Closed-Form Solutions of Third-Order Leonardo-Type Sequences: third-order Jacobsthal, Graham, and Jacobsthal-Narayana Families as Homogeneous Counterparts

Abstract. This paper develops closed-form formulas for third-order nonhomogeneous linear recurrence relations, designated as generalized Leonardo-type sequences, where the driving term is a polynomial. We analyze several notable cases of generalized Tribonacci numbers, which appear as the homogeneous counterparts of the original nonhomogeneous relations. In particular, our study covers classical examples including the third-order Jacobsthal sequence, modified third-order Jacobsthal sequence, third-order Jacobsthal-Lucas sequence, Graham sequence, Graham-Lucas sequence. Jacobsthal-Narayana sequence, Jacobsthal-Narayana-Lucas sequence.

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1. Introduction

Sequences arising from recurrence relations have long been central to mathematics, radiating outward into disciplines such as physics, engineering, architecture, biology, computer science, and even the arts. Though their definitions often appear elementary, they conceal profound richness, modeling processes of growth, oscillation, and symbolic construction. Classical second-order families—including the Fibonacci, Lucas, Pell, and Jacobsthal sequences—continue to serve as foundational examples of this tradition.

The domain, however, extends far beyond second-order cases. Higher-order recurrence sequences deepen both theoretical insight and practical application, enlarging the classical framework while exposing intricate algebraic and analytic structures. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences exemplify this progression, each governed by characteristic polynomials whose root configurations determine closed-form formulas. Homogeneous recurrences emphasize the relationship between characteristic polynomials and root multiplicities, whereas non-homogeneous forms introduce symbolic terms that interact with root structures to produce resonance effects. Collectively, these families establish a unified framework that links classical recurrence identities with the advancing field of symbolic recurrence theory.

This paper contributes to the ongoing discourse by establishing closed-form representations for third-order nonhomogeneous linear recurrence relations, formulated as generalized Leonardo-type sequences with polynomial inputs. The study further identifies distinguished manifestations of generalized Tribonacci numbers, which naturally arise as the homogeneous analogues of the corresponding nonhomogeneous relations. Our exposition encompasses several classical families, notably the third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas sequences; Graham sequence and Graham-Lucas sequences.

Let the third order nonhomogeneous linear recurrence relation, referred to as generalized Leonardo-type sequences, be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + p(n) \tag{1.1}$$

with initial conditions $W_0 = k_0, W_1 = k_1, W_2 = k_2$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i,$$

and the recurrence coefficients a_1, a_2, a_3 are complex scalars or polynomials in $\mathbb{C}[x]$. For more information on generalized Leonardo-type sequences, see Soykan [5] and [6].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} + a_3 V_{n-3} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2.$$

Suppose that θ_1, θ_2 and θ_3 are the roots of the characteristic equation

$$z^3 - a_1 z^2 - a_2 z - a_3 = 0 \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^3 - a_1z^2 - a_2z - a_3 = (z-1)^3 = z^3 - 3z^2 + 3z - 1 = 0$$

so that $a_1 = 3,\ a_2 = -3,\ a_3 = 1$ and (1.2) reduces to

$$V_n = 3V_{n-1} - 3V_{n-2} + V_{n-3}.$$

Our earlier study established particular solutions of third-order nonhomogeneous recurrences (1.1) with polynomial inputs for $s = 0, 1, 2, 3$; see Soykan [7]. Here, we extend those results to unified closed-form solutions by systematically applying Theorem 1.1. The method separates each recurrence into homogeneous and particular parts, with the particular solution obtained through iterative determination of coefficients.

The framework is governed by two parameters: the multiplicity r of 1 as a root of the characteristic equation (1.2), and the degree s of the polynomial $p(n)$. Explicit formulas are derived for all combinations $r = 0, 1, 2, 3$ and $s = 0, 1, 2, 3$, showing how resonance and multiplicity corrections arise when roots coincide. In this paper, however, only the non-resonant case $r = 0$ is needed, since the examples–generalized third-order Jacobsthal, Graham, and Jacobsthal-Narayana sequences–fall precisely into this category. Thus, the theorem is presented in detail for $r = 0$, with higher multiplicities noted but omitted.

