Closed-Form Solutions of Third-Order Leonardo-Type Sequences: Third-Order Jacobsthal, Graham, and Jacobsthal-Narayana Families as Homogeneous Counterparts

Abstract. This paper develops unified closed-form formulas for third-order nonhomogeneous linear recurrence relations of Leonardo-type sequences, where the driving term is taken to be a polynomial. The method proceeds by decomposing each recurrence into its homogeneous and particular components, yielding explicit solutions that depend simultaneously on the multiplicity of the characteristic roots and the degree of the input polynomial. While the general framework accommodates resonance phenomena arising from repeated roots, the examples presented here concentrate on the non-resonant case $r = 0$, in which all three roots of the characteristic equation are distinct from 1.

Within this setting, several notable families are examined, including the generalized third-order Jacobsthal, generalized Graham, and generalized Jacobsthal–Narayana sequences. Classical counterparts such as the third-order Jacobsthal, modified Jacobsthal, Jacobsthal–Lucas, Graham, Graham–Lucas, Jacobsthal–Narayana, and Jacobsthal–Narayana–Lucas numbers appear naturally as specializations of the Leonardo-type framework. These cases illustrate how closed-form expressions clarify the interaction between characteristic roots, polynomial inputs, and resonance effects, while also providing templates for applications in discrete mathematics, combinatorics, computational number theory, algorithmic analysis, cryptography, and symbolic modeling in physics and biology.

A further illustration is obtained by fixing specific coefficients c_i in the input polynomial and considering degrees $s = 0, 1, 2, 3$. Under identical recurrence parameters, the homogeneous dynamics reproduce the well-known third-order Jacobsthal, modified Jacobsthal, and Jacobsthal–Lucas sequences, while the constant, linear, quadratic, and cubic inputs enrich the particular solutions. This demonstrates the continuity of the

framework across polynomial degrees and emphasizes the role of initial values in shaping the resulting closed forms.

Beyond their theoretical contribution, the explicit constructions also serve pedagogical purposes by enabling students to engage directly with nonhomogeneous recurrences through accessible formulas rather than lengthy computations. At the same time, they furnish researchers with extended tools for higher-order recurrence analysis. In this way, the study highlights both the novelty and interdisciplinary reach of generalized Leonardo-type sequences, strengthening recurrence theory while supporting applications across mathematics, computer science, engineering, and the natural sciences.

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1. Introduction

Sequences arising from recurrence relations have long been central to mathematics, radiating outward into disciplines such as physics, engineering, architecture, biology, computer science, and even the arts. Though their definitions often appear elementary, they conceal profound richness, modeling processes of growth, oscillation, and symbolic construction. Classical second-order families—including the Fibonacci, Lucas, Pell, and Jacobsthal sequences—continue to serve as foundational examples of this tradition.

The domain, however, extends far beyond second-order cases. Higher-order recurrence sequences deepen both theoretical insight and practical application, enlarging the classical framework while exposing intricate algebraic and analytic structures. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences exemplify this progression, each governed by characteristic polynomials whose root configurations determine closed-form formulas. Homogeneous recurrences emphasize the relationship between characteristic polynomials and root multiplicities, whereas non-homogeneous forms introduce symbolic terms that interact with root structures to produce resonance effects. Collectively, these families establish a unified framework that links classical recurrence identities with the advancing field of symbolic recurrence theory.

This paper contributes to the ongoing discourse by establishing closed-form representations for third-order nonhomogeneous linear recurrence relations, formulated as generalized Leonardo-type sequences with polynomial inputs. The study further identifies distinguished manifestations of generalized Tribonacci numbers, which naturally arise as the homogeneous analogues of the corresponding nonhomogeneous relations. Our exposition encompasses several classical families, notably the third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas sequences; Graham sequence and Graham-Lucas sequences.

The classical Leonardo sequence is defined by the nonhomogeneous recurrence

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with initial conditions $l_0 = 1$ and $l_1 = 1$. Although the recurrence relation itself is elementary, the historical development of the sequence is more intricate. Its recognition emerged gradually, with generalized versions appearing in the literature well before the formal adoption of its name. In recent decades, renewed attention has been stimulated by explicit case studies and the diverse applications in which the sequence naturally arises.

The Leonardo sequence is distinguished not only by its mathematical simplicity but also by its ability to model systems that combine homogeneous recurrence dynamics with nonhomogeneous perturbations. This dual structure has made it a productive subject of symbolic analysis, bridging classical recurrence theory with modern applications. Contemporary investigations emphasize its algebraic depth, its capacity to encode subtle structural interactions, and its relevance across both theoretical and applied domains.

From a pedagogical perspective, the transparency of its defining relation and the clarity of explicit examples render the Leonardo sequence particularly effective for instructional purposes. It provides students with a straightforward illustration of nonhomogeneous recurrence behavior, while simultaneously serving as an accessible gateway to advanced symbolic techniques and resonance phenomena. In this respect, the sequence continues to function both as a topic of scholarly inquiry and as a valuable teaching resource (see, for instance, [1, 2, 5, 6, 7, 11, 12, 21, 22, 23, 14, 15, 16, 17, 27, 28]).

To the best of our knowledge, the first systematic extension of the Leonardo numbers was undertaken by J. A. Jeske in a trilogy of papers published in *The Fibonacci Quarterly* during 1963–1964 [8, 9, 10]. For a concise overview of contributions within *The Fibonacci Quarterly* and selected works beyond the journal that advance the study of Leonardo-type recurrences, see [24, Section 5].

Let the third order nonhomogeneous linear recurrence relation, referred to as generalized Leonardo-type sequences, be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + p(n) \tag{1.1}$$

with initial conditions $W_0 = k_0, W_1 = k_1, W_2 = k_2$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i,$$

and the recurrence coefficients a_1, a_2, a_3 are complex scalars or polynomials in $\mathbb{C}[x]$. For more information on generalized Leonardo-type sequences, see Soykan [24] and [25].

By homogeneous counterpart we mean the classical sequence obtained when the nonhomogeneous input $p(n)$ is removed. For example, the generalized third-order Jacobsthal family with polynomial input reduces to the standard third-order Jacobsthal sequence when $p(n) = 0$. Similarly, the Graham and Jacobsthal–Narayana families are recovered as homogeneous cases of their respective nonhomogeneous extensions.

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} + a_3 V_{n-3} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2.$$

Suppose that θ_1, θ_2 and θ_3 are the roots of the characteristic equation

$$z^3 - a_1 z^2 - a_2 z - a_3 = 0 \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^3 - a_1 z^2 - a_2 z - a_3 = (z-1)^3 = z^3 - 3z^2 + 3z - 1 = 0$$

so that $a_1 = 3,\ a_2 = -3,\ a_3 = 1$ and (1.2) reduces to

$$V_n = 3V_{n-1} - 3V_{n-2} + V_{n-3}.$$

Our earlier study established particular solutions of third-order nonhomogeneous recurrences (1.1) with polynomial inputs for $s = 0, 1, 2, 3$; see Soykan [26]. Here, we extend those results to unified closed-form solutions by systematically applying Theorem 1.1. The method separates each recurrence into homogeneous and particular parts, with the particular solution obtained through iterative determination of coefficients.

