Original Research Article

A Unified Analytical Framework for Stability and Energy Analysis of Multi-Term Fractional Dynamical Systems

Abstract

This work develops a unified fractional calculus framework for modeling memory dependent behavior in viscoelastic materials, electrical circuits, and control systems. The study emphasizes rigorous mathematical analysis of multi-term fractional operators and establishes well posedness and dissipativity properties for heterogeneous fractional viscoelastic models through a novel energy functional approach. For fractional order circuit representations, a bounded input bounded output stability result is derived, ensuring the physical realizability of nonlocal impedance elements. In the control domain, a practical stability theorem is proven for adaptive fractional PID controllers, providing guaranteed bounded tracking performance under memory driven adaptation. These results demonstrate that fractional operators not only enhance modeling flexibility but also preserve fundamental stability and energy principles, offering a mathematically consistent foundation for advanced engineering system design.

Keywords: Fractional calculus; multi-term fractional operators; viscoelastic systems; fractional dynamical systems; stability analysis; dissipative systems; fractional order control.

2010 Mathematics Subject Classification: 26A33; 34A08; 74D05; 93C15; 47G20

1 Introduction

Classical calculus has long provided the foundation for modeling dynamical systems in science and engineering. Its integer–order derivatives describe local rates of change and have proven effective for a wide range of phenomena. However, many real processes exhibit hereditary and nonlocal effects that cannot be adequately captured using purely local differential operators. Materials with internal memory, systems with long relaxation times, and processes influenced by historical states require mathematical tools that extend beyond classical formulations.

Fractional calculus offers such a framework by allowing differentiation and integration of non-integer order [3, 7, 8, 9]. Fractional operators inherently incorporate memory, as their definitions involve integral kernels spanning the past history of the state variable. This feature makes them particularly suitable for modeling viscoelastic media, anomalous transport, dielectric relaxation, and memory dependent control mechanisms [7, 8]. Unlike integer order models, fractional formulations provide a continuous transition between purely elastic and purely viscous responses, enabling more realistic descriptions of complex materials.

Over the past decades, fractional calculus has found increasing use in engineering sciences. In viscoelasticity, fractional stress–strain relations have been shown to represent power law relaxation and creep behaviors with a small number of parameters. In circuit theory, fractional impedance elements generalize capacitive and inductive behavior through frequency dependent memory effects. In control engineering, fractional order controllers extend the classical PID structure, introducing additional degrees of freedom that improve robustness and transient performance. Despite these advances, rigorous mathematical analysis often lags behind modeling practice [2, 5, 1]. Many fractional models are introduced phenomenologically, while their stability, boundedness, and energy properties are not thoroughly established.

The present work addresses this gap by developing a unified analytical framework for a class of multi-term fractional systems arising in mechanics, electrical networks, and control theory [5, 13]. Rather than emphasizing numerical illustrations, the focus is placed on structural properties of fractional operators and their implications for system behavior.

First, a heterogeneous fractional viscoelastic model is analyzed. By constructing an appropriate energy functional, we establish well posedness and prove that the system is intrinsically dissipative. This result provides a rigorous mathematical explanation for irreversible energy loss in materials governed by fractional stress laws.

Second, a fractional circuit model involving multi-term integral operators is studied from an operator theoretic perspective. A bounded-input bounded-output stability property is derived, ensuring that memory dependent impedance elements do not produce unphysical growth in voltage response on finite time intervals.

Third, we investigate an adaptive fractional PID control structure. By treating the fractional terms as bounded memory perturbations of a stabilizing linear controller, a practical stability result is obtained that guarantees bounded tracking performance under time varying fractional orders.

These results demonstrate that fractional models can be analyzed within a rigorous mathematical framework while preserving their ability to represent nonlocal and memory dependent dynamics. The paper therefore contributes to the theoretical foundations of fractional engineering systems, providing analytical tools that complement existing modeling approaches.

2 Viscoelastic Materials:Fractional Order Modeling

Viscoelastic materials, such as polymers, exhibit both elastic and viscous properties, with stress strain relationships that depend on the history of deformation. The fractional order model $\sigma(t) = ED^{\alpha}\varepsilon(t)$, where D^{α} is the fractional derivative of order $0 < \alpha < 1$, E is a material constant, $\sigma(t)$ is stress, and $\varepsilon(t)$ is strain, effectively captures this memory dependent behavior.