Theorem 1.1.

(a): *[5, Theorem 7.7. (a)] The case $r = 0$, i.e., all three roots of the characteristic equation of (1.2) is distinct from* 1.

The solution of (1.1) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = \sum_{i=0}^{s} A_i n^i = A_0 + \sum_{i=1}^{s} A_i n^i$$

is the particular solution of (1.1). For each $0 \leq i \leq s$, A_i can be calculated with the iteration

$$A_s = -\frac{c_s}{a_1 + a_2 + a_3 - 1},\ \textit{for } n = s$$

and

$$A_n = -\frac{1}{a_1 + a_2 + a_3 - 1}(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+1} \binom{k}{n} (a_1 + 2^{k-n}a_2 + 3^{k-n}a_3)A_k),\ \textit{for } n = s-1, s-2, ..., 2, 1, 0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s} A_i, \\ W_2^{(p)} &= \sum_{i=0}^{s} 2^i A_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(A_0, \sum_{i=0}^{s} A_i, \sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) W_2^{(p)} + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) W_1^{(p)} + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) W_0^{(p)}$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) W_2^{(p)} + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) W_1^{(p)} + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) W_0^{(p)}$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) W_2^{(p)} + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) W_1^{(p)} + a_3 (a_3 W_1^2 - a_3 W_0 W_2) W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

i.e.,

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) \sum_{i=0}^{s} 2^i A_i + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) \sum_{i=0}^{s} A_i + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) \sum_{i=0}^{s} 2^i A_i + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) \sum_{i=0}^{s} A_i + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) \sum_{i=0}^{s} 2^i A_i + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) \sum_{i=0}^{s} A_i + a_3 (a_3 W_1^2 - a_3 W_0 W_2) A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + \sum_{i=0}^{s} A_i n^i$$

(b): *The case $r=1$, i.e., 1 is a simple root of the characteristic equation of (1.2). The solution can be obtained in the same manner as case (a).*

(c): *The case $r=2$, i.e., 1 is a double root of the characteristic equation of (1.2). The procedure parallels case (a), but requires multiplicity corrections to account for the repeated root.*

(d): *The case $r=3$, i.e., 1 is a triple root of the characteristic equation of (1.2). The solution is constructed analogously to case (a), with adjustments reflecting the higher multiplicity.*

Proof. The result is obtained by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104-105), and Theorem 3.1 (pp. 88-89) for $m=3$, as presented in Soykan [5]. In principle, the theorem covers all multiplicities $r=0,1,2,3$: the case $r=0$ represents the non-resonant situation with three distinct roots different from 1, while $r=1,2,3$ correspond to simple, double, or triple roots of 1 in the characteristic equation (1.2). In this paper, however, only the non-resonant case $r=0$ is required, where the three roots are distinct from 1. The cases $r=1,2,3$ can be obtained in the same manner with appropriate multiplicity adjustments, but they are not needed for the examples under consideration. □

2. Examples: Closed-Form Solutions of Third Order nonhomogeneous Linear Recurrence Relations

In the following subsection, we consider special cases of Theorem 1.1 (a) for the special third order nonhomogeneous linear recurrence relations.

2.1. Generalized Third Order Jacobsthal Numbers. In this subsection, we consider the case $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. A generalized third order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0,V_1,V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + 2V_{n-3} \tag{2.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1,2,3,....$ Therefore, recurrence (2.1) holds for all integer n. For more information on generalized third order Jacobsthal sequence, see Soykan [2].

