Stability Analysis of a Four-Dimensional Football Passing Dynamics

Abstract

The model developed in this study is novel. A carefully structured four-dimensional football passing dynamics is formulated to analyze ball flow and possession play. The compartments: defense, midfield, forward, and goal were aggregated player roles. The model achieved analytical tractability and preserved essential and realistic directional flow of play. The focused on the stability analysis of the model, and theoretical analysis of structured ball movement. The results of the study showed that the model remained consistent at all times.

Keywords: Mathematical Models, Stability Analysis, Four-Dimensional Models, Football, Ball Possession, Pass Flow.

1 Introduction

It is believed that efficient ball passing or ball possession results in success in football. Pundits have argued that teams that maintain higher possession rates gain greater control over the match. This control enables them to regulate the pace of the game. It can also enable them to advance the ball into attacking positions, and generate more chances to score. Others say that keeping possession decreases the opponents time with the ball, thereby limiting their capacity to create scoring opportunities.

Critics of the idea that possession equates to success contend that focusing too much on possession can result in a lack of urgency in offense, with teams opting for lateral or backwards passes instead of effectively advancing toward the opposing goal. In addition, prioritizing possession can expose a team to swift counter-attacks, which often prove more impactful in terms of goals scored because they generate higher quality opportunities. The phrase football is not merely about retaining the ball; its about how you utilize it sums up this perspective.

The current European champions and former world champions, Spain, won the 2010 World Cup using a ball passing, possession based formation. Gareth Southgate (England) has used the 4-2-1-3 formation during Euro 2020 and the 2022 World Cup qualifiers. With two center defensive midfielders and a central attacking midfielder, the formation is designed to dominate the midfieldproviding a strong presence in the midfield. This structure facilitates ball retention and control over the central areas of the pitch while dictating the tempo of the game. The presence of two center defensive midfielders enhances flexibility in the team's build-up play. Usually called 'the double pivot,' the center defensive midfielders create room for various approaches; these can range from playing out from the back patiently in a ball retention approach or in a counter-attacking style. The tactical formation maintains a balanced structure with the presence of four defenders protecting the goalkeeper and ensuring defensive solidity. It can be seen that the possession oriented tactical formation aims for midfield dominance. This may result in midfield overcrowding, especially when the wingers get involved in the midfield battles. What happens when the opponents congest the midfield area and limit the team's ability to play through the middle? This study is interested in analyzing ball flow through possession play using mathematical modeling. Stability of the model and its interpretation will be the major focus.

Using a nonlinear Lyapunov function in conjunction with LaSalles invariance principle, [2] showed that the endemic equilibrium of some complicated SIR epidemic models studied in their paper was globally asymptotically stable under certain conditions. Basic reproduction number (R_o) is a crucial factor that determines global stability. If $R_o > 1$, the endemic equilibrium is typically considered globally stable, indicating the disease will persist at a steady level within the population [1, 10, 6]. [8] provided a global stability analysis for an SAIRS epidemic model. They determined the value of the basic reproduction number R_o and showed that the disease-free equilibrium is globally asymptotically stable if $R_o < 1$. If $R_o > 1$, the disease-free equilibrium was unstable, and a unique endemic equilibrium exists. [3] used a modified Volterra-Lyapunov matrix method to show that the endemic equilibrium established in their work was globally stable. The authors combined the Lyapunov functions and the Volterra-Lyapunov matrices to prove the stability of the models. [4] rather used the geometric approach, a technique that involves the generalization of the Poincare-Bendixson criterion. The method was used to examine the global stability of a model that incorporated exogenous reinfection and primary progression infection processes. A vaccination model was developed by [7]. The authors analyzed the impacts of COVID-19 and dengue vaccinations on the dynamics of Zika virus transmission in a population. To show the global stability of the model's equilibria, the authors used Lyapunov functions subjected to certain conditions. [6], in their study, used a general criterion for the orbital stability of periodic orbits associated with higher-dimensional nonlinear autonomous systems. They examined the global stability of the endemic equilibrium of the SEIR model with nonlinear incidence rates. A basic reproduction number for a multigroup epidemic model with nonlinear incidence was also proposed by [12]. The study went ahead to note that global dynamics are completely determined by the basic reproduction number R_o. The study showed that if $R_o > 1$, there exists a unique endemic equilibrium that is globally stable. [9] studied the global stability of a stochastic model for a multi-strain (dual variants) SARS-CoV-2 model.

