# Closed-Form Expressions of Leonardo-Type Sequences with Homogeneous Counterparts in Balancing and Oresme Numbers

**Abstract.** In this paper, we establish closed-form solutions for second-order nonhomogeneous linear recurrence relations, identified as generalized Leonardo-type sequences, where the input term is expressed as a polynomial. Our investigation highlights several distinguished cases of generalized Fibonacci numbers, which emerge as the homogeneous analogues of the original nonhomogeneous relations. Specifically, we examine classical families including the balancing numbers, balancing–Lucas numbers, Oresme numbers, and Oresme–Lucas numbers.

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## 1. Introduction

Recurrence relations have long stood at the foundation of mathematics, extending their influence into physics, engineering, architecture, biology, computer science, and even artistic inquiry. Though their definitions may appear elementary, they conceal profound depth, modeling growth processes, oscillatory behavior, and symbolic structures. Classical second-order families—such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences—remain enduring exemplars of this tradition.

Yet the framework extends far beyond second-order constructions. Higher-order recurrence sequences enrich both theory and application, broadening the classical foundation while unveiling intricate algebraic and analytic structures. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences illustrate this expansion, each governed by characteristic polynomials whose root configurations determine closed-form solutions. Homogeneous recurrences emphasize the relationship between characteristic polynomials and root multiplicities, while nonhomogeneous forms introduce symbolic terms that interact with root structures to generate resonance phenomena. Collectively, these families establish a coherent framework that unites classical recurrence identities with the evolving discipline of symbolic recurrence theory.

This paper seeks to illuminate the architecture of recurrence theory by constructing closed-form solutions for second-order nonhomogeneous linear relations, conceived as generalized Leonardo-type sequences with polynomial inputs. Within this structural edifice, generalized Fibonacci numbers emerge as the harmonious echoes of their nonhomogeneous origins, serving as the homogeneous analogues that mirror the underlying design. Our inquiry threads together venerable families—the the balancing and balancing–Lucas sequences, alongside the Oresme and Oresme–Lucas sequences-each contributing a distinct motif to the symphonic fabric of higher-order recurrence phenomena.

Let the second order nonhomogeneous linear recurrence relation be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + p(n) \tag{1.1}$$

with initial conditions $W_0 = u_0, W_1 = u_1$ where $p(n)$ is the polynomial with degree $s$, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i$$

and the recurrence coefficients $a_1, a_2$ are complex scalars or polynomials in $\mathbb{C}[x]$.

A sequence $\{W_n\}_{n\geq 0}$ defined by (1.1) is called a generalized Leonardo-type sequence with input function $p(n)$. For more information on generalized Leonardo-type sequences, see Soykan [4] and [3].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} \tag{1.2}$$

with the same initial conditions as $W_n$, i.e.,

$$V_0 = W_0, V_1 = W_1.$$

Suppose that $\theta_1$ and $\theta_2$ are the roots of the characteristic equation of (1.2)

$$z^2 - a_1 z - a_2 = 0, \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^2 - a_1 z - a_2 = (z-1)^2 = z^2 - 2z + 1 = 0$$

so that $a_1 = 2,\ a_2 = -1$ and (1.2) reduces to

$$V_n = 2V_{n-1} - V_{n-2}. \tag{1.4}$$

In earlier work, particular solutions of second-order nonhomogeneous recurrences (1.1) with polynomial inputs were obtained for $s = 0, 1, 2, 3, 4, 5, 6, 7$; see Soykan [5]. The present paper advances those results by deriving unified closed-form solutions through Theorem 1.1. Each recurrence is expressed as the sum of its homogeneous and particular components, with the particular solution determined iteratively. The framework is organized by the multiplicity $r$ of 1 as a root of the characteristic equation (1.3) and the polynomial degree $s$. Closed-form formulas are obtained for all values $r = 0, 1, 2$ and $s = 0, 1, 2, 3, 4, 5, 6, 7$, illustrating how these parameters jointly shape the solution and how resonance arises when roots coincide. In practice, however, our examples–generalized balancing, and Oresme sequences-require only the non-resonant case $r = 0$. Accordingly, the theorem is presented in detail for $r = 0$, while the cases $r = 1, 2$ are acknowledged but omitted as they are not needed here.

Theorem 1.1.

**(a):** *[3, Theorem 7.6. (a)] The case $r = 0$, i.e., all two roots of the characteristic equation of (1.3) is distinct from 1. The solution of (1.1) is given by*

$$\begin{aligned} W_n(W_0, W_1) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)}) + W_n^{(p)} \end{aligned}$$

*where $W_n^{(h)} = V_n(W_0, W_1) - V_n(W_0^{(p)}, W_1^{(p)})$ is the solution of (1.2) and*

$$W_n^{(p)} = \sum_{i=0}^{s} A_i n^i = A_0 + \sum_{i=1}^{s} A_i n^i$$

*is the particular solution of (1.1). For each $0 \le i \le s$, $A_i$ can be calculated with the iteration*

$$A_s = -\frac{c_s}{a_1 + a_2 - 1}, \text{ for } n = s$$

*and*

$$A_n = -\frac{1}{a_1 + a_2 - 1}\Big(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+1} \binom{k}{n} (a_1 + 2^{k-n} a_2) A_k\Big), \text{ for } n = s-1, s-2, ..., 2, 1, 0.$$