Binet formula of generalized third order Jacobsthal numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{2.2}$$

Here, α, β and $\gamma$are the roots of the cubic equation $z^3 - z^2 - z - 2 = 0$. Moreover

$$\begin{aligned}
\alpha &= 2, \\
\beta &= \frac{-1+i\sqrt{3}}{2}, \\
\gamma &= \frac{-1-i\sqrt{3}}{2}.
\end{aligned}$$

Third-order Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A077947, [1]), modified third-order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ (OEIS: A186575, [1]) and third-order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (OEIS: A226308, [1]) are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, J_2 = 1, \tag{2.3}$$

$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n, \quad K_0 = 3, K_1 = 1, K_2 = 3. \tag{2.4}$$

$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, j_2 = 5. \tag{2.5}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [**??**] and $\{K_n\}_{n\geq 0}$ is given in [**??**]. For more details on the generalized third-order Jacobsthal numbers and its special cases, see [2].

The sequences $\{J_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
J_{-n} &= -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)}, \\
K_{-n} &= -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)}, \\
j_{-n} &= -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.5) hold for all integer n.

Note that for all integers n, third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{2.6}$$

$$K_n = \alpha^n + \beta^n + \gamma^n, \tag{2.7}$$

$$j_n = \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{2.8}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned}
a_1 &= 1,\ a_2 = 1,\ a_3 = 2, \\
W_0 &= 0,\ W_1 = 1,\ W_2 = 1,
\end{aligned}$$

so that we have the case $V_n = J_n$ (third order Jacobsthal numbers).

EXAMPLE 2.1. *In this example, we consider the case* $a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ W_0 = 0,\ W_1 = 1,\ W_2 = 1,$ *in Theorem 1.1 (a). So* $V_n(0,1,1) = V_n$ *represents nth third order Jacobsthal number, i.e.,* $V_n = J_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(0,1,1) = \frac{1}{3}\left(c_0+3\right)J_n + \frac{2}{3}c_0J_{n-2} - \frac{1}{3}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_1n + c_0$$

is given by

$$W_n(0,1,1) = \frac{1}{3}\left(c_0+4c_1+3\right)J_n + \frac{1}{3}c_1J_{n-1} + \frac{2}{3}\left(c_0+3c_1\right)J_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{3}c_1, \\ A_0 &= -\frac{1}{3}(c_0+3c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} +_2 W_{n-2} + 2W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,1) = \frac{1}{9}(3c_0+12c_1+52c_2+9)J_n + \frac{1}{3}(c_1+9c_2)J_{n-1} + \frac{2}{9}(3c_0+9c_1+31c_2)J_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{3}(c_1+6c_2), \\ A_0 &= -\frac{1}{9}(3c_0+9c_1+31c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,1) = \frac{1}{9}(3c_0 + 12c_1 + 52c_2 + 258c_3 + 9)J_n + \frac{1}{3}(c_1 + 9c_2 + 65c_3)J_{n-1} + \frac{2}{9}(3c_0 + 9c_1 + 31c_2 + 135c_3)J_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{3}c_3, \\ A_2 &= -\frac{1}{3}(c_2 + 9c_3), \\ A_1 &= -\frac{1}{3}(c_1 + 6c_2 + 31c_3), \\ A_0 &= -\frac{1}{9}(3c_0 + 9c_1 + 31c_2 + 135c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 1,\ a_3 = 2, \\ W_0 &= 3,\ W_1 = 1,\ W_2 = 3, \end{aligned}$$

so that we have the case $V_n = K_n$ (modified third-order Jacobstha numbers).

EXAMPLE 2.2. *In this example, we consider the case* $a_1 = 1, a_2 = 1, a_3 = 2, W_0 = 3, W_1 = 1, W_2 = 3$ *in Theorem 1.1 (a). So* $V_n(3,1,3) = V_n$ *represents nth modified third-order Jacobstha number, i.e.,* $V_n = K_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(3,1,3) = \frac{1}{147}\left(17c_0 + 147\right)K_n + \frac{10}{147}c_0K_{n-1} - \frac{4}{147}c_0K_{n-2} - \frac{1}{3}c_0$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_1n + c_0$$

is given by

$$W_n(3,1,3) = \frac{1}{441}\left(51c_0 + 176c_1 + 441\right)K_n + \frac{1}{147}\left(10c_0 + 47c_1\right)K_{n-1} - \frac{2}{441}\left(6c_0 - 11c_1\right)K_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{3}c_1, \\ A_0 &= -\frac{1}{3}(c_0 + 3c_1). \end{aligned}$$