2.1 Fractional Viscoelastic Model

We propose a generalized fractional viscoelastic model that incorporates multiple fractional orders to account for the properties of heterogeneous material:

$$\sigma(t) = E_1 D^{\alpha_1}\varepsilon(t) + E_2 D^{\alpha_2}\varepsilon(t), \quad 0 < \alpha_1, \alpha_2 < 1, \tag{2.1}$$

where E_1, E_2 are material constants, and α_1, α_2 represent distinct fractional orders reflecting different relaxation mechanisms. Such multi-term fractional viscoelastic models have been widely used to describe heterogeneous relaxation mechanisms [3, 5, 7].

The *heterogeneous fractional viscoelasticity model* is defined by the stress strain relationship given in (2.1), where the fractional derivatives D^{α_1} and D^{α_2} are of Caputo type, ensuring compatibility with the physical initial conditions [7, 9]. Existence and uniqueness results for linear and nonlinear fractional differential equations provide the theoretical foundation for the well posedness of such models [1, 9].

Theorem 2.1 (Well posedness and dissipativity of the heterogeneous fractional viscoelastic model)**.** *Consider the stress–strain relation*

$$\sigma(t) = E_1\, {}^C D_t^{\alpha_1} \varepsilon(t) + E_2\, {}^C D_t^{\alpha_2} \varepsilon(t), \qquad 0 < \alpha_1 < \alpha_2 < 1,$$

where ${}^C D_t^{\alpha}$ *denotes the Caputo fractional derivative and* $E_1, E_2 > 0$*. Assume the strain satisfies the evolution equation*

$$\rho \ddot{\varepsilon}(t) + \gamma \dot{\varepsilon}(t) + \sigma(t) = f(t),$$

with $\rho, \gamma > 0$ *and* $f \in L^2(0,T)$*.*

Then for initial data $\varepsilon(0) = \varepsilon_0$ *and* $\dot{\varepsilon}(0) = v_0$*, there exists a unique solution*

$$\varepsilon \in C^1([0,T]).$$

Moreover, the system is dissipative in the sense that the energy functional

$$\mathcal{E}(t) = \frac{\rho}{2}|\dot{\varepsilon}(t)|^2 + \frac{E_1}{2} I^{1-\alpha_1}(\dot{\varepsilon}^2)(t) + \frac{E_2}{2} I^{1-\alpha_2}(\dot{\varepsilon}^2)(t)$$

satisfies

$$\mathcal{E}(t) + \gamma \int_0^t |\dot{\varepsilon}(\tau)|^2 d\tau \le \mathcal{E}(0) + \int_0^t f(\tau)\dot{\varepsilon}(\tau)\, d\tau.$$

In particular, if $f = 0$*, then* $\mathcal{E}(t)$ *is non increasing on* $[0,T]$*.*

Proof. We rewrite the governing equation as

$$\rho \ddot{\varepsilon}(t) + \gamma \dot{\varepsilon}(t) + E_1\, {}^C D_t^{\alpha_1} \varepsilon(t) + E_2\, {}^C D_t^{\alpha_2} \varepsilon(t) = f(t).$$

Step 1: Existence and uniqueness. Using the definition of the Caputo derivative [5, 7, 1],

$${}^C D_t^{\alpha} \varepsilon(t) = I^{1-\alpha} \dot{\varepsilon}(t),$$

the equation can be expressed as a Volterra integro differential equation:

$$\rho \ddot{\varepsilon}(t) + \gamma \dot{\varepsilon}(t) + E_1 I^{1-\alpha_1} \dot{\varepsilon}(t) + E_2 I^{1-\alpha_2} \dot{\varepsilon}(t) = f(t).$$

The fractional integral operators are bounded linear operators from $L^2(0,T)$ into $L^2(0,T)$. Hence the equation is equivalent to a second order equation with memory kernel of convolution type. The Caputo derivatives can be written as convolution operators

$${}^C D_t^{\alpha_i} \varepsilon(t) = \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t-\tau)^{-\alpha_i} \dot{\varepsilon}(\tau)\, d\tau, \quad i = 1, 2.$$

Since the kernels $(t-\tau)^{-\alpha_i}$ are weakly singular and integrable on $(0,t)$ [4, 6], for every $t > 0$ when $0 < \alpha_i < 1$, the governing equation can be rewritten as a linear Volterra integro differential equation with weakly singular memory kernels. Such systems are known to admit unique solutions in $C^1([0,T])$ by classical resolvent kernel theory for Volterra equations.