*Here*

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \end{aligned}$$

*and*

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}) &= V_n(A_0, \sum_{i=0}^{s} A_i) \\ &= \frac{(W_1 - a_1 W_0)\sum_{i=0}^{s} A_i - a_2 W_0 A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_n(W_0, W_1) + \frac{-a_2(W_0 \sum_{i=0}^{s} A_i - A_0 W_1)}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n-1}(W_0, W_1) \\ &= -\frac{W_0 \sum_{i=0}^{s} A_i - A_0 W_1}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_{n+1}(W_0, W_1) + \frac{W_1 \sum_{i=0}^{s} A_i - (W_0 a_2 + W_1 a_1) A_0}{W_1^2 - a_2 W_0^2 - a_1 W_0 W_1} V_n(W_0, W_1). \end{aligned}$$

*In summary, the solution of (1.1) is given by*

$$W_n(W_0,W_1)=\left(-\frac{(W_1-a_1W_0)\sum_{i=0}^{s}A_i-a_2W_0A_0}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+\frac{a_2(W_0\sum_{i=0}^{s}A_i-A_0W_1)}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n-1}(W_0,W_1)+\sum_{i=0}^{s}A_in^i$$

*i.e.,*

$$W_n(W_0,W_1)=\frac{W_0\sum_{i=0}^{s}A_i-A_0W_1}{W_1^2-a_2W_0^2-a_1W_0W_1}V_{n+1}(W_0,W_1)+\left(-\frac{W_1\sum_{i=0}^{s}A_i-(W_0a_2+W_1a_1)A_0}{W_1^2-a_2W_0^2-a_1W_0W_1}+1\right)V_n(W_0,W_1)+\sum_{i=0}^{s}A_in^i$$

**(b):** *The case $r=1$, i.e., 1 is a simple root of the characteristic equation of (1.3). Its solution can be obtained in direct analogy with case (a).*

**(c):** *The case $r=2$, i.e., 1 is a double root of the characteristic equation of (1.3). The construction follows the same method as in case (a), with appropriate multiplicity corrections.*

**Proof.** The derivation is achieved by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104–105), and Theorem 3.1 (pp. 88–89) for the case $m=2$, as given in Soykan [3]. The theorem in full generality encompasses all multiplicities $r=0,1,2$: the non-resonant case $r=0$ arises when all two roots are distinct from 1, whereas $r=1,2$ correspond to situations where 1 is a simple or double root of the characteristic equation (1.3). Since the examples treated in this paper involve only $r=0$, the explicit formulations for $r=1,2$ are omitted. They can be derived in analogy with case (a), incorporating multiplicity corrections, but are not required for the present applications.

## 2. Examples: Closed-Form Solutions of Second Order nonhomogeneous Linear Recurrence Relations

In the following subsection, we consider special cases of Theorem 1.1 (a) for the special second order nonhomogeneous linear recurrence relations.

### 2.1. Generalized balancing Numbers: The case $a_1=6$ and $a_2=-1$.

Next, we consider the case $a_1=6, a_2=-1$. A generalized balancing sequence $\{V_n\}_{n\geq0}=\{V_n(V_0,V_1)\}_{n\geq0}$ is defined by the second-order recurrence relation

$$V_n=6V_{n-1}-V_{n-2} \tag{2.1}$$

with the initial values $V_0=v_0, V_1=v_1$ not all being zero.

The sequence $\{V_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$V_{-n}=6V_{-(n-1)}-V_{-(n-2)}$$

for $n=1,2,3,....$ Therefore, recurrence (2.1) holds for all integer $n$. For more information on generalized balancing numbers, see Soykan [1].

The Binet formula of generalized balancing numbers can be written as

$$V_n=\frac{V_1-\beta V_0}{\alpha-\beta}\alpha^n-\frac{V_1-\alpha V_0}{\alpha-\beta}\beta^n$$

where $\alpha$ and $\beta$ are the roots of the quadratic equation $z^2 - 6z + 1 = 0$. Moreover

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2} \\ \beta &= 3 - 2\sqrt{2} \end{aligned}$$

So

$$V_n = \frac{V_1 - (3 - 2\sqrt{2})V_0}{4\sqrt{2}}(3 + 2\sqrt{2})^n - \frac{V_1 - (3 + 2\sqrt{2})V_0}{4\sqrt{2}}(3 - 2\sqrt{2})^n. \tag{2.2}$$

Now we define three special cases of the sequence $\{V_n\}$. balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$B_n = 6B_{n-1} - B_{n-2}, \qquad B_0 = 0, B_1 = 1, \tag{2.3}$$

$$H_n = 6H_{n-1} - H_{n-2}, \qquad H_0 = 2, H_1 = 6, \tag{2.4}$$

$$C_n = 6C_{n-1} - C_{n-2}, \qquad C_0 = 1, C_1 = 3. \tag{2.5}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.5) hold for all integer $n$.