(c): *The case $s = 2$). The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,1,3) = \frac{1}{441}(51c_0 + 176c_1 + 708c_2 + 441)K_n + \frac{1}{441}(30c_0 + 141c_1 + 673c_2)K_{n-1} - \frac{2}{441}(6c_0 - 11c_1 - 179c_2)K_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{3}c_2, \\ A_1 &= -\frac{1}{3}(c_1 + 6c_2), \\ A_0 &= -\frac{1}{9}(3c_0 + 9c_1 + 31c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,1,3) = \frac{1}{441}(51c_0 + 176c_1 + 708c_2 + 3478c_3 + 441)K_n + \frac{1}{441}(30c_0 + 141c_1 + 673c_2 + 3513c_3)K_{n-1} - \frac{2}{441}(6c_0 - 11c_1 - 179c_2 - 1375c_3)K_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{3}c_3, \\ A_2 &= -\frac{1}{3}(c_2 + 9c_3), \\ A_1 &= -\frac{1}{3}(c_1 + 6c_2 + 31c_3), \\ A_0 &= -\frac{1}{9}(3c_0 + 9c_1 + 31c_2 + 135c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 1,\ a_3 = 2, \\ W_0 &= 2,\ W_1 = 1,\ W_2 = 5, \end{aligned}$$

so that we have the case $V_n = j_n$ (third-order Jacobsthal-Luca numbers).

EXAMPLE 2.3. *In this example, we consider the case $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $W_0 = 2$, $W_1 = 1$, $W_2 = 5$ in Theorem 1.1 (a). So $V_n(2,1,5) = V_n$ represents nth third-order Jacobsthal-Luca number, i.e., $V_n = j_n$.*

(a): *The case $s = 0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_0$$

is given by

$$W_n(2,1,5) = \frac{1}{18}\left(c_0 + 18\right) j_n + \frac{1}{6} c_0 j_{n-1} - \frac{1}{18} c_0 j_{n-2} - \frac{1}{3} c_0.$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(2,1,5) = \frac{1}{18}(c_0 + 4c_1 + 18) j_n + \frac{1}{18}(3c_0 + 10c_1) j_{n-1} - \frac{1}{18} c_0 j_{n-2} + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{3} c_1, \\ A_0 &= -\frac{1}{3}(c_0 + 3c_1). \end{aligned}$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,1,5) = \frac{1}{54}(3c_0 + 12c_1 + 58c_2 + 54) j_n + \frac{1}{18}(3c_0 + 10c_1 + 36c_2) j_{n-1} - \frac{1}{54}(3c_0 - 38c_2) j_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{3} c_2, \\ A_1 &= -\frac{1}{3}(c_1 + 6c_2), \\ A_0 &= -\frac{1}{9}(3c_0 + 9c_1 + 31c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(2,1,5) = \frac{1}{54}(3c_0 + 12c_1 + 58c_2 + 330c_3 + 54) j_n + \frac{1}{18}(3c_0 + 10c_1 + 36c_2 + 152c_3) j_{n-1} - \frac{1}{54}(3c_0 - 38c_2 - 306c_3) j_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$
\begin{aligned}
A_3 &= -\frac{1}{3}c_3, \\
A_2 &= -\frac{1}{3}(c_2 + 9c_3), \\
A_1 &= -\frac{1}{3}(c_1 + 6c_2 + 31c_3), \\
A_0 &= -\frac{1}{9}(3c_0 + 9c_1 + 31c_2 + 135c_3).
\end{aligned}
$$

2.2. Generalized Graham Numbers. In this subsection, we consider the case $a_1 = 2$, $a_2 = 3$ and $a_3 = 5$. A generalized Graham sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} \tag{2.9}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{3}{5}V_{-(n-1)} - \frac{2}{5}V_{-(n-2)} + \frac{1}{5}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.9) holds for all integer n. For more information on generalized Graham numbers, see Soykan [3].