Step 2: Energy identity. Multiply the governing equation by $\dot{\varepsilon}(t)$ and integrate from 0 to t:

$$\rho \int_0^t \ddot{\varepsilon}\dot{\varepsilon}\, d\tau + \gamma \int_0^t |\dot{\varepsilon}|^2 d\tau + E_1 \int_0^t (I^{1-\alpha_1}\dot{\varepsilon})\dot{\varepsilon}\, d\tau + E_2 \int_0^t (I^{1-\alpha_2}\dot{\varepsilon})\dot{\varepsilon}\, d\tau = \int_0^t f(\tau)\dot{\varepsilon}(\tau)\, d\tau.$$

The first term yields

$$\rho \int_0^t \ddot{\varepsilon}\dot{\varepsilon}\, d\tau = \frac{\rho}{2}|\dot{\varepsilon}(t)|^2 - \frac{\rho}{2}|\dot{\varepsilon}(0)|^2.$$

Using the positivity of fractional integrals,

$$\int_0^t (I^{1-\alpha}\dot{\varepsilon})\dot{\varepsilon}\,d\tau = \frac{1}{2}I^{1-\alpha}(\dot{\varepsilon}^2)(t),$$

we obtain

$$\mathcal{E}(t) - \mathcal{E}(0) + \gamma\int_0^t |\dot{\varepsilon}|^2 d\tau = \int_0^t f(\tau)\dot{\varepsilon}(\tau)\,d\tau,$$

where

$$\mathcal{E}(t) = \frac{\rho}{2}|\dot{\varepsilon}(t)|^2 + \frac{E_1}{2}I^{1-\alpha_1}(\dot{\varepsilon}^2)(t) + \frac{E_2}{2}I^{1-\alpha_2}(\dot{\varepsilon}^2)(t).$$

Step 3: Dissipativity. If $f = 0$, the right hand side vanishes and

$$\mathcal{E}(t) + \gamma\int_0^t |\dot{\varepsilon}|^2 d\tau = \mathcal{E}(0),$$

which shows $\mathcal{E}(t)$ is non increasing. The construction of such fractional energy functionals is consistent with modern stability analysis for fractional systems [2, 4]. Hence the system is dissipative.

□

2.2 Physical Interpretation of the Dissipative Behavior

The energy inequality established in Theorem 2.1 provides a direct physical interpretation of the heterogeneous fractional viscoelastic model. The presence of Caputo fractional derivatives introduces memory dependent damping, which naturally leads to irreversible energy loss over time.

In classical viscoelasticity, damping is typically modeled using integer order derivatives [3, 7], resulting in exponential decay of oscillations. In contrast, the fractional operators in the present model produce a history dependent response, where the rate of energy dissipation is governed by the fractional orders α_1 and α_2. Smaller fractional orders correspond to stronger memory fading and enhanced damping [4, 5], while larger orders retain elastic memory for longer durations.

Theorem 2.1 rigorously confirms that, in the absence of external forcing, the total system energy is non increasing. This establishes the model as intrinsically dissipative and consistent with the thermodynamic behavior of real viscoelastic materials. Consequently, the fractional parameters serve as tunable quantities that regulate the balance between energy storage and energy dissipation in complex polymeric structures.

3 Fractional Order Electrical Circuits

Fractional order circuits incorporate elements like capacitors and inductors with non-integer order impedance, such as $Z(s) = \frac{1}{Cs^\alpha}$, where $0 < \alpha < 1$. These models account for frequency dependent losses in real world components.

3.1 Fractional Circuit Model

We introduce a fractional order RC circuit with a composite impedance:

$$Z(s) = R + \frac{1}{C_1 s^{\alpha_1} + C_2 s^{\alpha_2}}, \quad 0 < \alpha_1, \alpha_2 < 1, \tag{3.1}$$

where R is resistance, and C_1, C_2 are fractional capacitance with orders α_1, α_2. Fractional impedance models of this type are commonly used to represent dielectric and electrochemical memory effects [7, 8].

Theorem 3.1 (BIBO stability of a multi-term fractional RC circuit)**.** *Consider the fractional circuit governed by [2, 12],*

$$V(t) = RI(t) + \frac{1}{C_1} I^{\alpha_1} I(t) + \frac{1}{C_2} I^{\alpha_2} I(t), \qquad 0 < \alpha_1 < \alpha_2 < 1,$$

where I^{α} denotes the Riemann–Liouville fractional integral.

If the input current satisfies $I \in L^{\infty}(0, \infty)$, then the output voltage $V(t)$ is bounded on every finite interval $[0, T]$. More precisely,

$$\|V\|_{L^{\infty}(0,T)} \leq R\|I\|_{L^{\infty}} + \frac{T^{\alpha_1}}{C_1 \Gamma(1+\alpha_1)} \|I\|_{L^{\infty}} + \frac{T^{\alpha_2}}{C_2 \Gamma(1+\alpha_2)} \|I\|_{L^{\infty}}.$$

Furthermore, the system defines a causal bounded linear operator from L^{∞} to $C([0, T])$.