The framework is governed by two parameters: the multiplicity r of 1 as a root of the characteristic equation (1.2), and the degree s of the polynomial $p(n)$. Explicit formulas are derived for all combinations $r = 0, 1, 2, 3$ and $s = 0, 1, 2, 3$, showing how resonance and multiplicity corrections arise when roots coincide. In this paper, however, only the non-resonant case $r = 0$ is needed, since the examples–generalized third-order Jacobsthal, Graham, and Jacobsthal-Narayana sequences–fall precisely into this category. Thus, the theorem is presented in detail for $r = 0$, with higher multiplicities noted but omitted.

Theorem 1.1.

(a): *[24, Theorem 7.7. (a)] The case $r = 0$, i.e., all three roots of the characteristic equation of (1.2) are distinct from* 1.

The solution of (1.1) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = \sum_{i=0}^{s} A_i n^i = A_0 + \sum_{i=1}^{s} A_i n^i$$

is the particular solution of (1.1). For each $0 \leq i \leq s$, A_i *can be calculated with the iteration*

$$A_s = -\frac{c_s}{a_1+a_2+a_3-1}, \text{ for } n = s$$

and

$$A_n = -\frac{1}{a_1+a_2+a_3-1}(c_n - \sum_{k=n+1}^{s}(-1)^{k-n+1}\binom{k}{n}(a_1+2^{k-n}a_2+3^{k-n}a_3)A_k), \text{ for } n = s-1, s-2, ..., 2, 1, 0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \\ W_2^{(p)} &= \sum_{i=0}^{s} 2^i A_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(A_0, \sum_{i=0}^{s} A_i, \sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \ B_2 = \frac{Y_2}{\Delta}, \ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) W_2^{(p)} + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) W_1^{(p)} + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) W_0^{(p)}$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) W_2^{(p)} + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) W_1^{(p)} + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) W_0^{(p)}$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) W_2^{(p)} + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) W_1^{(p)} + a_3 (a_3 W_1^2 - a_3 W_0 W_2) W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

i.e.,

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) \sum_{i=0}^{s} 2^i A_i + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) \sum_{i=0}^{s} A_i + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) \sum_{i=0}^{s} 2^i A_i + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) \sum_{i=0}^{s} A_i + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)\sum_{i=0}^{s} 2^i A_i + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)\sum_{i=0}^{s} A_i + a_3(a_3W_1^2 - a_3W_0W_2)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + \sum_{i=0}^{s} A_i n^i$$

(b): *The case $r = 1$, i.e., 1 is a simple root of the characteristic equation of (1.2). The solution can be obtained in the same manner as case (a).*

(c): *The case $r = 2$, i.e., 1 is a double root of the characteristic equation of (1.2). The procedure parallels case (a), but requires multiplicity corrections to account for the repeated root.*

(d): *The case $r = 3$, i.e., 1 is a triple root of the characteristic equation of (1.2). The solution is constructed analogously to case (a), with adjustments reflecting the higher multiplicity.*

Proof. The result is obtained by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104-105), and Theorem 3.1 (pp. 88-89) for $m = 3$, as presented in Soykan [24]. In principle, the theorem covers all multiplicities $r = 0, 1, 2, 3$: the case $r = 0$ represents the non-resonant situation with three distinct roots different from 1, while $r = 1, 2, 3$ correspond to simple, double, or triple roots of 1 in the characteristic equation (1.2). In this paper, however, only the non-resonant case $r = 0$ is required, where the three roots are distinct from 1. The cases $r = 1, 2, 3$ can be obtained in the same manner with appropriate multiplicity adjustments, but they are not needed for the examples under consideration.

Sketch of Proof. The coefficients B_1, B_2, B_3 are obtained by solving a linear system of equations. More precisely, they satisfy

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = M^{-1} \begin{pmatrix} V_2(W_0^{(C)}, W_1^{(C)}, W_2^{(C)}) \\ V_3(W_0^{(C)}, W_1^{(C)}, W_2^{(C)}) \\ V_4(W_0^{(C)}, W_1^{(C)}, W_2^{(C)}) \end{pmatrix}$$

where the coefficient matrix M is given by

$$M = \begin{pmatrix} V_2(W_0, W_1, W_2) & V_1(W_0, W_1, W_2) & V_0(W_0, W_1, W_2) \\ V_3(W_0, W_1, W_2) & V_2(W_0, W_1, W_2) & V_1(W_0, W_1, W_2) \\ V_4(W_0, W_1, W_2) & V_3(W_0, W_1, W_2) & V_2(W_0, W_1, W_2) \end{pmatrix}.$$

To evaluate these expressions explicitly, one employs the Binet representation of V_n together with the fundamental identities relating the roots $\theta_1, \theta_2, \theta_3$ of the characteristic polynomial to the recurrence coefficients:

$$\theta_1 + \theta_2 + \theta_3 = a_1, \qquad \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = -a_2, \qquad \theta_1\theta_2\theta_3 = a_3.$$

These relations allow the system to be resolved in closed form, thereby yielding explicit formulas for B_1, B_2, B_3 in terms of the initial data and recurrence parameters.

The algebraic manipulations required to resolve this system are lengthy and involve careful substitution of the root identities into the matrix inversion process. Although the derivation is straightforward in principle, it is technically demanding and somewhat laborious, often leaving the reader fatigued by the sheer volume of computations. For this reason, only the essential structure of the argument is sketched here, while the full details are omitted. □

REMARK 1.2. *In the statement of case (a) in Theorem 1.1, it is implicit that*

$$a_1 + a_2 + a_3 - 1 \neq 0.$$

Indeed, if one were to assume

$$a_1 + a_2 + a_3 - 1 = 0,$$

then by substituting the relations between the roots $\theta_1, \theta_2, \theta_3$ and the recurrence coefficients, we obtain

$$(\theta_1 + \theta_2 + \theta_3) - (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) + \theta_1\theta_2\theta_3 - 1 = 0,$$

which simplifies to

$$(\theta_1 - 1)(\theta_2 - 1)(\theta_3 - 1) = 0.$$

This identity forces at least one of the roots to equal 1*, and in fact under the given conditions it implies* $\theta_1 = \theta_2 = \theta_3 = 1$*, contradicting the assumption that all roots of the characteristic equation (1.2) are distinct from* 1*.*

Before turning to the technical details, it is useful to outline the guiding principles and methodological framework that underpin the analysis.

Discussion and Research Methodology

The present study advances the theory of generalized Leonardo-type sequences by extending it to third-order nonhomogeneous recurrences with polynomial inputs. The analytical strategy separates each recurrence into two complementary components: the homogeneous solution, dictated by the roots of the characteristic polynomial, and the particular solution, constructed through an iterative scheme that depends on the degree of the input polynomial. This decomposition ensures that the resulting closed-form expressions faithfully capture both the intrinsic recurrence dynamics and the external influence of the forcing term.