Binet formula of generalized Graham numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{2.10}$$

Here, α, β and γ are the roots of the cubic equation $z^3 - 2z^2 - 3z - 5 = 0$. Moreover

$$
\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega^2\left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega\left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3},
\end{aligned}
$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Now we define two special cases of the sequence $\{V_n\}$. Graham sequence $\{G_n\}_{n\geq 0}$ and Graham-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = 2G_{n+2}+3G_{n+1}+5G_n, \qquad G_0=0, G_1=1, G_2=2, \tag{2.11}$$

$$H_{n+3} = 2H_{n+2}+3H_{n+1}+5H_n, \qquad H_0=3, H_1=2, H_2=10. \tag{2.12}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{3}{5}G_{-(n-1)}-\frac{2}{5}G_{-(n-2)}+\frac{1}{5}G_{-(n-3)}, \\ H_{-n} &= -\frac{3}{5}H_{-(n-1)}-\frac{2}{5}H_{-(n-2)}+\frac{1}{5}H_{-(n-3)} \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (2.11) and (2.12) hold for all integer n.For all integers n, Graham and Graham-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ H_n &= \alpha^n+\beta^n+\gamma^n, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 2,\ a_2=3,\ a_3=5, \\ W_0 &= 0,\ W_1=1,\ W_2=2, \end{aligned}$$

so that we have the case $V_n=G_n$ (Graham numbers).

Example 2.4. *In this example, we consider the case $a_1=2,\ a_2=3,\ a_3=5, W_0=0,\ W_1=1,\ W_2=2$ in Theorem 1.1 (a). So $V_n(0,1,2)=V_n$ represents nth Graham number, i.e., $V_n=G_n$.*

- **(a):** *The case $s=0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1}+3W_{n-2}+5W_{n-3}+c_0$$

is given by

$$W_n(0,1,2)=\frac{1}{9}(c_0+9)G_n-\frac{1}{9}c_0G_{n-1}+\frac{5}{9}c_0G_{n-2}-\frac{1}{9}c_0.$$

- **(b):** *The case $s=1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1}+3W_{n-2}+5W_{n-3}+c_1n+c_0$$

is given by

$$W_n(0,1,2)=\frac{1}{81}(9c_0+32c_1+81)G_n-\frac{1}{81}(9c_0+23c_1)G_{n-1}+\frac{5}{81}(9c_0+23c_1)G_{n-2}+A_1n+A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{9}c_1, \\ A_0 &= -\frac{1}{81}(9c_0 + 23c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(0,1,2) = \frac{1}{729}(81c_0 + 288c_1 + 1022c_2 + 729)G_n - \frac{1}{729}(81c_0 + 207c_1 + 365c_2)G_{n-1} + \frac{5}{729}(81c_0 + 207c_1 + 527c_2)G_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{9}c_2, \\ A_1 &= -\frac{1}{81}(9c_1 + 46c_2), \\ A_0 &= -\frac{1}{729}(81c_0 + 207c_1 + 527c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(0,1,2) = \frac{1}{2187}(243c_0 + 864c_1 + 3066c_2 + 11104c_3 + 2187)G_n - \frac{1}{2187}(243c_0 + 621c_1 + 1095c_2 - 929c_3)G_{n-1} + \frac{5}{2187}(243c_0 + 621c_1 + 1581c_2 + 4255c_3)G_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{9}c_3, \\ A_2 &= -\frac{1}{27}(3c_2 + 23c_3), \\ A_1 &= -\frac{1}{243}(27c_1 + 138c_2 + 527c_3), \\ A_0 &= -\frac{1}{2187}(243c_0 + 621c_1 + 1581c_2 + 4255c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 2,\ a_2 = 3,\ a_3 = 5, \\ W_0 &= 3,\ W_1 = 2,\ W_2 = 10, \end{aligned}$$

so that we have the case $V_n = H_n$ (Graham Lucas numbers).