For all integers $n$, balancing, modified Lucas-balancing and Lucas-balancing numbers can be expressed using Binet's formulas as

$$\begin{aligned} B_n &= \frac{\alpha^n}{(\alpha - \beta)} + \frac{\beta^n}{(\beta - \alpha)}, \\ H_n &= \alpha^n + \beta^n, \\ C_n &= \frac{\alpha^n + \beta^n}{2}, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 6, \\ a_2 &= -1, \\ W_0 &= 0, \\ W_1 &= 1, \end{aligned}$$

so that we have the case $V_n = B_n$ (balancing numbers).

Example 2.1. *In this example, we consider the case* $a_1 = 6, a_2 = -1, W_0 = 0, W_1 = 1$ *in Theorem 1.1 (a). So* $V_n(0,1) = B_n$ *represents nth balancing number i.e.,* $V_n = B_n$.

**(a):** *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_0$$

*is given by*

$$W_n(0,1) = (\frac{1}{4}c_0 + 1)B_n - \frac{1}{4}c_0 B_{n-1} - \frac{1}{4}c_0,$$

*i.e.,*

$$W_n(0,1) = \frac{1}{4}c_0 B_{n+1} + (1 - \frac{5}{4}c_0)B_n - \frac{1}{4}c_0.$$

**(b):** *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_1 n + c_0$$

*is given by*

$$W_n(0,1) = \frac{1}{4}(c_0 + 2c_1 + 4)B_n - \frac{1}{4}(c_0 + c_1)B_{n-1} - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1),$$

*i.e.,*

$$W_n(0,1) = \frac{1}{4}(c_0 + c_1)B_{n+1} + \frac{1}{4}(4 - 4c_1 - 5c_0)B_n - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1).$$

**(c):** *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_2 n^2 + c_1 n + c_0$$

*is given by*

$$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2)B_{n-1} + A_2 n^2 + A_1 n + A_0,$$

*i.e.,*

$$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 - 8)B_n + A_2 n^2 + A_1 n + A_0,$$

*where*

$$\begin{aligned} A_2 &= -\frac{1}{4}c_2, \\ A_1 &= -\frac{1}{4}(c_1 + 2c_2), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2). \end{aligned}$$

**(d):** *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

*is given by*

$$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 22c_3 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3)B_{n-1} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

*i.e.,*

$$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 + 8c_3 - 8)B_n + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

*where*

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{4}(c_2 + 3c_3), \\ A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3). \end{aligned}$$

**(e):** *The case $s = 4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

*is given by*

$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 22c_3 + 60c_4 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4)B_{n-1} + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0,$

*i.e.,*

$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 + 8c_3 + 12c_4 - 8)B_n + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0,$

*where*

$$\begin{aligned} A_4 &= -\frac{1}{4}c_4, \\ A_3 &= -\frac{1}{4}(c_3 + 4c_4), \\ A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4), \\ A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3 + 20c_4), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4). \end{aligned}$$

**(f):** *The case $s=5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(0,1) = \frac{1}{8}(2c_0 + 4c_1 + 9c_2 + 22c_3 + 60c_4 + 184c_5 + 8)B_n - \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4 + 32c_5)B_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

$W_n(0,1) = \frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4 + 32c_5)B_{n+1} - \frac{1}{8}(10c_0 + 8c_1 + 9c_2 + 8c_3 + 12c_4 + 8c_5 - 8)B_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*where*

$$\begin{aligned}
A_5 &= -\frac{1}{4}c_5, \\
A_4 &= -\frac{1}{4}(c_4 + 5c_5), \\
A_3 &= -\frac{1}{4}(c_3 + 4c_4 + 15c_5), \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4 + 25c_5), \\
A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3 + 20c_4 + 60c_5), \\
A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3 + 12c_4 + 32c_5).
\end{aligned}$$

**(g):** *The case $s=6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(0,1) = \frac{1}{16}(4c_0 + 8c_1 + 18c_2 + 44c_3 + 120c_4 + 368c_5 + 1293c_6 + 16)B_n - \frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6)B_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

$W_n(0,1) = \frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6)B_{n+1} - \frac{1}{16}(20c_0 + 16c_1 + 18c_2 + 16c_3 + 24c_4 + 16c_5 + 93c_6 - 16)B_n + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

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*where*

$$\begin{aligned}
A_6 &= -\frac{1}{4}c_6, \\
A_5 &= -\frac{1}{4}(c_5 + 6c_6), \\
A_4 &= -\frac{1}{8}(2c_4 + 10c_5 + 45c_6), \\
A_3 &= -\frac{1}{4}(c_3 + 4c_4 + 15c_5 + 50c_6), \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4 + 25c_5 + 90c_6), \\
A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3 + 20c_4 + 60c_5 + 192c_6), \\
A_0 &= -\frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6).
\end{aligned}$$

**(h):** *The case $s = 7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(0,1) = \frac{1}{16}(4c_0+8c_1+18c_2+44c_3+120c_4+368c_5+1293c_6+5174c_7+16)B_n - \frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6+865c_7)B_{n-1}+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

*i.e.,*

$W_n(0,1) = \frac{1}{16}(4c_0+4c_1+6c_2+10c_3+24c_4+64c_5+231c_6+865c_7)B_{n+1} - \frac{1}{16}(20c_0+16c_1+18c_2+16c_3+24c_4+16c_5+93c_6+16c_7-16)B_n+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0$