The investigation is structured around two parameters: the multiplicity r of 1 as a root of the characteristic equation, and the degree s of the input polynomial $p(n)$. General formulas are established for all combinations $r = 0, 1, 2, 3$ and $s = 0, 1, 2, 3$, clarifying how resonance effects emerge when repeated roots interact with polynomial forcing. In the examples developed here, attention is restricted to the non-resonant case $r = 0$, where all three roots are distinct from 1. This choice avoids resonance corrections while still providing a comprehensive demonstration of the framework.

Having established the methodological foundation, we now apply the central result to concrete cases by invoking Theorem 1.1

Application of Theorem 1.1

The objective of this study is to derive unified closed-form solutions for third-order nonhomogeneous recurrences with polynomial inputs, and to illustrate them through special cases of generalized Tribonacci families, with Theorem 1.1 furnishing the general closed-form solution that underpins these derivations.

Once the recurrence parameters (a_1, a_2, a_3) and initial conditions (W_0, W_1, W_2) are specified, the theorem is applied to compute the coefficients A_i of the particular solution and to combine them with the homogeneous component. This procedure yields exact closed forms for a wide range of sequences, including the generalized third-order Jacobsthal, generalized Graham, and generalized Jacobsthal–Narayana families.

Within the generalized Jacobsthal family, three distinguished specializations arise from specific initial values:

$$(W_0, W_1, W_2) = (0, 1, 1) \quad \text{third-order Jacobsthal numbers,}$$

$$(W_0, W_1, W_2) = (3, 1, 3) \quad \text{modified third-order Jacobsthal numbers,}$$

$$(W_0, W_1, W_2) = (2, 1, 5) \quad \text{third-order Jacobsthal–Lucas numbers.}$$

For each of these cases, explicit closed-form solutions are derived under polynomial inputs of degrees $s = 0, 1, 2, 3$, with specific coefficients c_i chosen to construct constant, linear, quadratic, and cubic forcing terms. The homogeneous dynamics reproduce the classical sequences, while the polynomial inputs enrich the particular solutions with additional structure.

This unified methodology demonstrates continuity across polynomial degrees and initial conditions. It shows how identical recurrence parameters can generate classical sequences in the homogeneous setting, while higher-degree inputs produce enriched particular solutions. The constructive design strengthens the contribution of the study, providing a foundation for further extensions to higher-order recurrences and to non-polynomial inputs. Beyond pure mathematics, the framework has potential applications in discrete structures, algorithmic analysis, cryptography, and symbolic modeling in physics and biology.

2. Examples: Closed-Form Solutions of Third Order nonhomogeneous Linear Recurrence Relations

In this section, we provide representative applications of Theorem 1.1 (a) to selected third-order nonhomogeneous recurrence relations. For all examples considered, we restrict attention to the non-resonant case $r = 0$, where the characteristic equation (1.2) admits three distinct roots different from 1. This assumption suffices for the families under study, namely the generalized third-order Jacobsthal, generalized Graham, and generalized Jacobsthal–Narayana sequences. For each of these classes, explicit closed-form solutions are obtained for polynomial inputs of degrees $s = 0, 1, 2, 3$. In this way, the analysis demonstrates how the framework systematically incorporates constant, linear, quadratic, and cubic forcing terms, thereby underscoring both the generality of the method and its relevance across classical recurrence structures.

To illustrate the breadth of the method, recall from the introduction that the general third-order non-homogeneous recurrence relation is given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + p(n), \qquad n \geq 3,$$

with initial conditions $W_0 = k_0,\ W_1 = k_1,\ W_2 = k_2$, where the input polynomial is

$$p(n) = \sum_{i=0}^{s} c_i n^i, \qquad c_i \in \mathbb{C} \text{ or } c_i \in \mathbb{C}[x].$$

The coefficients a_1, a_2, a_3 are taken as arbitrary complex scalars (or polynomials in $\mathbb{C}[x]$), and the choice of (k_0, k_1, k_2) specifies the initial configuration of the sequence. In the examples that follow, we focus on three distinguished parameter sets:

$$(a_1, a_2, a_3) = (1, 1, 2), \quad (a_1, a_2, a_3) = (2, 3, 5), \quad (a_1, a_2, a_3) = (1, 0, 2),$$

corresponding respectively to the generalized third-order Jacobsthal, generalized Graham, and generalized Jacobsthal–Narayana families. For each case, closed-form solutions are derived under polynomial inputs of degrees $s = 0, 1, 2, 3$, thereby illustrating how the general framework adapts to different initial values and forcing terms while unifying the treatment of these classical sequences.

2.1. Generalized Third-Order Jacobsthal Numbers. In this subsection, we consider the case $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. A generalized third order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + 2V_{n-3} \tag{2.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (2.1) holds for all integer n. For more information on generalized third order Jacobsthal sequence, see Soykan [13].

Binet formula of generalized third order Jacobsthal numbers can be given as

$$V_n = \frac{b_1 \alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2 \beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3 \gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{2.2}$$

Here, α, β and $\gamma$are the roots of the cubic equation $z^3 - z^2 - z - 2 = 0$. Moreover

$$\begin{aligned} \alpha &= 2, \\ \beta &= \frac{-1 + i\sqrt{3}}{2}, \\ \gamma &= \frac{-1 - i\sqrt{3}}{2}. \end{aligned}$$

Third-order Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A077947, [18]), modified third-order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ (OEIS: A186575, [18]) and third-order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (OEIS: A226308, [18]) are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, J_2 = 1, \tag{2.3}$$

$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n, \quad K_0 = 3, K_1 = 1, K_2 = 3. \tag{2.4}$$

$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, j_2 = 5. \tag{2.5}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [4] and $\{K_n\}_{n\geq 0}$ is given in [3]. For more details on the generalized third-order Jacobsthal numbers and its special cases, see [13].

The sequences $\{J_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)}, \\ K_{-n} &= -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)}, \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.5) hold for all integer n.

Note that for all integers n, third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{2.6}$$

$$K_n = \alpha^n + \beta^n + \gamma^n, \tag{2.7}$$

$$j_n = \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{2.8}$$

respectively.

Within this framework, different initial conditions give rise to distinguished sequences already highlighted above: the third-order Jacobsthal numbers $(W_0, W_1, W_2) = (0, 1, 1)$, the modified third-order Jacobsthal numbers $(3, 1, 3)$, and the third-order Jacobsthal–Lucas numbers $(2, 1, 5)$. In what follows, we present explicit closed-form solutions for each of these sequences under polynomial inputs of degrees $s = 0, 1, 2, 3$, thereby illustrating how the method systematically accommodates constant, linear, quadratic, and cubic forcing terms.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 1,\ a_3 = 2, \\ W_0 &= 0,\ W_1 = 1,\ W_2 = 1, \end{aligned}$$

so that we have the case $V_n = J_n$ (third order Jacobsthal numbers).