EXAMPLE 2.5. *In this example, we consider the case* $a_1 = 2,\ a_2 = 3,\ a_3 = 5, W_0 = 3,\ W_1 = 2,\ W_2 = 10$ *in Theorem 1.1 (a). So* $V_n(3,2,10) = V_n$ *represents nth Graham Lucas number, i.e.,* $V_n = H_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_0$$

is given by

$$W_n(3,2,10) = \frac{1}{11079}\left(320c_0 + 11079\right)H_n + \frac{58}{11079}c_0H_{n-1} - \frac{695}{11079}c_0H_{n-2} - \frac{1}{9}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_1n + c_0$$

is given by

$$W_n(3,2,10) = \frac{1}{99711}(2880c_0+8323c_1+99711)H_n + \frac{1}{99711}(522c_0+4763c_1)H_{n-1} - \frac{5}{99711}(1251c_0+2819c_1)H_{n-2} + A_1n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{9}c_1, \\ A_0 &= -\frac{1}{81}(9c_0 + 23c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_2n^2 + c_1n + c_0$$

is given by

$$W_n(3,2,10) = \frac{1}{897399}(25920c_0+74\,907c_1+228409c_2+897399)H_n + \frac{1}{897399}(4698c_0+42867c_1+222887c_2)H_{n-1} - \frac{5}{897399}(11259c_0 + 25371c_1 + 44201c_2)H_{n-2} + A_2n^2 + A_1n + A_0$$

where

$$\begin{aligned} A_2 &= -\frac{1}{9}c_2, \\ A_1 &= -\frac{1}{81}(9c_1 + 46c_2), \\ A_0 &= -\frac{1}{729}(81c_0 + 207c_1 + 527c_2). \end{aligned}$$

(d): *The case* $s = 3$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 2W_{n-1} + 3W_{n-2} + 5W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(3,2,10) = \frac{1}{2692197}(77760c_0+224721c_1+685227c_2+2312171c_3+2692197)H_n + \frac{1}{2692197}(14094c_0+128601c_1+668661c_2+2975491c_3)H_{n-1} - \frac{5}{2692197}(33777c_0+76113c_1+132603c_2+39403c_3)H_{n-2} + A_3n^3+A_2n^2+A_1n+A_0$

where

$$\begin{aligned}
A_3 &= -\frac{1}{9}c_3,\\
A_2 &= -\frac{1}{27}(3c_2+23c_3),\\
A_1 &= -\frac{1}{243}(27c_1+138c_2+527c_3),\\
A_0 &= -\frac{1}{2187}(243c_0+621c_1+1581c_2+4255c_3).
\end{aligned}$$

2.3. Generalized Jacobsthal Narayana Numbers. In this subsection, we consider the case $a_1 = 1$, $a_2 = 0$, $a_3 = 2$. The generalized Jacobsthal-Narayana numbers

$$\{V_n(V_0, V_1, V_2; 1, 0, 2)\}_{n\geq 0}$$

(or $\{V_n\}_{n\geq 0}$ or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n = V_{n-1} + 2V_{n-3}, \qquad V_0 = a, V_1 = b, V_2 = c, \quad n \geq 3 \tag{2.13}$$

where V_0, V_1, V_2 are arbitrary complex (or real) numbers.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2.13) holds for all integers n. For more information on generalized Jacobsthal-Narayana numbers, see Soykan [4].