*where*

$$\begin{aligned}
A_7 &= -\frac{1}{4}c_7, \\
A_6 &= -\frac{1}{4}(c_6 + 7c_7), \\
A_5 &= -\frac{1}{8}(2c_5 + 12c_6 + 63c_7), \\
A_4 &= -\frac{1}{8}(2c_4 + 10c_5 + 45c_6 + 175c_7), \\
A_3 &= -\frac{1}{4}(c_3 + 4c_4 + 15c_5 + 50c_6 + 210c_7), \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4 + 25c_5 + 90c_6 + 336c_7), \\
A_1 &= -\frac{1}{16}(4c_1 + 8c_2 + 18c_3 + 40c_4 + 120c_5 + 384c_6 + 1617c_7), \\
A_0 &= -\frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6 + 865c_7).
\end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 6, \\ a_2 &= -1, \\ W_0 &= 2, \\ W_1 &= 6, \end{aligned}$$

so that we have the case $V_n = H_n$ (modified Lucas-balancing numbers).

EXAMPLE 2.2. *In this example, we consider the case* $a_1 = 6, a_2 = -1, W_0 = 2, W_1 = 6$ *in Theorem 1.1 (a). So* $V_n(2,6) = H_n$ *represents nth modified Lucas-balancing number i.e.,* $V_n = H_n$.

**(a):** *The case* $s = 0$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_0$$

*is given by*

$$W_n(2,6) = \frac{1}{32}(c_0 + 32)H_n + \frac{1}{32}c_0 H_{n-1} - \frac{1}{4}c_0,$$

*i.e.,*

$$W_n(2,6) = -\frac{1}{32}c_0 H_{n+1} + \frac{1}{32}(7c_0 + 32)H_n - \frac{1}{4}c_0.$$

**(b):** *The case* $s = 1$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_1 n + c_0$$

*is given by*

$$W_n(2,6) = \frac{1}{64}(2c_0 + 5c_1 + 64)H_n + \frac{1}{64}(2c_0 + c_1)H_{n-1} - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1),$$

*i.e.,*

$$W_n(2,6) = -\frac{1}{64}(2c_0 + c_1)H_{n+1} + \frac{1}{64}(14c_0 + 11c_1 + 64)H_n - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1).$$

**(c):** *The case* $s = 2$*. The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_2 n^2 + c_1 n + c_0$$

*is given by*

$$W_n(2,6) = \frac{1}{64}(2c_0 + 5c_1 + 12c_2 + 64)H_n + \frac{1}{64}(2c_0 + c_1)H_{n-1} + A_2 n^2 + A_1 n + A_0$$

*i.e.,*

$$W_n(2,6) = -\frac{1}{64}(2c_0 + c_1)H_{n+1} + \frac{1}{64}(14c_0 + 11c_1 + 12c_2 + 64)H_n + A_2 n^2 + A_1 n + A_0$$

*where*

$$\begin{aligned} A_2 &= -\frac{1}{4}c_2, \\ A_1 &= -\frac{1}{4}(c_1 + 2c_2), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2). \end{aligned}$$

**(d):** *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$$W_n(2,6) = \frac{1}{128}(4c_0 + 10c_1 + 24c_2 + 61c_3 + 128)H_n + \frac{1}{128}(4c_0 + 2c_1 - 7c_3)H_{n-1} + A_3n^3 + A_2n^2 + A_1n + A_0$$

*i.e.,*

$$W_n(2,6) = -\frac{1}{128}(4c_0 + 2c_1 - 7c_3)H_{n+1} + \frac{1}{128}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 128)H_n + A_3n^3 + A_2n^2 + A_1n + A_0$$

*where*

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{4}(c_2 + 3c_3), \\ A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3), \\ A_0 &= -\frac{1}{8}(2c_0 + 2c_1 + 3c_2 + 5c_3). \end{aligned}$$

**(e):** *The case $s = 4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(2,6) = \frac{1}{128}(4c_0 + 10c_1 + 24c_2 + 61c_3 + 168c_4 + 128)H_n + \frac{1}{128}(4c_0 + 2c_1 - 7c_3 - 24c_4)H_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

$W_n(2,6) = -\frac{1}{128}(4c_0 + 2c_1 - 7c_3 - 24c_4)H_{n+1} + \frac{1}{128}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 24c_4 + 128)H_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*where*

$$
\begin{aligned}
A_4 &= -\frac{1}{4}c_4,\\
A_3 &= -\frac{1}{4}(c_3+4c_4),\\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4),\\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4),\\
A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4).
\end{aligned}
$$

**(f):** *The case $s=5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(2,6) = \frac{1}{128}(4c_0+10c_1+24c_2+61c_3+168c_4+520c_5+128)H_n + \frac{1}{128}(4c_0+2c_1-7c_3-24c_4-88c_5)H_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

$W_n(2,6) = -\frac{1}{128}(4c_0+2c_1-7c_3-24c_4-88c_5)H_{n+1} + \frac{1}{128}(28c_0+22c_1+24c_2+19c_3+24c_4-8c_5+128)H_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*where*

$$
\begin{aligned}
A_5 &= -\frac{1}{4}c_5,\\
A_4 &= -\frac{1}{4}(c_4+5c_5),\\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5),\\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5),\\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5),\\
A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4+32c_5).
\end{aligned}
$$