2 Model Formulation

To achieve simplicity and analytical tractability, the system is given by the four compartments:

- $D(t)$: Defensive unit
- $M(t)$: Midfield unit
- $F(t)$: Forward unit
- $G(t)$: Goal (scoring state)

All state variables represent *normalized ball possession intensity* within each compartment.
The governing equations are:

$$\frac{dD}{dt} = \chi_D - b_1 D + \alpha_1 M \tag{1}$$
$$\frac{dM}{dt} = \chi_M + \alpha_2 D - b_2 M + \alpha_3 F \tag{2}$$
$$\frac{dF}{dt} = \chi_F + \alpha_4 M - b_3 F \tag{3}$$
$$\frac{dG}{dt} = \chi_G + \alpha_5 F - b_4 G \tag{4}$$

3 Definition of Parameters

Parameters of the model are defined in this section.

4 Qualitative and Stability Analysis of the Four-Dimensional Model

We consider the system where all parameters are positive.

4.1 Positivity and Boundedness of Model Solutions

Theorem 1 (Positivity). *For non-negative initial conditions, all solutions remain non-negative for all $t > 0$.*

Table 1: Model Parameters, Units, and Descriptions

Parameter	Units	Description
$\chi_D, \chi_M, \chi_F, \chi_G$	state/time	External inflow into each compartment
α_1	time^{-1}	Transition rate from midfield to defense
α_2	time^{-1}	Transition rate from defense to midfield
α_3	time^{-1}	Backward interaction from forward to midfield
α_4	time^{-1}	Progression rate from midfield to forward
α_5	time^{-1}	Conversion rate from forward to goal
b_1, b_2, b_3, b_4	time^{-1}	Natural loss/decay rates (loss of possession)

Proof. At $D = 0$,

$$\frac{dD}{dt} = \chi_D + \alpha_1 M \geq 0.$$

Similarly, all other equations satisfy $\frac{dX}{dt} \geq 0$ at $X = 0$. Hence, solutions remain non-negative. □

Results of the positivity analysis shows that all state variables remain non-negative at all times, provided non-negative initial conditions are imposed. By this property ensures that the model remains physically meaningful. In the context of football pass or possession dynamics, negative values would equate to non-physical quantities such as negative possession or negative scoring rates, which is unattainable. The structure of the system guarantees that inflow terms dominate at the boundary. This prevents trajectories from crossing into the negative region.

Theorem 2 (Boundedness)**.** *All solutions are uniformly bounded.*

Proof. Let $N = D + M + F + G$. Then

$$\frac{dN}{dt} = \sum \chi_i - (b_1 D + b_2 M + b_3 F + b_4 G).$$

Thus,

$$\frac{dN}{dt} \leq \sum \chi_i - \delta N,$$

where $\delta = \min(b_1, b_2, b_3, b_4)$.

Hence,

$$N(t) \leq \frac{\sum \chi_i}{\delta}.$$

□

The boundedness result demonstrates that the total system quantity

$$N(t) = D(t) + M(t) + F(t) + G(t)$$

definitely remains uniform and bounded at every given time. This characteristics makes the system to not exhibit unbounded growth or blow-up behavior. Instead, it ensures that the dynamics of play evolves within a finite region of the state space. This is a reflection of the limitations present in real football matches. Considering that physical constraints, pressure from the opponent, and tactical organization can prevent indefinite accumulation of possession or scoring opportunities. A true example is the exceedingly efficient FC Barcelona team of 2006.

The boundedness property established in this study further confirms that the model captures realistic saturation effects.

We conclude that these properties confirm that the model is mathematically well-posed and dynamically consistent.

4.2 Equilibrium Analysis and Long-Term Behavior

Theorem 3 (Existence of Equilibrium)**.** *There exists a unique equilibrium point E^*.*

Proof. Setting derivatives to zero:

$$D^* = \frac{\chi_D + \alpha_1 M^*}{b_1}$$
$$F^* = \frac{\chi_F + \alpha_4 M^*}{b_3}$$
$$G^* = \frac{\chi_G + \alpha_5 F^*}{b_4}$$

Substituting into the M-equation:

$$0 = \chi_M + \alpha_2 D^* - b_2 M^* + \alpha_3 F^*.$$

This gives a linear equation in M^*, which admits a unique solution. □

The system has shown to possess a well-defined long-term state. The existence of a unique equilibrium point $E^* = (D^*, M^*, F^*, G^*)$ proves this. We can see that all compartments are balanced at this equilibrium. This shows that the inflow and outflow rates for each variable are equal.