For all integers n, Binet's formula of generalized Jacobsthal-Narayana numbers is given as follows.

$$\begin{aligned}
V_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n,
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\\
p_2 &= V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\\
p_3 &= V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{V_2-(\beta+\gamma)V_1+\beta\gamma V_0}{(\alpha-\beta)(\alpha-\gamma)} \\
&= \frac{(\alpha V_2+\alpha(-1+\alpha)V_1+2V_0)}{\alpha^2+6}, \\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0}{(\beta-\alpha)(\beta-\gamma)} \\
&= \frac{(\beta V_2+\beta(-1+\beta)V_1+2V_0)}{\beta^2+6}, \\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{V_2-(\alpha+\beta)V_1+\alpha\beta V_0}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{(\gamma V_2+\gamma(-1+\gamma)V_1+2V_0)}{\gamma^2+6}.
\end{aligned}$$

Here, α,β and γ are the roots of the cubic equation

$$z^3-z^2-2=(z-\alpha)(z-\beta)(z-\gamma)=0.$$

Moreover,

$$\begin{aligned}
\alpha &= \frac{1}{3}+\left(\frac{28}{27}+\sqrt{\frac{29}{27}}\right)^{1/3}+\left(\frac{28}{27}-\sqrt{\frac{29}{27}}\right)^{1/3}, \\
\beta &= \frac{1}{3}+\omega\left(\frac{28}{27}+\sqrt{\frac{29}{27}}\right)^{1/3}+\omega^2\left(\frac{28}{27}-\sqrt{\frac{29}{27}}\right)^{1/3}, \\
\gamma &= \frac{1}{3}+\omega^2\left(\frac{28}{27}+\sqrt{\frac{29}{27}}\right)^{1/3}+\omega\left(\frac{28}{27}-\sqrt{\frac{29}{27}}\right)^{1/3},
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

Note that

$$\left\{\begin{array}{c}
\alpha+\beta+\gamma=1, \\
\alpha\beta+\alpha\gamma+\beta\gamma=0, \\
\alpha\beta\gamma=2.
\end{array}\right.$$

Now, we define two special cases of the sequence $\{V_n\}$. Jacobsthal-Narayana sequence$\{B_n\}_{n\geq 0}$, Jacobsthal-Narayana-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$B_n = B_{n-1}+2B_{n-3}, \qquad B_0=0, B_1=1, B_2=1, \tag{2.14}$$

$$C_n = C_{n-1}+2C_{n-3}, \qquad C_0=3, C_1=1, C_2=1. \tag{2.15}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$\begin{aligned}
B_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)}, \\
C_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.14)-(2.15) hold for all integer n.

For all integers n, Binet's formulas of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are

$$\begin{aligned} B_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{\alpha^{n+2}}{\alpha^2+6} + \frac{\beta^{n+2}}{\beta^2+6} + \frac{\gamma^{n+2}}{\gamma^2+6}, \\ C_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

respectively.

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 0,\ a_3 = 2, \\ W_0 &= 0,\ W_1 = 1,\ W_2 = 1, \end{aligned}$$

so that we have the case $V_n = B_n$ (Jacobsthal-Narayana numbers).

Example 2.6. *In this example, we consider the case* $a_1 = 1,\ a_2 = 0,\ a_3 = 2, W_0 = 0,\ W_1 = 1,\ W_2 = 1$ *in Theorem 1.1 (a). So* $V_n(0,1,1) = V_n$ *represents nth Jacobsthal-Narayana number, i.e.,* $V_n = B_n$.

(a): *The case* $s = 0$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_0$$

is given by

$$W_n(0,1,1) = \frac{1}{2}\left(c_0+2\right)B_n + c_0 B_{n-2} - \frac{1}{2}c_0.$$

(b): *The case* $s = 1$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(0,1,1) = \frac{1}{4}(2c_0+9c_1+4)B_n + \frac{1}{2}c_1 B_{n-1} + \frac{1}{2}(2c_0+7c_1)B_{n-2} + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0+7c_1). \end{aligned}$$

(c): *The case* $s = 2$. *The solution (the closed-form solution) of the sequence* $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(0,1,1) = \frac{1}{4}(2c_0+9c_1+46c_2+4)B_n + \frac{1}{2}(c_1+10c_2)B_{n-1} + \frac{1}{2}(2c_0+7c_1+30c_2)B_{n-2} + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1+7c_2), \\ A_0 &= -\frac{1}{4}(2c_0+7c_1+30c_2). \end{aligned}$$