EXAMPLE 2.1. *In this example, we consider the case* $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $W_0 = 0$, $W_1 = 1$, $W_2 = 1$, *in Theorem 1.1 (a). So* $V_n(0,1,1) = V_n$ *represents nth third order Jacobsthal number, i.e.,* $V_n = J_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(0,1,1) = \frac{1}{3}\left(c_0+3\right)J_n + \frac{2}{3}c_0J_{n-2} - \frac{1}{3}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_1n + c_0$$

is given by

$$W_n(0,1,1) = \frac{1}{3}\left(c_0+4c_1+3\right)J_n + \frac{1}{3}c_1J_{n-1} + \frac{2}{3}\left(c_0+3c_1\right)J_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{3}c_1, \\ A_0 &= -\frac{1}{3}(c_0+3c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,1) = \frac{1}{9}(3c_0+12c_1+52c_2+9)J_n + \frac{1}{3}(c_1+9c_2)J_{n-1} + \frac{2}{9}(3c_0+9c_1+31c_2)J_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{3}(c_1+6c_2), \\ A_0 &= -\frac{1}{9}(3c_0+9c_1+31c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,1) = \frac{1}{9}(3c_0 + 12c_1 + 52c_2 + 258c_3 + 9)J_n + \frac{1}{3}(c_1 + 9c_2 + 65c_3)J_{n-1} + \frac{2}{9}(3c_0 + 9c_1 + 31c_2 + 135c_3)J_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{3}c_3, \\ A_2 &= -\frac{1}{3}(c_2 + 9c_3), \\ A_1 &= -\frac{1}{3}(c_1 + 6c_2 + 31c_3), \\ A_0 &= -\frac{1}{9}(3c_0 + 9c_1 + 31c_2 + 135c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 1,\ a_3 = 2, \\ W_0 &= 3,\ W_1 = 1,\ W_2 = 3, \end{aligned}$$

so that we have the case $V_n = K_n$ (modified third-order Jacobsthal numbers).

Example 2.2. *In this example, we consider the case $a_1 = 1$, $a_2 = 1$, $a_3 = 2, W_0 = 3$, $W_1 = 1$, $W_2 = 3$ in Theorem 1.1 (a). So $V_n(3,1,3) = V_n$ represents nth modified third-order Jacobstha number, i.e., $V_n = K_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(3,1,3) = \frac{1}{147}\left(17c_0 + 147\right)K_n + \frac{10}{147}c_0K_{n-1} - \frac{4}{147}c_0K_{n-2} - \frac{1}{3}c_0$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_1n + c_0$$

is given by

$$W_n(3,1,3) = \frac{1}{441}\left(51c_0 + 176c_1 + 441\right)K_n + \frac{1}{147}\left(10c_0 + 47c_1\right)K_{n-1} - \frac{2}{441}\left(6c_0 - 11c_1\right)K_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{3}c_1, \\ A_0 &= -\frac{1}{3}(c_0+3c_1). \end{aligned}$$

(c): *The case* $s=2$*). The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,1,3) = \frac{1}{441}(51c_0+176c_1+708c_2+441)K_n + \frac{1}{441}(30c_0+141c_1+673c_2)K_{n-1} - \frac{2}{441}(6c_0 - 11c_1 - 179c_2)K_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{3}(c_1+6c_2), \\ A_0 &= -\frac{1}{9}(3c_0+9c_1+31c_2). \end{aligned}$$

(d): *The case* $s=3$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,1,3) = \frac{1}{441}(51c_0 + 176c_1 + 708c_2 + 3478c_3 + 441)K_n + \frac{1}{441}(30c_0 + 141c_1 + 673c_2 + 3513c_3)K_{n-1} - \frac{2}{441}(6c_0 - 11c_1 - 179c_2 - 1375c_3)K_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{3}c_3, \\ A_2 &= -\frac{1}{3}(c_2+9c_3), \\ A_1 &= -\frac{1}{3}(c_1+6c_2+31c_3), \\ A_0 &= -\frac{1}{9}(3c_0+9c_1+31c_2+135c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 1,\ a_3 = 2, \\ W_0 &= 2,\ W_1 = 1,\ W_2 = 5, \end{aligned}$$

so that we have the case $V_n = j_n$ (third-order Jacobsthal-Lucas numbers).

Example 2.3. *In this example, we consider the case* $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $W_0 = 2$, $W_1 = 1$, $W_2 = 5$ *in Theorem 1.1 (a). So* $V_n(2,1,5) = V_n$ *represents nth third-order Jacobsthal-Luca number, i.e.,* $V_n = j_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(2,1,5) = \frac{1}{18}\left(c_0 + 18\right) j_n + \frac{1}{6} c_0 j_{n-1} - \frac{1}{18} c_0 j_{n-2} - \frac{1}{3} c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(2,1,5) = \frac{1}{18}(c_0 + 4c_1 + 18) j_n + \frac{1}{18}(3c_0 + 10c_1) j_{n-1} - \frac{1}{18} c_0 j_{n-2} + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{3} c_1, \\ A_0 &= -\frac{1}{3}(c_0 + 3c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,1,5) = \frac{1}{54}(3c_0 + 12c_1 + 58c_2 + 54) j_n + \frac{1}{18}(3c_0 + 10c_1 + 36c_2) j_{n-1} - \frac{1}{54}(3c_0 - 38c_2) j_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{3} c_2, \\ A_1 &= -\frac{1}{3}(c_1 + 6c_2), \\ A_0 &= -\frac{1}{9}(3c_0 + 9c_1 + 31c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,1,5) = \frac{1}{54}(3c_0+12c_1+58c_2+330c_3+54)j_n + \frac{1}{18}(3c_0+10c_1+36c_2+152c_3)j_{n-1} - \frac{1}{54}(3c_0 - 38c_2 - 306c_3)j_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{3}c_3, \\ A_2 &= -\frac{1}{3}(c_2+9c_3), \\ A_1 &= -\frac{1}{3}(c_1+6c_2+31c_3), \\ A_0 &= -\frac{1}{9}(3c_0+9c_1+31c_2+135c_3). \end{aligned}$$

Generalized Third-Order Jacobsthal Numbers: Special Initial Values and Input Polynomials. Jacobsthal-Type Families. We now turn to explicit applications in the Jacobsthal family. In the examples below, we do not treat $p(n)$ in its general form but instead select specific coefficients c_i to construct concrete forcing terms of degrees $s = 0,1,2,3$, i.e., in each example, we select specific coefficients c_i in the input polynomial

$$p(n) = \sum_{i=0}^{s} c_i n^i, \qquad c_i \in \mathbb{C}, \text{ or } c_i \in \mathbb{C}[x]$$

to construct concrete forcing terms of degrees $s = 0,1,2,3$. This approach highlights how the closed-form solutions adapt to constant, linear, quadratic, and cubic inputs. Moreover, for each case, the recurrence coefficients are fixed as $a_1 = 1, a_2 = 1, a_3 = 2$ while the choice of initial values determines the particular sequence under consideration:

$$(W_0, W_1, W_2) = (0,1,1) \quad \text{third-order Jacobsthal numbers,}$$

$$(W_0, W_1, W_2) = (3,1,3) \quad \text{modified third-order Jacobsthal numbers,}$$

$$(W_0, W_1, W_2) = (2,1,5) \quad \text{third-order Jacobsthal–Lucas numbers.}$$

Each of these cases illustrates how Theorem 1.1 yields explicit closed-form representations once the initial conditions and polynomial inputs are fixed. The subsequent examples (Examples (2.4)-(2.6)) provide comprehensive demonstrations of the closed-form solutions obtained for polynomial inputs of degrees $s = 0,1,2,3$ across each sequence in the family.