Proof. Assume $I(t)$ is bounded: $|I(t)| \leq M$ for all $t \geq 0$.

The fractional integrals satisfy [7],

$$|I^{\alpha} I(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} I(\tau)\, d\tau \right| \leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau = \frac{Mt^{\alpha}}{\Gamma(1+\alpha)}.$$

Thus,

$$|V(t)| \leq RM + \frac{Mt^{\alpha_1}}{C_1 \Gamma(1+\alpha_1)} + \frac{Mt^{\alpha_2}}{C_2 \Gamma(1+\alpha_2)}.$$

For any finite interval $[0, T]$,

$$\sup_{0 \leq t \leq T} |V(t)| \leq M \left(R + \frac{T^{\alpha_1}}{C_1 \Gamma(1+\alpha_1)} + \frac{T^{\alpha_2}}{C_2 \Gamma(1+\alpha_2)} \right),$$

which proves boundedness.

Linearity and causality follow directly from the convolution form of the fractional integrals. Hence the system is BIBO stable on finite horizons. □

Remark: Although boundedness of fractional integrals is a classical analytical property, its formulation in Theorem 3.1 has an important structural interpretation. The multi-term fractional impedance relation can be viewed as a convolution operator [4, 6] with weakly singular memory kernels acting on the input current. The theorem therefore ensures that such fractional circuit elements define bounded, causal operators on signal spaces over finite time intervals. This guarantees that memory dependent impedance models remain physically realizable and do not introduce artificial energy growth due to nonlocal effects.

3.2 Interpretation of the Fractional Circuit Model

The boundedness result established in Theorem 3.1 provides a theoretical foundation for fractional order circuit elements. Unlike ideal capacitors, fractional capacitors exhibit memory dependent impedance that evolves with time. The theorem guarantees that, for bounded current inputs, the voltage response remains bounded on finite time intervals, ensuring physical realizability and preventing unbounded energy accumulation.

This property confirms that multi-term fractional impedance models are mathematically well posed and suitable for describing lossy dielectric materials and electrochemical systems where nonlocal charge dynamics play a significant role.

4 Fractional Order Control Systems

Fractional order PID controllers, defined as [8, 12]:

$$u(t) = K_p e(t) + K_i D^{-\lambda} e(t) + K_d D^{\mu} e(t), \quad 0 < \lambda, \mu < 1,$$

offer enhanced flexibility in controlling complex systems. We propose a novel adaptive fractional PID controller with dynamic order adjustment.

4.1 Adaptive Fractional PID Controller

The controller adjusts λ and μ based on the error dynamics:

$$\lambda(t) = \lambda_0 + k_1|e(t)|, \quad \mu(t) = \mu_0 + k_2|e(t)|, \tag{4.1}$$

where λ_0, μ_0 are nominal orders, and k_1, k_2 are tuning parameters. Adaptive adjustment of fractional orders has been explored as a mechanism for improving robustness in memory dependent controllers [4, 11].

Theorem 4.1 (Practical stability of an adaptive fractional PID controller)**.** *Consider the linear system [10, 11, 13],*

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad e(t) = r(t) - x(t),$$

and the adaptive fractional PID controller

$$u(t) = K_p e(t) + K_i\, I^{\lambda(t)} e(t) + K_d\, {}^C D_t^{\mu(t)} e(t),$$

where $I^{\lambda(t)}$ denotes the Riemann–Liouville fractional integral of order $\lambda(t)$ and

$$\lambda(t) = \lambda_0 + k_1|e(t)|, \quad \mu(t) = \mu_0 + k_2|e(t)|,$$

with $0 < \lambda_0, \mu_0 < 1$ and $k_1, k_2 > 0$.

Assume (A, B) is controllable, the reference $r(t)$ is bounded, and full state feedback is available. Then there exist controller gains K_p, K_i, K_d such that the closed loop error satisfies

$$\limsup_{t\to\infty} |e(t)| \le \delta,$$

for some $\delta > 0$ depending on k_1, k_2. Hence the system is practically stable and achieves bounded tracking with memory dependent adaptation.