(d): *The case $s=3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0$$

is given by

$W_n(0,1,1) = \frac{1}{8}(4c_0+18c_1+92c_2+567c_3+8)B_n + \frac{1}{4}(2c_1+20c_2+167c_3)B_{n-1} + \frac{1}{4}(4c_0+14c_1+60c_2+341c_3)B_{n-2} + A_3n^3 + A_2n^2 + A_1n + A_0$

where

$$\begin{aligned} A_3 &= -\frac{1}{2}c_3, \\ A_2 &= -\frac{1}{4}(2c_2+21c_3), \\ A_1 &= -\frac{1}{2}(c_1+7c_2+45c_3), \\ A_0 &= -\frac{1}{8}(4c_0+14c_1+60c_2+341c_3). \end{aligned}$$

In the following Example, we consider Theorem 1.1 (a) for

$$\begin{aligned} a_1 &= 1,\ a_2 = 0,\ a_3 = 2, \\ W_0 &= 3,\ W_1 = 1,\ W_2 = 1, \end{aligned}$$

so that we have the case $V_n = C_n$ (Jacobsthal-Narayana-Lucas numbers).

EXAMPLE 2.7. *In this example, we consider the case $a_1 = 1,\ a_2 = 0,\ a_3 = 2, W_0 = 3,\ W_1 = 1,\ W_2 = 1$ in Theorem 1.1 (a). So $V_n(3,1,1) = V_n$ represents nth Jacobsthal-Narayana-Lucas number, i.e., $V_n = C_n$.*

(a): *The case $s=0$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{58}(11c_0+58)C_n + \frac{3}{29}c_0C_{n-1} + \frac{2}{29}c_0C_{n-2} - \frac{1}{2}c_0.$$

(b): *The case $s = 1$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_1 n + c_0$$

is given by

$$W_n(3,1,1) = \frac{1}{116}(22c_0 + 87c_1 + 116)C_n + \frac{1}{58}(6c_0 + 29c_1)C_{n-1} + \frac{1}{58}(4c_0 + 29c_1)C_{n-2} + A_1 n + A_0$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2}c_1, \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1). \end{aligned}$$

(c): *The case $s = 2$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,1,1) = \frac{1}{116}(22c_0 + 87c_1 + 422c_2 + 116)C_n + \frac{1}{58}(6c_0 + 29c_1 + 152c_2)C_{n-1} + \frac{1}{58}(4c_0 + 29c_1 + 198c_2)C_{n-2} + A_2 n^2 + A_1 n + A_0$

where

$$\begin{aligned} A_2 &= -\frac{1}{2}c_2, \\ A_1 &= -\frac{1}{2}(c_1 + 7c_2), \\ A_0 &= -\frac{1}{4}(2c_0 + 7c_1 + 30c_2). \end{aligned}$$

(d): *The case $s = 3$. The solution (the closed-form solution) of the sequence $\{W_n\}$ defined by non-homogeneous linear recurrence relation*

$$W_n = W_{n-1} + 2W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0$$

is given by

$W_n(3,1,1) = \frac{1}{232}(44c_0 + 174c_1 + 844c_2 + 5205c_3 + 232)C_n + \frac{1}{116}(12c_0 + 58c_1 + 304c_2 + 1873c_3)C_{n-1} + \frac{1}{116}(8c_0 + 58c_1 + 396c_2 + 2863c_3)C_{n-2} + A_3 n^3 + A_2 n^2 + A_1 n + A_0$

where

$$
\begin{aligned}
A_3 &= -\frac{1}{2}c_3, \\
A_2 &= -\frac{1}{4}(2c_2 + 21c_3), \\
A_1 &= -\frac{1}{2}(c_1 + 7c_2 + 45c_3), \\
A_0 &= -\frac{1}{8}(4c_0 + 14c_1 + 60c_2 + 341c_3).
\end{aligned}
$$

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