)O_n + A_3n^3 + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_3 &= 4c_3, \\ A_2 &= 4(c_2 - 6c_3), \\ A_1 &= 4(c_1 - 4c_2 + 24c_3), \\ A_0 &= 4(c_0 - 2c_1 + 8c_2 - 44c_3). \end{aligned}$$

Conclusion

This paper has presented explicit closed-form expressions for second-order nonhomogeneous linear recurrence relations, formulated as generalized Leonardo-type sequences with polynomial inputs. The analysis was structured around the multiplicity $r = 0, 1, 2$ of 1 as a root of the characteristic equation, with solutions obtained for input polynomials of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$. These results clarify how root multiplicity and polynomial degree jointly determine the particular solution, while the homogeneous component remains governed by the same recurrence relation. In the examples, emphasis was placed on the non-resonant case $r = 0$, where both roots differ from 1, thereby showing how classical integer sequences naturally appear as homogeneous analogues within the unified framework.

Applications were carried out for two principal families: the generalized balancing and generalized Oresme sequences. Within the balancing family, the choices $(W_0, W_1) = (0, 1), (2, 6), (1, 3)$ yielded the classical balancing, modified Lucas-balancing, and Lucas-balancing numbers. For each case, closed-form solutions were derived under polynomial inputs of degrees $s = 0, 1, 2, 3, 4, 5, 6, 7$, with coefficients c_i selected to generate constant, linear, quadratic, cubic, quartic, quintic, sextic, and septic forcing terms. These examples demonstrate how the general theorem adapts smoothly to both generalized families and their classical specializations.

Beyond theoretical contributions, the results hold pedagogical and practical importance. For teaching, they provide accessible models that allow students to explore nonhomogeneous recurrences with polynomial inputs of varying degrees without excessive computation. For research, they offer resonance-aware formulas and explicit derivations that can be applied in mathematics, computer science, engineering, and physics. In this way, the study strengthens instructional practice while extending the reach of recurrence theory across multiple disciplines.

The unified framework consolidates recurrence theory and opens pathways for modern applications. By supplying resonance-aware formulas and explicit derivations, the method can be adapted to optimization problems in transportation, coding theory, and discrete modeling. Closed-form solutions are particularly valuable because they yield exact predictability for complex systems, avoiding the computational burden of iterative evaluation. The framework developed here extends recurrence theory, clarifies resonance phenomena, and provides tools relevant for algorithm analysis, numerical modeling, and optimization problems where efficiency and precision are essential.

At the same time, the iterative scheme introduced in this paper produces explicit polynomial-type particular solutions, but its scope is currently limited to polynomial inputs. Extending the framework to accommodate non-polynomial functions—such as trigonometric or other analytic inputs—would require further methodological refinements and may introduce new resonance effects. Moreover, while the approach successfully clarifies the role of root multiplicity and resonance corrections, computational complexity grows rapidly with higher-order recurrences. This suggests that practical implementation for advanced cases will benefit from integration with computer algebra systems or symbolic computation platforms. In this respect, the current framework provides a solid foundation while pointing toward natural directions for further generalization and computational support.

Finally, the methodology opens promising avenues for future research. It can be incorporated into symbolic computation platforms, applied to verify classical identities in recurrence theory, and extended to interdisciplinary contexts such as cryptography, coding theory, and discrete modeling in physics and biology. By acknowledging current limitations while outlining directions for extension, the study offers a balanced perspective: consolidating the contribution of the present results while pointing toward further exploration and development.

It should be emphasized that in Theorem 1.1, the parameter r refers specifically to the multiplicity of the root 1 in the characteristic equation, rather than to repeated roots in general. This distinction is crucial in the Oresme case, since its characteristic equation possesses a repeated root at $\frac{1}{2}$. Because the root 1 does not occur, the Oresme family remains in the non-resonant setting $r = 0$. In other words, resonance phenomena arise only when the root 1 appears with multiplicity, and the presence of a repeated root different from 1 does not alter the classification. This clarification ensures that the applicability of Theorem 1.1 to the Oresme sequences is fully transparent and avoids possible misinterpretation.

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