**(g):** *The case $s=6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(2,6) = \frac{1}{128}(4c_0+10c_1+24c_2+61c_3+168c_4+520c_5+1824c_6+128)H_n + \frac{1}{128}(4c_0+2c_1-7c_3-24c_4-88c_5-300c_6)H_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

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$W_n(2,6) = -\frac{1}{128}(4c_0 + 2c_1 - 7c_3 - 24c_4 - 88c_5 - 300c_6)H_{n+1} + \frac{1}{128}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 24c_4 - 8c_5 + 24c_6 + 128)H_n + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*where*

$$
\begin{aligned}
A_6 &= -\frac{1}{4}c_6,\\
A_5 &= -\frac{1}{4}(c_5 + 6c_6),\\
A_4 &= -\frac{1}{8}(2c_4 + 10c_5 + 45c_6),\\
A_3 &= -\frac{1}{4}(c_3 + 4c_4 + 15c_5 + 50c_6),\\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4 + 25c_5 + 90c_6),\\
A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3 + 20c_4 + 60c_5 + 192c_6),\\
A_0 &= -\frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6).
\end{aligned}
$$

**(h):** *The case $s = 7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_7n^7 + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(2,6) = \frac{1}{256}(8c_0+20c_1+48c_2+122c_3+336c_4+1040c_5+3648c_6+14\,657c_7+256)H_n + \frac{1}{256}(8c_0 + 4c_1 - 14c_3 - 48c_4 - 176c_5 - 600c_6 - 2579c_7)H_{n-1} + A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

$W_n(2,6) = -\frac{1}{256}(8c_0 + 4c_1 - 14c_3 - 48c_4 - 176c_5 - 600c_6 - 2579c_7)H_{n+1} + \frac{1}{256}(56c_0 + 44c_1 + 48c_2 + 38c_3 + 48c_4 - 16c_5 + 48c_6 - 817c_7 + 256)H_n + A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*where*

$$
\begin{aligned}
A_7 &= -\frac{1}{4}c_7,\\
A_6 &= -\frac{1}{4}(c_6 + 7c_7),\\
A_5 &= -\frac{1}{8}(2c_5 + 12c_6 + 63c_7),\\
A_4 &= -\frac{1}{8}(2c_4 + 10c_5 + 45c_6 + 175c_7),\\
A_3 &= -\frac{1}{4}(c_3 + 4c_4 + 15c_5 + 50c_6 + 210c_7),\\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4 + 25c_5 + 90c_6 + 336c_7),\\
A_1 &= -\frac{1}{16}(4c_1 + 8c_2 + 18c_3 + 40c_4 + 120c_5 + 384c_6 + 1617c_7),\\
A_0 &= -\frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6 + 865c_7).
\end{aligned}
$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 6, \\ a_2 &= -1, \\ W_0 &= 1, \\ W_1 &= 3, \end{aligned}$$

so that we have the case $V_n = C_n$ (Lucas-balancing numbers).

Example 2.3. *In this example, we consider the case $a_1 = 6, a_2 = -1, W_0 = 1, W_1 = 3$ in Theorem 1.1 (a). So $V_n(1,3) = C_n$ represents nth Lucas-balancing number i.e., $V_n = C_n$.*

**(a):** *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_0$$

*is given by*

$$W_n(1,3) = \frac{1}{16}(c_0 + 16)C_n + \frac{1}{16}c_0 C_{n-1} - \frac{1}{4}c_0,$$

*i.e.,*

$$W_n(1,3) = -\frac{1}{16}c_0 C_{n+1} + \frac{1}{16}(7c_0 + 16)C_n - \frac{1}{4}c_0.$$

**(b):** *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_1 n + c_0$$

*is given by*

$$W_n(1,3) = \frac{1}{32}(2c_0 + 5c_1 + 32)C_n + \frac{1}{32}(2c_0 + c_1)C_{n-1} - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1),$$

*i.e.,*

$$W_n(1,3) = -\frac{1}{32}(2c_0 + c_1)C_{n+1} + \frac{1}{32}(14c_0 + 11c_1 + 32)C_n - \frac{1}{4}c_1 n - \frac{1}{4}(c_0 + c_1).$$

**(c):** *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_2 n^2 + c_1 n + c_0$$

*is given by*

$$W_n(1,3) = \frac{1}{32}(2c_0 + 5c_1 + 12c_2 + 32)C_n + \frac{1}{32}(2c_0 + c_1)C_{n-1} + A_2 n^2 + A_1 n + A_0$$

*i.e.,*

$$W_n(1,3) = -\frac{1}{32}(2c_0 + c_1)C_{n+1} + \frac{1}{32}(14c_0 + 11c_1 + 12c_2 + 32)C_n + A_2 n^2 + A_1 n + A_0$$

*where*

$$\begin{aligned} A_2 &= -\frac{1}{4}c_2, \\ A_1 &= -\frac{1}{4}(c_1+2c_2), \\ A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2). \end{aligned}$$