Proof. Let the tracking error be $e(t) = r(t) - x(t)$ with bounded reference $r(t)$. For linear systems with continuous inputs, the state and hence the error remain continuously differentiable on finite intervals; therefore we assume

$$e \in C^1([0, T]).$$

The closed-loop system can be written as

$$\dot{x}(t) = Ax(t) + B\big[K_p e(t) + K_i I^{\lambda(t)} e(t) + K_d\, {}^C D_t^{\mu(t)} e(t)\big].$$

Step 1: Boundedness of adaptive orders. Since $e(t)$ is continuous and bounded on finite intervals,

$$\lambda(t) = \lambda_0 + k_1|e(t)|, \quad \mu(t) = \mu_0 + k_2|e(t)|$$

remain in a compact subset of $(0, 1)$.

Step 2: Bounded control signal. Since fractional integral and Caputo derivative operators of order less than one map $C^1([0, T])$ functions into bounded functions on $[0, T]$ [2, 6] (by continuity of weakly singular convolution operators), there exist constants $C_1, C_2 > 0$ such that for each $t \in [0, T]$,

$$|I^{\lambda(t)} e(t)| \le C_1 \sup_{0\le\tau\le t} |e(\tau)|, \qquad |{}^C D_t^{\mu(t)} e(t)| \le C_2 \sup_{0\le\tau\le t} |e(\tau)|.$$

Hence, for some constant $C_3 > 0$,

$$|u(t)| \le C_3 \sup_{0\le\tau\le t} |e(\tau)|.$$

Step 3: Practical stability. Because (A, B) is controllable, choose K_p such that $A - BK_p$ is Hurwitz. Define

$$\Delta(t) = K_i I^{\lambda(t)} e(t) + K_d\, {}^C D_t^{\mu(t)} e(t).$$

Then the closed-loop dynamics can be written as

$$\dot{x}(t) = (A - BK_p)x(t) + B\Delta(t),$$

where $|\Delta(t)| \le C_4 \sup_{0\le\tau\le t} |e(\tau)|$ for some constant $C_4 > 0$.

Taking norms and using the bound on $\Delta(t)$ gives

$$|x(t)| \le Me^{-\eta t}|x(0)| + M\int_0^t e^{-\eta(t-s)}|\Delta(s)|\, ds,$$

for some $M, \eta > 0$. Using the bound on $\Delta(t)$ and the boundedness of $r(t)$, the integral term can be estimated by a constant depending on k_1, k_2. Applying the variation of constants formula and estimating the perturbation term using Grönwall's inequality yields [6, 10],

$$|e(t)| \le \beta e^{-\eta t}|x(0)| + \delta,$$

where δ depends on k_1, k_2. Therefore,

$$\limsup_{t\to\infty} |e(t)| \le \delta,$$

which proves practical stability. □

Remark 4.1. The use of practical stability in Theorem 4.1 is natural in the context of fractional and adaptive control systems. Fractional operators introduce long memory effects, while the adaptive laws make the controller parameters time varying. These features generally prevent strict asymptotic convergence to zero from being guaranteed under realistic assumptions. Instead, ensuring that the tracking error remains confined within a small neighborhood of the origin provides a more appropriate and physically meaningful stability notion. The result therefore reflects the inherent nonlocal dynamics of fractional systems while still guaranteeing reliable bounded performance.

4.2 Control Interpretation of the Practical Stability Result

Theorem 4.1 establishes practical stability for linear systems controlled by an adaptive fractional order PID structure. In the context of motor speed regulation, the state vector represents the electromechanical dynamics of the motor, while the control input corresponds to the applied voltage or torque signal.

The adaptive fractional orders $\lambda(t)$ and $\mu(t)$ introduce memory dependent correction terms [8, 4], that adjust according to the tracking error. This mechanism enhances transient performance by allowing stronger corrective action during large deviations and smoother behavior near steady state.

The theorem guarantees that, provided the plant is controllable and the reference signal is bounded, the tracking error remains confined within a small neighborhood of zero. This ensures reliable speed regulation without the risk of instability caused by parameter variation or external disturbances.

Thus, the adaptive fractional controller offers a mathematically justified framework for improving robustness and flexibility in motor control applications, while maintaining guaranteed bounded performance.

5 Conclusion and Future Scope

This study established a mathematically grounded framework for analyzing engineering systems governed by fractional dynamics. By focusing on the structural properties of multi-term fractional operators, the work clarified how memory effects influence stability, boundedness, and energy evolution in viscoelastic, electrical, and control models. The developed results show that fractional formulations can be rigorously analyzed within classical functional analysis and system theory, providing dependable foundations for nonlocal modeling.

Future research may extend these results to nonlinear fractional systems, distributed parameter models, and variable order operators where memory effects evolve with time or state. Another promising direction involves coupling the present theoretical framework with high accuracy numerical schemes to bridge analysis and simulation. Such developments could broaden the applicability of fractional models in advanced materials, smart structures, and adaptive control technologies.

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