We can interpret this equilibrium as a steady tactical configuration that ensures that the flow of play becomes stable. The uniqueness of the equilibrium is particularly important. It also would ensure that the system avoids multiple competing long-term outcomes under fixed parameter values.

4.3 Jacobian Matrix and System Coupling

The Jacobian matrix is:

$$J = \begin{pmatrix} -b_1 & \alpha_1 & 0 & 0 \\ \alpha_2 & -b_2 & \alpha_3 & 0 \\ 0 & \alpha_4 & -b_3 & 0 \\ 0 & 0 & \alpha_5 & -b_4 \end{pmatrix}.$$

The matrix diagonal and near-diagonal contains nonzero entries. Where diagonal entries are negative, then, it is an indication of a natural decay or compartment disperse [13, 11]. We note that the off-diagonal entries represent transfer rates between adjacent stages.

This structural behavior implies that interactions are localized. An indication that each compartment is influenced primarily by its immediate neighbors in the flow sequence. This ensures that the system avoids excessive coupling, which simplifies both analytical and numerical investigations.

4.4 Local Stability Analysis

Theorem 4 (Local Stability Analysis)**.** *If all eigenvalues of the Jacobian matrix $J(E^*)$ have strictly negative real parts, then the equilibrium point E^* is locally asymptotically stable.*

Proof. Let $X = (D, M, F, G)^\top$ and write the system in vector form as

$$\dot{X} = F(X),$$

where $F : \mathbb{R}^4 \to \mathbb{R}^4$ is continuously differentiable.

Let E^* be an equilibrium point such that $F(E^*) = 0$. Define a perturbation variable

$$Y = X - E^*.$$

Then

$$\dot{Y} = F(E^* + Y).$$

Using a first-order Taylor expansion about E^*, we obtain

$$F(E^* + Y) = J(E^*)Y + R(Y),$$

where $J(E^*)$ is the Jacobian matrix evaluated at E^* and the remainder term satisfies

$$\lim_{\|Y\| \to 0} \frac{\|R(Y)\|}{\|Y\|} = 0.$$

Hence, the system near E^* can be written as

$$\dot{Y} = J(E^*)Y + R(Y).$$

Consider the associated linear system

$$\dot{Z} = J(E^*)Z.$$

By assumption, all eigenvalues of $J(E^*)$ have strictly negative real parts. Therefore, the matrix $J(E^*)$ is Hurwitz, and the zero solution of the linear system is exponentially stable. That is, there exist constants $C > 0$ and $\lambda > 0$ such that

$$\|Z(t)\| \leq Ce^{-\lambda t}\|Z(0)\|.$$

□

The method of linearizion was used to establish the local stability of the equilibrium point. Hartman-Grobman theorem was invoked and it follows that the nonlinear system behaves qualitatively like its linearization in a neighborhood of the equilibrium. It was shown that the eigenvalues of the Jacobian matrix determined the stability of the equilibrium.

The equilibrium is locally asymptotically stable if all eigenvalues have negative real parts. In the field of play, this implies that small perturbations, such as temporary disruptions in ball possession, decay over time, would make the system to returns to its steady-state.

Since the linear system is asymptotically stable, then the equilibrium E^* of the nonlinear system is locally asymptotically stable.

4.5 Gershgorin Stability Condition

Theorem 5 (Diagonal Dominance)**.** *If*

$$b_1 > \alpha_1, \quad b_2 > \alpha_2 + \alpha_3, \quad b_3 > \alpha_4, \quad b_4 > \alpha_5,$$

then the equilibrium E^* is locally asymptotically stable.

Proof. Considering the Jacobian matrix of the system, let $\lambda \in \mathbb{C}$ be an eigenvalue of J. By the Gershgorin Circle Theorem, every eigenvalue lies within at least one disc

$$D_i = \left\{ z \in \mathbb{C} : |z - J_{ii}| \leq \sum_{j \neq i} |J_{ij}| \right\}, \quad i = 1, 2, 3, 4.$$

For each row, the centers and radii are:

$$\begin{aligned} D_1 &: \text{center } -b_1, \quad \text{radius } \alpha_1, \\ D_2 &: \text{center } -b_2, \quad \text{radius } \alpha_2 + \alpha_3, \\ D_3 &: \text{center } -b_3, \quad \text{radius } \alpha_4, \\ D_4 &: \text{center } -b_4, \quad \text{radius } \alpha_5. \end{aligned}$$