**(d):** *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$$W_n(1,3) = \frac{1}{64}(4c_0+10c_1+24c_2+61c_3+64)C_n + \frac{1}{64}(4c_0+2c_1-7c_3)C_{n-1} + A_3n^3 + A_2n^2 + A_1n + A_0$$

*i.e.,*

$$W_n(1,3) = -\frac{1}{64}(4c_0+2c_1-7c_3)C_{n+1} + \frac{1}{64}(28c_0+22c_1+24c_2+19c_3+64)C_n + A_3n^3 + A_2n^2 + A_1n + A_0$$

*where*

$$\begin{aligned} A_3 &= -\frac{1}{4}c_3, \\ A_2 &= -\frac{1}{4}(c_2+3c_3), \\ A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3), \\ A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3). \end{aligned}$$

**(e):** *The case $s=4$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(1,3) = \frac{1}{64}(4c_0+10c_1+24c_2+61c_3+168c_4+64)C_n + \frac{1}{64}(4c_0+2c_1-7c_3-24c_4)C_{n-1} + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

$W_n(1,3) = -\frac{1}{64}(4c_0+2c_1-7c_3-24c_4)C_{n+1} + \frac{1}{64}(28c_0+22c_1+24c_2+19c_3+24c_4+64)C_n + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*where*

$$\begin{aligned}
A_4 &= -\frac{1}{4}c_4,\\
A_3 &= -\frac{1}{4}(c_3+4c_4),\\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4),\\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4),\\
A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4).
\end{aligned}$$

**(f):** *The case $s=5$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(1,3) = \frac{1}{64}(4c_0+10c_1+24c_2+61c_3+168c_4+520c_5+64)C_n + \frac{1}{64}(4c_0+2c_1-7c_3-24c_4-88c_5)C_{n-1} + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

*i.e.,*

$W_n(1,3) = -\frac{1}{64}(4c_0+2c_1-7c_3-24c_4-88c_5)C_{n+1} + \frac{1}{64}(28c_0+22c_1+24c_2+19c_3+24c_4-8c_5+64)C_n + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0,$

*where*

$$\begin{aligned}
A_5 &= -\frac{1}{4}c_5,\\
A_4 &= -\frac{1}{4}(c_4+5c_5),\\
A_3 &= -\frac{1}{4}(c_3+4c_4+15c_5),\\
A_2 &= -\frac{1}{4}(c_2+3c_3+9c_4+25c_5),\\
A_1 &= -\frac{1}{8}(2c_1+4c_2+9c_3+20c_4+60c_5),\\
A_0 &= -\frac{1}{8}(2c_0+2c_1+3c_2+5c_3+12c_4+32c_5).
\end{aligned}$$

**(g):** *The case $s=6$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_6n^6 + c_5n^5 + c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$W_n(1,3) = \frac{1}{64}(4c_0+10c_1+24c_2+61c_3+168c_4+520c_5+1824c_6+64)C_n + \frac{1}{64}(4c_0+2c_1-7c_3-24c_4-88c_5-300c_6)C_{n-1} + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0$

*i.e.,*

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$W_n(1,3) = -\frac{1}{64}(4c_0 + 2c_1 - 7c_3 - 24c_4 - 88c_5 - 300c_6)C_{n+1} + \frac{1}{64}(28c_0 + 22c_1 + 24c_2 + 19c_3 + 24c_4 - 8c_5 + 24c_6 + 64)C_n + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

*where*

$$\begin{aligned}
A_6 &= -\frac{1}{4}c_6, \\
A_5 &= -\frac{1}{4}(c_5 + 6c_6), \\
A_4 &= -\frac{1}{8}(2c_4 + 10c_5 + 45c_6), \\
A_3 &= -\frac{1}{4}(c_3 + 4c_4 + 15c_5 + 50c_6), \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4 + 25c_5 + 90c_6), \\
A_1 &= -\frac{1}{8}(2c_1 + 4c_2 + 9c_3 + 20c_4 + 60c_5 + 192c_6), \\
A_0 &= -\frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6).
\end{aligned}$$

**(h):** *The case $s = 7$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = 6W_{n-1} - W_{n-2} + c_7 n^7 + c_6 n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

*is given by*

$W_n(1,3) = \frac{1}{128}(8c_0 + 20c_1 + 48c_2 + 122c_3 + 336c_4 + 1040c_5 + 3648c_6 + 14657c_7 + 128)C_n + \frac{1}{128}(8c_0 + 4c_1 - 14c_3 - 48c_4 - 176c_5 - 600c_6 - 2579c_7)C_{n-1} + A_7 n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

*i.e.,*

$W_n(1,3) = -\frac{1}{128}(8c_0 + 4c_1 - 14c_3 - 48c_4 - 176c_5 - 600c_6 - 2579c_7)C_{n+1} + \frac{1}{128}(56c_0 + 44c_1 + 48c_2 + 38c_3 + 48c_4 - 16c_5 + 48c_6 - 817c_7 + 128)C_n + A_7 n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