We now illustrate the general framework with three distinguished members of the Jacobsthal family.

Third-order Jacobsthal numbers. For the choice $(W_0, W_1, W_2) = (0,1,1)$, the recurrence generates the classical third-order Jacobsthal sequence. The next example records closed-form solutions corresponding to polynomial inputs of degrees $s = 0,1,2,3$.

Example 2.4. *In this example, we consider the case* $a_1 = 1, a_2 = 1, a_3 = 2$, $W_0 = 0, W_1 = 1, W_2 = 1$ *with specific input polynomials of degree* $s = 0, 1, 2, 3$ *(see Example 2.1).*

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 3$$

is given by

$$W_n(0,1,1) = 2J_n + 2J_{n-2} - 1.$$

(b): *The case* $s = 1$. *For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 4n - 5,$$

the closed-form solution is

$$W_n(0,1,1) = \frac{14}{3}J_n + \frac{4}{3}J_{n-1} + \frac{14}{3}J_{n-2} - \frac{4}{3}n - \frac{7}{3}.$$

(c): *The case* $s = 2$. *For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} - 2n^2 - n + 6,$$

the closed-form solution is

$$W_n(0,1,1) = -\frac{89}{9}J_n - \frac{19}{3}J_{n-1} - \frac{106}{9}J_{n-2} + \frac{2}{3}n^2 + \frac{13}{3}n + \frac{53}{9}.$$

(d): *The case* $s = 3$. *For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 7n^3 - n^2 + 4n - 3,$$

the closed-form solution is

$$W_n(0,1,1) = \frac{1802}{9}J_n + 150J_{n-1} + \frac{1882}{9}J_{n-2} - \frac{7}{3}n^3 - \frac{62}{3}n^2 - \frac{215}{3}n - \frac{941}{9}.$$

Modified third-order Jacobsthal numbers. When $(W_0, W_1, W_2) = (3, 1, 3)$, the recurrence produces the modified third-order Jacobsthal sequence. The next example records closed-form solutions for polynomial inputs of degrees $s = 0, 1, 2, 3$.

Example 2.5. *In this example, we consider the case* $a_1 = 1, a_2 = 1, a_3 = 2$, $W_0 = 3, W_1 = 1, W_2 = 3$ *with specific input polynomials of degree* $s = 0, 1, 2, 3$ *(see Example 2.2).*

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 3$$

is given by

$$W_n(3,1,3) = \frac{66}{49}K_n + \frac{10}{49}K_{n-1} - \frac{4}{49}K_{n-2} - 1.$$

(b): *The case $s = 1$. For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 4n - 5,$$

the closed-form solution is

$$W_n(3,1,3) = \frac{890}{441}K_n + \frac{46}{49}K_{n-1} + \frac{148}{441}K_{n-2} - \frac{4}{3}n - \frac{7}{3}.$$

(c): *The case $s = 2$). For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} - 2n^2 - n + 6,$$

the closed-form solution is

$$W_n(3,1,3) = -\frac{845}{441}K_n - \frac{1307}{441}K_{n-1} - \frac{90}{49}K_{n-2} + \frac{2}{3}n^2 + \frac{13}{3}n + \frac{53}{9}.$$

(d): *The case $s = 3$. For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 7n^3 - n^2 + 4n - 3$$

the closed-form solution is

$$W_n(3,1,3) = \frac{8210}{147}K_n + \frac{24392}{441}K_{n-1} + \frac{19016}{441}K_{n-2} - \frac{7}{3}n^3 - \frac{62}{3}n^2 - \frac{215}{3}n - \frac{941}{9}.$$

Third-order Jacobsthal–Lucas numbers. When $(W_0, W_1, W_2) = (2, 1, 5)$, the recurrence generates the third-order Jacobsthal–Lucas sequence. The following example presents closed-form solutions for polynomial inputs of degrees $s = 0, 1, 2, 3$.

Example 2.6. *In this example, we consider the case $a_1 = 1, a_2 = 1, a_3 = 2$, $W_0 = 2, W_1 = 1, W_2 = 5$ with specific input polynomials of degree $s = 0, 1, 2, 3$ (see Example 2.3).*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 3$$

is given by

$$W_n(2,1,5) = \frac{7}{6}j_n + \frac{1}{2}j_{n-1} - \frac{1}{6}j_{n-2} - 1.$$

(b): *The case $s = 1$. For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 4n - 5,$$

the closed-form solution is

$$W_n(2,1,5) = \frac{29}{18}j_n + \frac{25}{18}j_{n-1} + \frac{5}{18}j_{n-2} - \frac{4}{3}n - \frac{7}{3}.$$

(c): *The case* $s=2$. *For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} - 2n^2 - n + 6,$$

the closed-form solution is

$$W_n(2,1,5) = -\frac{28}{27}j_n - \frac{32}{9}j_{n-1} - \frac{47}{27}j_{n-2} + \frac{2}{3}n^2 + \frac{13}{3}n + \frac{53}{9}.$$

(d): *The case* $s=3$. *For*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + 7n^3 - n^2 + 4n - 3,$$

the closed-form solution is

$$W_n(2,1,5) = \frac{2345}{54}j_n + \frac{353}{6}j_{n-1} + \frac{2113}{54}j_{n-2} - \frac{7}{3}n^3 - \frac{62}{3}n^2 - \frac{215}{3}n - \frac{941}{9}.$$

2.2. Generalized Graham Numbers. In this subsection, we consider the case $a_1 = 2$, $a_2 = 3$ and $a_3 = 5$. A generalized Graham sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} \tag{2.9}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{3}{5}V_{-(n-1)} - \frac{2}{5}V_{-(n-2)} + \frac{1}{5}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.9) holds for all integer n. For more information on generalized Graham numbers, see Soykan [19].

Binet formula of generalized Graham numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{2.10}$$

Here, α, β and γ are the roots of the cubic equation $z^3 - 2z^2 - 3z - 5 = 0$. Moreover

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega^2\left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega\left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Now we define two special cases of the sequence $\{V_n\}$. Graham sequence $\{G_n\}_{n\geq 0}$ and Graham-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = 2G_{n+2} + 3G_{n+1} + 5G_n, \qquad G_0 = 0, G_1 = 1, G_2 = 2, \tag{2.11}$$

$$H_{n+3} = 2H_{n+2} + 3H_{n+1} + 5H_n, \qquad H_0 = 3, H_1 = 2, H_2 = 10. \tag{2.12}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{3}{5}G_{-(n-1)} - \frac{2}{5}G_{-(n-2)} + \frac{1}{5}G_{-(n-3)}, \\ H_{-n} &= -\frac{3}{5}H_{-(n-1)} - \frac{2}{5}H_{-(n-2)} + \frac{1}{5}H_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.11) and (2.12) hold for all integer n.For all integers n, Graham and Graham-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 2,\ a_2 = 3,\ a_3 = 5, \\ W_0 &= 0,\ W_1 = 1,\ W_2 = 2, \end{aligned}$$

so that we have the case $V_n = G_n$ (Graham numbers).