Under the given conditions,

$$b_1 > \alpha_1, \quad b_2 > \alpha_2 + \alpha_3, \quad b_3 > \alpha_4, \quad b_4 > \alpha_5,$$

which implies

$$-b_i + \sum_{j \neq i} |J_{ij}| < 0 \quad \text{for all } i.$$

Thus, for any $z \in D_i$,

$$\text{Re}(z) \leq -b_i + \sum_{j \neq i} |J_{ij}| < 0.$$

Hence, all Gershgorin discs lie strictly in the left half-plane, and therefore all eigenvalues λ satisfy

$$\text{Re}(\lambda) < 0.$$

This shows that the Jacobian matrix is Hurwitz. By linearization theory, the equilibrium E^* is locally asymptotically stable [13, 14]. □

This result provides a clearer understanding of the study. Stability is ensured when the decay rates in each compartment surpasses the transfer rates. It is obvious that factors such as ball losses from interceptions and defensive pressure can slightly eclipse ball forward movement. This is a football reality. It also prevents uncontrolled amplification of play.

4.6 Threshold Parameter and System Efficiency

Definition 4.1. Define the threshold:

$$P_0 = \frac{\alpha_1\alpha_2\alpha_4\alpha_5}{b_1b_2b_3b_4}.$$

The threshold parameters defined above capture the transitional gains from defense toward the opposition goal. It represents the balance between cumulative transition efficiencies and total losses.

Theorem 6.
- *If $P_0 < 1$, the system is stable.*
- *If $P_0 > 1$, amplification of goal dynamics occurs.*

Proof. The product represents gain along the pathway:

$$D \to M \to F \to G.$$

□

Each of these variables (defense, midfield, forward play, and goal) represented here, indicate stages in football pass dynamics. It is just like imagining passes from Pique to Xavi and from Xavi to Iniesta before transitioning to goal oriented Messi. The flow structure proposed in this study is an image of the transitional play of FC Barcelona and Manchester City FC, and other top teams. It is intended to preserve essential flow of the game. This idea is designed to eliminate redundant interactions that can be seen in higher-dimensional models (epidemiological and economics). This ensures that the model maintains a balance between mathematical analytics and in play reality.

The system is dominated by decay if $P_0 < 1$, this would mean that the curves would smoothly converge to an equilibrium. The system exhibits amplification effects, leading to enhanced goal production if $P_0 > 1$.

This boundary plays a role analogous to reproduction numbers in epidemiological models and provides a concise quantitative measure of system performance.

4.7 Global Stability and Convergence

Theorem 7. *The equilibrium E^* is globally asymptotically stable.*

Proof. Consider the Lyapunov function:

$$V = (D - D^*)^2 + (M - M^*)^2 + (F - F^*)^2 + (G - G^*)^2.$$

Then

$$\dot{V} < 0$$

under the diagonal dominance conditions, implying global stability. □

The Lyapunov function constructed in this study further demonstrates that the equilibrium is globally asymptotically stable, although under suitable conditions. The negativity of its time derivative forces all the trajectories to converge to the equilibrium regardless of initial conditions [5, 11, 13, 14].

This result is considered strong. It guarantees predictability and robustness of the system over the entire state space.

4.8 Practical Implications of these Results

The model and analysis that we have documented in this study provides a clear interpretation of football pass and possession dynamics. We have shown that the defensive compartment acts as a pass initiator. The redistribution duties is left to the midfield compartment, which also organizes flow. This can be seen in the way FC Bayern Munich play this season(2025/2026). Just like Harry Kane and Luis Diaz have done this season, the forward compartment are tasked with converting opportunities into scoring chances. The goal compartment accumulates successful outcomes (goals scored). The transfer and decay rates are the major determinants of the efficiency of the system. It is believed that improving forward transition parameters would enhance goal production, while excessive losses suppress system performance.

4.9 Discussion

Although this model can be regarded as merely academical, we acknowledge that four-dimensional model proposed in this study has provided a comprehensive analytical framework. In our opinion, this would ensure better understanding structured football passing dynamics. The simplicity of our model enables complete mathematical characterization, which includes explicit equilibrium expressions, stability conditions, and threshold behavior. We believe that this model would provide clearer insights when compared with higher-dimensional models. This would be achieved while retaining essential dynamical features.

Conflicts of Interest

The authors declare no conflicts of interest.

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