*where*

$$\begin{aligned}
A_7 &= -\frac{1}{4}c_7, \\
A_6 &= -\frac{1}{4}(c_6 + 7c_7), \\
A_5 &= -\frac{1}{8}(2c_5 + 12c_6 + 63c_7), \\
A_4 &= -\frac{1}{8}(2c_4 + 10c_5 + 45c_6 + 175c_7), \\
A_3 &= -\frac{1}{4}(c_3 + 4c_4 + 15c_5 + 50c_6 + 210c_7), \\
A_2 &= -\frac{1}{4}(c_2 + 3c_3 + 9c_4 + 25c_5 + 90c_6 + 336c_7), \\
A_1 &= -\frac{1}{16}(4c_1 + 8c_2 + 18c_3 + 40c_4 + 120c_5 + 384c_6 + 1617c_7), \\
A_0 &= -\frac{1}{16}(4c_0 + 4c_1 + 6c_2 + 10c_3 + 24c_4 + 64c_5 + 231c_6 + 865c_7).
\end{aligned}$$

**2.2. Generalized Oresme Numbers: The case $a_1 = 1$ and $a_2 = -\frac{1}{4}$.** Now, we consider the case $a_1 = 1, a_2 = -\frac{1}{4}$. A generalized Oresme sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$V_n = V_{n-1} - \frac{1}{4}V_{n-2} \tag{2.6}$$

with the initial values $V_0 = v_0, V_1 = v_1$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = 4V_{-(n-1)} - 4V_{-(n-2)}$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (2.6) holds for all integer $n$. For more information on generalized Oresme numbers, see Soykan [2].

Binet formula of generalized Oresme numbers can be given as

$$V_n = (D_1 + D_2 n)\alpha^n \tag{2.7}$$

where

$$\begin{aligned} D_1 &= V_0, \\ D_2 &= \frac{1}{\alpha}\left(V_1 - \alpha V_0\right). \end{aligned}$$

i.e.,

$$V_n = (V_0 + \frac{1}{\alpha}\left(V_1 - \alpha V_0\right) n)\alpha^n$$

Here, $\alpha = \beta = \frac{1}{2}$ are the roots of the quadratic equation

$$z^2 - z + \frac{1}{4} = 0. \tag{2.8}$$

i.e. the roots of characteristic equation (2.8) are equal. Note that

$$V_n = (V_0 + 2\left(V_1 - \frac{1}{2}V_0\right) n) \times \frac{1}{2^n}.$$

Now we define three special cases of the sequence $\{V_n\}$. Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence

$$G_{n+2} = G_{n+1} - \frac{1}{4}G_n, \qquad G_0 = 0, G_1 = 1, \tag{2.9}$$

$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n, \qquad H_0 = 2, H_1 = 1, \tag{2.10}$$

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \qquad O_0 = 0, O_1 = \frac{1}{2}. \tag{2.11}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= 4G_{-(n-1)} - 4G_{-(n-2)}, \\ H_{-n} &= 4H_{-(n-1)} - 4H_{-(n-2)}, \\ O_{-n} &= 4O_{-(n-1)} - 4O_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.9)-(2.11) hold for all integer $n$.

For all integers $n$, modified Oresme, Oresme-Lucas and Oresme numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1} = \frac{n}{2^{n-1}}, \\ H_n &= 2\alpha^n = \frac{1}{2^{n-1}}, \\ O_n &= n\alpha^n = \frac{n}{2^n}, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1, \\ a_2 &= -\frac{1}{4}, \\ W_0 &= 0, \\ W_1 &= 1, \end{aligned}$$

so that we have the case $V_n = G_n$ (modified Oresme numbers).

Example 2.4. *In this example, we consider the case* $a_1 = 1, a_2 = -\frac{1}{4}, W_0 = 0, W_1 = 1$ *in Theorem 1.1 (a). So* $V_n(0,1) = V_n$ *represents nth modified Oresme number, i.e.,* $V_n = G_n$.

**(a):** *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_0$$

*is given by*

$$W_n(0,1) = -\left(4c_0 - 1\right)G_n + c_0 G_{n-1} + 4c_0$$

*i.e.,*

$$W_n(0,1) = -4c_0 G_{n+1} + G_n + 4c_0.$$

**(b):** *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_1 n + c_0$$

*is given by*

$$W_n(0,1) = -(4c_0 - 4c_1 - 1)G_n + (c_0 - 2c_1)G_{n-1} + A_1 n + A_0$$

*i.e.,*

$$W_n(0,1) = -4(c_0 - 2c_1)G_{n+1} - (4c_1 - 1)G_n + A_1 n + A_0$$

*where*

$$\begin{aligned} A_1 &= 4c_1, \\ A_0 &= 4(c_0 - 2c_1). \end{aligned}$$

**(c):** *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_2 n^2 + c_1 n + c_0$$

*is given by*

$$W_n(0,1) = -(4c_0 - 4c_1 + 20c_2 - 1)G_n + (c_0 - 2c_1 + 8c_2)G_{n-1} + A_2 n^2 + A_1 n + A_0$$

*i.e.,*

$$W_n(0,1) = -4(c_0 - 2c_1 + 8c_2)G_{n+1} - (4c_1 - 12c_2 - 1)G_n + A_2 n^2 + A_1 n + A_0$$

*where*

$$\begin{aligned} A_2 &= 4c_2, \\ A_1 &= 4(c_1 - 4c_2), \\ A_0 &= 4(c_0 - 2c_1 + 8c_2). \end{aligned}$$