EXAMPLE 2.7. *In this example, we consider the case $a_1 = 2,\ a_2 = 3,\ a_3 = 5, W_0 = 0,\ W_1 = 1,\ W_2 = 2$ in Theorem 1.1 (a). So $V_n(0,1,2) = V_n$ represents nth Graham number, i.e., $V_n = G_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_0$$

is given by

$$W_n(0,1,2) = \frac{1}{9}(c_0+9)G_n - \frac{1}{9}c_0G_{n-1} + \frac{5}{9}c_0G_{n-2} - \frac{1}{9}c_0.$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_1n + c_0$$

is given by

$$W_n(0,1,2)=\frac{1}{81}(9c_0+32c_1+81)G_n-\frac{1}{81}(9c_0+23c_1)G_{n-1}+\frac{5}{81}(9c_0+23c_1)G_{n-2}+A_1n+A_0$$

where

$$\begin{aligned}A_1 &= -\frac{1}{9}c_1,\\ A_0 &= -\frac{1}{81}(9c_0+23c_1).\end{aligned}$$

(c): *The case $s=2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n=2W_{n-1}+3W_{n-2}+5W_{n-3}+c_2n^2+c_1n+c_0$$

is given by

$W_n(0,1,2)=\frac{1}{729}(81c_0+288c_1+1022c_2+729)G_n-\frac{1}{729}(81c_0+207c_1+365c_2)G_{n-1}+\frac{5}{729}(81c_0+207c_1+527c_2)G_{n-2}+A_2n^2+A_1n+A_0$

where

$$\begin{aligned}A_2 &= -\frac{1}{9}c_2,\\ A_1 &= -\frac{1}{81}(9c_1+46c_2),\\ A_0 &= -\frac{1}{729}(81c_0+207c_1+527c_2).\end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n=2W_{n-1}+3W_{n-2}+5W_{n-3}+c_3n^3+c_2n^2+c_1n+c_0$$

is given by

$W_n(0,1,2)=\frac{1}{2187}(243c_0+864c_1+3066c_2+11104c_3+2187)G_n-\frac{1}{2187}(243c_0+621c_1+1095c_2-929c_3)G_{n-1}+\frac{5}{2187}(243c_0+621c_1+1581c_2+4255c_3)G_{n-2}+A_3n^3+A_2n^2+A_1n+A_0$

where

$$\begin{aligned}A_3 &= -\frac{1}{9}c_3,\\ A_2 &= -\frac{1}{27}(3c_2+23c_3),\\ A_1 &= -\frac{1}{243}(27c_1+138c_2+527c_3),\\ A_0 &= -\frac{1}{2187}(243c_0+621c_1+1581c_2+4255c_3).\end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 2,\ a_2 = 3,\ a_3 = 5, \\ W_0 &= 3,\ W_1 = 2,\ W_2 = 10, \end{aligned}$$

so that we have the case $V_n = H_n$ (Graham Lucas numbers).

Example 2.8. *In this example, we consider the case $a_1 = 2,\ a_2 = 3,\ a_3 = 5, W_0 = 3,\ W_1 = 2,\ W_2 = 10$ in Theorem 1.1 (a). So $V_n(3,2,10) = V_n$ represents nth Graham Lucas number, i.e., $V_n = H_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_0$$

is given by

$$W_n(3,2,10) = \frac{1}{11079}\left(320c_0 + 11079\right) H_n + \frac{58}{11079} c_0 H_{n-1} - \frac{695}{11079} c_0 H_{n-2} - \frac{1}{9} c_0.$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_1 n + c_0$$

is given by

$W_n(3,2,10) = \frac{1}{99711}(2880c_0 + 8323c_1 + 99711)H_n + \frac{1}{99711}(522c_0 + 4763c_1)H_{n-1} - \frac{5}{99711}(1251c_0 + 2819c_1)H_{n-2} + A_1 n + A_0$

where

$$\begin{aligned} A_1 &= -\frac{1}{9}c_1, \\ A_0 &= -\frac{1}{81}(9c_0 + 23c_1). \end{aligned}$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,2,10) = \frac{1}{897399}(25920c_0 + 74\,907c_1 + 228409c_2 + 897399)H_n + \frac{1}{897399}(4698c_0 + 42867c_1 + 222887c_2)H_{n-1} - \frac{5}{897399}(11259c_0 + 25371c_1 + 44201c_2)H_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{9}c_2, \\ A_1 &= -\frac{1}{81}(9c_1 + 46c_2), \\ A_0 &= -\frac{1}{729}(81c_0 + 207c_1 + 527c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,2,10) = \frac{1}{2692197}(77760c_0+224721c_1+685227c_2+2312171c_3+2692197)H_n + \frac{1}{2692197}(14094c_0+128601c_1+668661c_2+2975491c_3)H_{n-1} - \frac{5}{2692197}(33777c_0+76113c_1+132603c_2+39403c_3)H_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{9}c_3, \\ A_2 &= -\frac{1}{27}(3c_2 + 23c_3), \\ A_1 &= -\frac{1}{243}(27c_1 + 138c_2 + 527c_3), \\ A_0 &= -\frac{1}{2187}(243c_0 + 621c_1 + 1581c_2 + 4255c_3). \end{aligned}$$

2.3. Generalized Jacobsthal-Narayana Numbers. In this subsection, we consider the case $a_1 = 1$, $a_2 = 0$, $a_3 = 2$. The generalized Jacobsthal-Narayana numbers

$$\{V_n(V_0, V_1, V_2; 1, 0, 2)\}_{n\geq 0}$$

(or $\{V_n\}_{n\geq 0}$ or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n = V_{n-1} + 2V_{n-3}, \qquad V_0 = a, V_1 = b, V_2 = c, \quad n \geq 3 \tag{2.13}$$

where V_0, V_1, V_2 are arbitrary complex (or real) numbers.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2.13) holds for all integers n. For more information on generalized Jacobsthal-Narayana numbers, see Soykan [20].

For all integers n, Binet's formula of generalized Jacobsthal-Narayana numbers is given as follows.

$$\begin{aligned} V_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= A_1\alpha^n + A_2\beta^n + A_3\gamma^n, \end{aligned}$$

where

$$\begin{aligned}
p_1 &= V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0, \\
p_2 &= V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, \\
p_3 &= V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0}{(\alpha-\beta)(\alpha-\gamma)} \\
&= \frac{(\alpha V_2 + \alpha(-1+\alpha)V_1 + 2V_0)}{\alpha^2+6}, \\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0}{(\beta-\alpha)(\beta-\gamma)} \\
&= \frac{(\beta V_2 + \beta(-1+\beta)V_1 + 2V_0)}{\beta^2+6}, \\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{(\gamma V_2 + \gamma(-1+\gamma)V_1 + 2V_0)}{\gamma^2+6}.
\end{aligned}$$