**(d):** *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

*is given by*

$$W_n(0,1) = -(4c_0 - 4c_1 + 20c_2 - 100c_3 - 1)G_n + (c_0 - 2c_1 + 8c_2 - 44c_3)G_{n-1} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

*i.e.,*

$$W_n(0,1) = -4(c_0 - 2c_1 + 8c_2 - 44c_3)G_{n+1} - (4c_1 - 12c_2 + 76c_3 - 1)G_n + A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

*where*

$$\begin{aligned} A_3 &= 4c_3, \\ A_2 &= 4(c_2 - 6c_3), \\ A_1 &= 4(c_1 - 4c_2 + 24c_3), \\ A_0 &= 4(c_0 - 2c_1 + 8c_2 - 44c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1, \\ a_2 &= -\frac{1}{4}, \\ W_0 &= 0, \\ W_1 &= \frac{1}{2}, \end{aligned}$$

so that we have the case $V_n = O_n$ (Oresme numbers).

Example 2.5. *In this example, we consider the case $a_1 = 1, a_2 = -\frac{1}{4}, W_0 = 0, W_1 = \frac{1}{2}$ in Theorem 1.1 (a). So $V_n(0, \frac{1}{2}) = V_n$ represents nth Oresme number, i.e., $V_n = O_n$.*

**(a):** *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_0$$

*is given by*

$$W_n(0, \frac{1}{2}) = -(8c_0 - 1)O_n + 2c_0O_{n-1} + 4c_0$$

*i.e.,*

$$W_n(0, \frac{1}{2}) = -8c_0O_{n+1} + O_n + 4c_0.$$

**(b):** *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_1n + c_0$$

*is given by*

$$W_n(0, \frac{1}{2}) = -(8c_0 - 8c_1 - 1)O_n + 2(c_0 - 2c_1)O_{n-1} + A_1n + A_0$$

*i.e.,*

$$W_n(0, \frac{1}{2}) = -8(c_0 - 2c_1)O_{n+1} - (8c_1 - 1)O_n + A_1n + A_0$$

*where*

$$\begin{aligned} A_1 &= 4c_1, \\ A_0 &= 4(c_0 - 2c_1). \end{aligned}$$

**(c):** *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_2n^2 + c_1n + c_0$$

*is given by*

$$W_n(0, \frac{1}{2}) = -(8c_0 - 8c_1 + 40c_2 - 1)O_n + 2(c_0 - 2c_1 + 8c_2)O_{n-1} + A_2n^2 + A_1n + A_0$$

*i.e.,*

$$W_n(0, \frac{1}{2}) = -8(c_0 - 2c_1 + 8c_2)O_{n+1} - (8c_1 - 24c_2 - 1)O_n + A_2n^2 + A_1n + A_0$$

*where*

$$\begin{aligned} A_2 &= 4c_2, \\ A_1 &= 4(c_1 - 4c_2), \\ A_0 &= 4(c_0 - 2c_1 + 8c_2). \end{aligned}$$

**(d):** *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} + c_3n^3 + c_2n^2 + c_1n + c_0$$

*is given by*

$$W_n(0, \frac{1}{2}) = -(8c_0 - 8c_1 + 40c_2 - 200c_3 - 1)O_n + 2(c_0 - 2c_1 + 8c_2 - 44c_3)O_{n-1} + A_3n^3 + A_2n^2 + A_1n + A_0$$

*i.e.,*

$$W_n(0, \frac{1}{2}) = -8(c_0 - 2c_1 + 8c_2 - 44c_3)O_{n+1} - (8c_1 - 24c_2 + 152c_3 - 1)O_n + A_3n^3 + A_2n^2 + A_1n + A_0$$

*where*

$$\begin{aligned} A_3 &= 4c_3, \\ A_2 &= 4(c_2 - 6c_3), \\ A_1 &= 4(c_1 - 4c_2 + 24c_3), \\ A_0 &= 4(c_0 - 2c_1 + 8c_2 - 44c_3). \end{aligned}$$

## References

[1] Soykan, Y., A Study on Generalized Balancing Numbers, Asian Journal of Advanced Research and Reports, 15(5), 78-100, 2021. DOI: 10.9734/AJARR/2021/v15i530401

[2] Soykan, Y., Generalized Oresme Numbers, Earthline Journal of Mathematical Sciences, 7(2), 333-367, 2021. https://doi.org/10.34198/ejms.7221.333367

[3] Soykan, Y., An Extensive Study on Generalized Leonardo Numbers and Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 13(3), 51–250, 2026. ISSN: 2347-2529

[4] Soykan, Y., Leonardo Polynomials and Numbers: Solutions, Linearizations, and Generating Functions, International Journal of Advances in Applied Mathematics and Mechanics, 13(4), 80-222, 2026. https://doi.org/10.26541/ijaamm.2026.130408

[5] Soykan, Y., Explicit Formulas for Polynomial-type Particular Solutions of Generalized Leonardo-type Sequences, Asian Journal of Advanced Research and Reports, 20(5), 164-208, 2026. https://doi.org/10.9734/ajarr/2026/v20i51361