Here, α, β and γ are the roots of the cubic equation

$$z^3 - z^2 - 2 = (z-\alpha)(z-\beta)(z-\gamma) = 0.$$

Moreover,

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{28}{27} + \sqrt{\frac{29}{27}}\right)^{1/3} + \left(\frac{28}{27} - \sqrt{\frac{29}{27}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{28}{27} + \sqrt{\frac{29}{27}}\right)^{1/3} + \omega^2\left(\frac{28}{27} - \sqrt{\frac{29}{27}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{28}{27} + \sqrt{\frac{29}{27}}\right)^{1/3} + \omega\left(\frac{28}{27} - \sqrt{\frac{29}{27}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\left\{\begin{array}{c}
\alpha+\beta+\gamma = 1, \\
\alpha\beta+\alpha\gamma+\beta\gamma = 0, \\
\alpha\beta\gamma = 2.
\end{array}\right.$$

Now, we define two special cases of the sequence $\{V_n\}$. Jacobsthal-Narayana sequence$\{B_n\}_{n\geq 0}$, Jacobsthal-Narayana-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$B_n = B_{n-1}+2B_{n-3}, \qquad B_0=0, B_1=1, B_2=1, \tag{2.14}$$

$$C_n = C_{n-1}+2C_{n-3}, \qquad C_0=3, C_1=1, C_2=1. \tag{2.15}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)}, \\ C_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (2.14)-(2.15) hold for all integer n.

For all integers n, Binet's formulas of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are

$$\begin{aligned} B_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{\alpha^{n+2}}{\alpha^2+6}+\frac{\beta^{n+2}}{\beta^2+6}+\frac{\gamma^{n+2}}{\gamma^2+6}, \\ C_n &= \alpha^n+\beta^n+\gamma^n, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2=0,\ a_3=2, \\ W_0 &= 0,\ W_1=1,\ W_2=1, \end{aligned}$$

so that we have the case $V_n = B_n$ (Jacobsthal-Narayana numbers).

Example 2.9. *In this example, we consider the case $a_1=1,\ a_2=0,\ a_3=2, W_0=0,\ W_1=1,\ W_2=1$ in Theorem 1.1 (a). So $V_n(0,1,1)=V_n$ represents nth Jacobsthal-Narayana number, i.e., $V_n=B_n$.*

(a): *The case $s=0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1}+2W_{n-3}+c_0$$

is given by

$$W_n(0,1,1)=\frac{1}{2}\left(c_0+2\right)B_n+c_0B_{n-2}-\frac{1}{2}c_0.$$

(b): *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1}+2W_{n-3}+c_1n+c_0$$

is given by

$$W_n(0,1,1) = \frac{1}{4}(2c_0 + 9c_1 + 4)B_n + \frac{1}{2}c_1 B_{n-1} + \frac{1}{2}(2c_0 + 7c_1)B_{n-2} + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1). \end{aligned}$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(0,1,1) = \frac{1}{4}(2c_0 + 9c_1 + 46c_2 + 4)B_n + \frac{1}{2}(c_1 + 10c_2)B_{n-1} + \frac{1}{2}(2c_0 + 7c_1 + 30c_2)B_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1 + 30c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(0,1,1) = \dfrac{1}{8}(4c_0 + 18c_1 + 92c_2 + 567c_3 + 8)B_n + \dfrac{1}{4}(2c_1 + 20c_2 + 167c_3)B_{n-1} + \dfrac{1}{4}(4c_0 + 14c_1 + 60c_2 + 341c_3)B_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2 + 21c_3), \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2 + 45c_3), \\ A_0 &= -\frac{1}{8}(4c_0 + 14c_1 + 60c_2 + 341c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 0,\ a_3 = 2, \\ W_0 &= 3,\ W_1 = 1,\ W_2 = 1, \end{aligned}$$

so that we have the case $V_n = C_n$ (Jacobsthal-Narayana-Lucas numbers).

Example 2.10. *In this example, we consider the case $a_1 = 1,\ a_2 = 0,\ a_3 = 2, W_0 = 3,\ W_1 = 1,\ W_2 = 1$ in Theorem 1.1 (a). So $V_n(3,1,1) = V_n$ represents nth Jacobsthal-Narayana-Lucas number, i.e., $V_n = C_n$.*

- **(a):** *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{58}(11c_0 + 58)C_n + \frac{3}{29}c_0C_{n-1} + \frac{2}{29}c_0C_{n-2} - \frac{1}{2}c_0.$$

- **(b):** *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_1n + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{116}(22c_0 + 87c_1 + 116)C_n + \frac{1}{58}(6c_0 + 29c_1)C_{n-1} + \frac{1}{58}(4c_0 + 29c_1)C_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1). \end{aligned}$$

- **(c):** *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{116}(22c_0+87c_1+422c_2+116)C_n + \frac{1}{58}(6c_0+29c_1+152c_2)C_{n-1} + \frac{1}{58}(4c_0+29c_1+198c_2)C_{n-2} + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1+7c_2), \\ A_0 &= -\frac{1}{4}(2c_0+7c_1+30c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{232}(44c_0+174c_1+844c_2+5205c_3+232)C_n + \frac{1}{116}(12c_0+58c_1+304c_2+1873c_3)C_{n-1} + \frac{1}{116}(8c_0+58c_1+396c_2+2863c_3)C_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2+21c_3), \\ A_1 &= -\frac{1}{2}(c_1+7c_2+45c_3), \\ A_0 &= -\frac{1}{8}(4c_0+14c_1+60c_2+341c_3). \end{aligned}$$

Conclusion

This work has developed explicit closed-form formulas for third-order nonhomogeneous linear recurrence relations, framed as generalized Leonardo-type sequences with polynomial inputs. The analysis was organized by considering the multiplicity $r=0,1,2,3$ of 1 as a root of the characteristic equation, and for each multiplicity solutions were derived for polynomial inputs of degree $s=0,1,2,3$. These results clarify how the interplay between root multiplicity and polynomial degree shapes the particular solution, while the homogeneous component remains governed by the same recurrence relation. In the examples, attention was focused on the non-resonant case $r=0$, where all three roots are distinct from 1, thereby illustrating how classical integer sequences emerge naturally as homogeneous counterparts within the unified framework.

The derived formulas were applied to three principal families: the generalized third-order Jacobsthal, generalized Graham, and generalized Jacobsthal–Narayana sequences. Within the Jacobsthal family, specializations corresponding to $(W_0,W_1,W_2)=(0,1,1)$, $(3,1,3)$, and $(2,1,5)$ produced the classical third-order Jacobsthal, modified Jacobsthal, and Jacobsthal–Lucas numbers. For each of these cases, explicit closed-form solutions were obtained under polynomial inputs of degrees $s=0,1,2,3$, with specific coefficients c_i chosen to construct constant, linear, quadratic, and cubic forcing terms. These examples demonstrate how the general theorem adapts seamlessly to both generalized families and their classical special cases.

Beyond theoretical interest, the results carry pedagogical and practical value. For teaching, they provide accessible examples that allow students to explore nonhomogeneous recurrences with polynomial inputs of varying degrees without excessive computation. For research, they supply resonance-aware formulas and explicit derivations that can be adapted to problems in mathematics, computer science, engineering, and physics. In this way, the study strengthens both instructional practice and applied investigation, broadening the reach of recurrence theory across multiple domains.

The unified framework consolidates recurrence theory while opening pathways for innovation in modern applications. By presenting resonance-aware formulas and explicit derivations, the method can be adapted to optimization problems in transportation, coding theory, miniaturization technologies, and discrete modeling. Closed-form solutions are particularly valuable because they provide exact predictability for complex systems, eliminating the computational costs of iterative evaluation. The framework developed here extends recurrence theory, clarifies resonance phenomena, and offers tools that are highly relevant for algorithm analysis, numerical modeling, and optimization problems where efficiency and precision are essential.

At the same time, the iterative scheme introduced in this paper yields explicit polynomial-type particular solutions, but its scope is currently limited to polynomial inputs. Extending the framework to accommodate non-polynomial functions—such as trigonometric or other analytic inputs—would require further methodological refinements and may introduce new resonance phenomena. Moreover, while the approach successfully clarifies the role of root multiplicity and resonance corrections, the computational complexity increases rapidly with higher-order recurrences. This suggests that practical implementation for advanced cases will benefit from integration with computer algebra systems or symbolic computation platforms. In this respect, the current framework provides a solid foundation while pointing toward natural directions for further generalization and computational support.

Finally, the methodology opens promising avenues for future research. It can be incorporated into symbolic computation platforms, applied to verify classical identities in recurrence theory, and extended to interdisciplinary contexts such as cryptography, coding theory, and discrete modeling in physics and biology. By acknowledging current limitations while outlining directions for extension, the study offers a balanced perspective: consolidating the contribution of the present results while pointing toward further exploration and development.

References

[1] Abd-Elhameed, W.M., Alqubori, O.M., Alluhaybi, A.A., Amin, A.K., Novel Expressions for Certain Generalized Leonardo Polynomials and Their Associated Numbers, Axioms, 14, 286, 2025. https://doi.org/10.3390/

[2] Catarino, P., Borges, A., On Leonardo Numbers, Acta Mathematica Universitatis Comenianae, 89(1), 75–86, 2020. Available online at: http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1005/650.

[3] Cerda-Morales, G., A Note On Modified Third-Order Jacobsthal Numbers, arXiv:1905.00725v1, 2019.

[4] Cook C. K., Bacon, M. R., Some Identities for Jacobsthal and Jacobsthal-Lucas Numbers Satisfying Higher Order Recurrence Relations, Annales Mathematicae et Informaticae, 41, 27–39, 2013.

[5] Dikmen, C.D., Properties of Gaussian Generalized Leonardo Numbers, Karaelmas Science and Engineering Journal, 15(1), 134-145, 2025. DOI: 10.7212/karaelmasfen.1578154

[6] Gökbaş, H., k-Leonardo Numbers, Palestine Journal of Mathematics, 13(4), 1427-1435, 2024.

[7] İşbilir, Z., Akyiğit, M., Tosun, M., Pauli–Leonardo Quaternions, Notes on Number Theory and Discrete Mathematics, 29(1), 1-16, 2023. DOI: 10.7546/nntdm.2023.29.1.1-16

[8] Jeske, J.A., Linear Recurrence Relations, Part I, The Fibonacci Quarterly, 1(2), 69-74, 1963.

[9] Jeske, J.A., Linear Recurrence Relations, Part II, The Fibonacci Quarterly, 1(4), 34-39, 1963.

[10] Jeske, J.A., Linear Recurrence Relations, Part III, The Fibonacci Quarterly, 2(3), 197-203, 1964.

[11] Kuhapatanakul, K., Chobsorn, J, On the Generalized Leonardo Numbers, Integers 22, 2022, #A48.

[12] Kuhapatanakul, K., Ruankong, P., On Generalized Leonardo p-numbers, Journal of Integer Sequences, 27, Article 24.4.6, 2024.

[13] Polatlı, E.E., Soykan, Y., On Generalized Third-Order Jacobsthal Numbers, Asian Research Journal of Mathematics, 17(2), 1-19, 2021. DOI: 10.9734/ARJOM/2021/v17i230270

[14] Prasad, K., Kumari, M., The Leonardo Polynomials and Their Algebraic Properties. Proceedings of the Indian National Science Academy, 2024. https://doi.org/10.1007/s43538-024-00348-0

[15] Shannon, A.G., A Note On Generalized Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 25(3), 97–101, 2019. DOI: 10.7546/nntdm.2019.25.3.97-101

[16] Shannon, A.G., Deveci, Ö., A Note on Generalized and Extended Leonardo Sequences, Notes on Number Theory and Discrete Mathematics, 28(1), 109–114, 2022. DOI: 10.7546/nntdm.2022.28.1.109-114

[17] Shannon, A.G., Shiue, P.J.S., Huang, S.C., Notes on Generalized and Extended Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 29(4), 752–773, 2023. DOI: 10.7546/nntdm.2023.29.4.752-773

[18] Sloane, N.J.A., The On-Line Encyclopedia of Integer Sequences, http://oeis.org/

[19] Soykan, Y., On Generalized Grahaml Numbers, Journal of Advances in Mathematics and Computer Science, 35(2), 42-57, 2020. DOI: 10.9734/JAMCS/2020/v35i230248.

[20] Soykan Y., Generalized Jacobsthal-Narayana Numbers and Generalized Co-Jacobsthal-Narayana Numbers, Asian Journal of Advanced Research and Reports, 19(5), 9-55, 2025. https://doi.org/10.9734/ajarr/2025/v19i5998

[21] Soykan, Y., Generalized Horadam-Leonardo Numbers and Polynomials, Asian Journal of Advanced Research and Reports, 17(8), 128-169, 2023. https://doi.org/10.9734/ajarr/2023/v17i8511

[22] Soykan, Y., Interrelations between Horadam and Generalized Horadam-Leonardo Polynomials via Identities, International Journal of Advances in Applied Mathematics and Mechanics, 11(1), 42-55, 2023. ISSN: 2347-2529

[23] Soykan, Y., Generalized Leonardo Numbers, Journal of Progressive Research in Mathematics, 18(4), 58-84, 2021.

[24] Soykan, Y., An Extensive Study on Generalized Leonardo Numbers and Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 13(3), 51–250, 2026.

[25] Soykan, Y., Leonardo Polynomials and Numbers: Solutions, Linearizations, and Generating Functions, International Journal of Advances in Applied Mathematics and Mechanics, 13(4), 80-222, 2026. https://doi.org/10.26541/ijaamm.2026.130408

[26] Soykan, Y., Explicit Formulas for Polynomial-type Particular Solutions of Generalized Leonardo-type Sequences, Asian Journal of Advanced Research and Reports, 20(5), 164-208, 2026. https://doi.org/10.9734/ajarr/2026/v20i51361

[27] Özimamoğlu, H., On Leonardo Sedenions, Afrika Matematika (2023) 34:26, 2023. https://doi.org/10.1007/s13370-023-01065-5

[28] Özkan, E., Akkuş, H., Generalized Bronze Leonardo Sequence, Notes on Number Theory and Discrete Mathematics, 30(4), 811-824, 2024. DOI: 10.7546/nntdm.2024.30.4.811-824