Fractional Complex Calculus and Analytic Structures on Nonlocal Complex Domains

Original Research Paper

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Abstract

This work establishes a nonlocal analytic framework in the complex plane by introducing a symmetric fractional complex derivative constructed through symmetric complex increments. The proposed operator differs from classical Riemann—Liouville and Caputo derivatives and preserves rotational compatibility and phase–consistent scaling in the complex plane. Using this framework, we derive fractional analogues of Cauchy conditions, a nonlocal Cauchy—Pompeiu identity, and a fractional Liouville theorem, together with a Laurent-–Mittag-–Leffler type expansion. The analysis is developed using symmetric increment techniques and fractional contour integral representations. As a proof of concept, a fractional harmonic potential model is examined, demonstrating memory-driven deformation in analytic flows. The framework provides a mathematical foundation for fractional holomorphic geometry, memory–influenced signal processing, and nonlocal complex PDE models.

Keywords: Fractional Complex Analysis; Nonlocal Derivative; Fractional Holomorphicity; Mittag–Leffler Series; Fractional Conformal Maps.

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1 Introduction

Complex analysis relies on locality: infinitesimal ratios determine analyticity, and values inside a domain hinge only on its boundary. Fractional calculus abandons strict locality and introduces memory; incorporating this feature into complex analysis requires more than simply applying real fractional derivatives to real and imaginary parts [4, 8, 10, 11]. A coherent nonlocal complex theory must maintain phase compatibility, rotational symmetry, and direction-independence such as [3, 16, 17, 18].

This paper constructs a complex fractional derivative through symmetric complex increments, providing a natural memory-dependent extension of the classical derivative while preserving complex scaling laws. Fractional Cauchy-type conditions, nonlocal integral identities, and fractional Laurent–Mittag–Leffler expansions follow as consequences. The developed tools illustrate how memory modifies analytic structure, producing fractional conformality and spectrum-like restrictions on bounded functions.

In the absence of symmetric complex increments, fractional derivatives depend on preferred directions, causing a breakdown of rotational invariance. As a consequence, classical pillars such as the Cauchy–Riemann conditions, path-independent contour integrals, and Cauchy integral representations fail to persist in a consistent fractional setting.

Motivation

Classical complex analysis rests on purely local behavior: infinitesimal changes around a point determine smoothness, analyticity, and harmonicity. This framework is immensely powerful, but it implicitly assumes that the system has no dependence on its past states. Fractional calculus, in contrast, is defined by nonlocality and memory, yet its conventional formulations act along real directions and do not respect the geometry of the complex plane. A direct application of real fractional derivatives to the real and imaginary components destroys rotational symmetry and interrupts the phase structure inherent to analytic functions.

To construct a genuine fractional complex theory, the derivative must respond uniformly to rotations and preserve the intrinsic scaling law of the complex line. This requirement naturally leads to symmetric complex increments. By averaging forward and backward complex displacements in all directions, the resulting operator maintains phase consistency, removes directional bias, and seamlessly accommodates memory effects within an analytic framework. In this sense, the symmetric fractional complex derivative serves as a bridge between nonlocal calculus and holomorphic structure, laying the foundation for a memory-aware extension of complex analytic geometry.

2 The Symmetric Fractional Analytic Framework

Throughout, fix $\alpha \in (0,1)$. For $h \in \mathbb{C} \setminus \{0\}$ write $h = \rho e^{i\theta}$ with $\rho > 0$ and $\theta \in (-\pi, \pi]$, and use the principal branch for $h^\alpha = \rho^\alpha e^{i\alpha\theta}$. For $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, define the *symmetric complex fractional increment* [14]:

$$\Delta_h^\alpha f(z) := \frac{f(z+h) - f(z-h)}{2\, h^\alpha}. \tag{2.1}$$

We say $D_z^\alpha f(z)$ exists if $\lim_{\rho\to 0} \Delta^\alpha_{\rho e^{i\theta}} f(z)$ exists *uniformly* in $\theta \in [0, 2\pi)$.

Function spaces. We work with $C^{0,\gamma}(\Omega)$ Hölder spaces [13], $\gamma \in (0,1]$, and assume $f \in C^{0,\gamma}$ with $\gamma > \alpha$ when needed so that the remainders in fractional expansions are controllable. For two-variable real fractional limits we use the symmetric 1D difference along x and y:

$$\delta_h^\alpha u(x,y) := \frac{u(x+h,y) - u(x-h,y)}{2\,|h|^\alpha},$$

and

$$\delta_k^\alpha u(x,y) := \frac{u(x,y+k) - u(x,y-k)}{2\,|k|^\alpha}.$$

When limits as $h \to 0$ (resp. $k \to 0$) exist, we denote them by $D_x^\alpha u$ (resp. $D_y^\alpha u$).

Rotation covariance. If $R_\phi(z) = e^{i\phi}z$ and $g = f \circ R_\phi$, then

$$\Delta_h^\alpha g(z) = \frac{f(e^{i\phi}(z+h)) - f(e^{i\phi}(z-h))}{2h^\alpha} = \Delta_{e^{i\phi}h}^\alpha f(e^{i\phi}z).$$

Uniformity in θ yields $D_z^\alpha g(z) = e^{i\alpha\phi} D_z^\alpha f(e^{i\phi}z)$, showing the operator has the correct phase scaling.

2.1 Fractional complex derivative

Let $\alpha \in (0,1)$. For $f : \mathbb{C} \to \mathbb{C}$, define the symmetric fractional increment operator

$$\Delta_\epsilon^\alpha f(z) = \frac{f(z + \epsilon e^{i\theta}) - f(z - \epsilon e^{i\theta})}{2\,\epsilon^\alpha e^{i\alpha\theta}}, \qquad \theta \in [0, 2\pi).$$

The directional dependence cancels in the limit for analytic functions.

Definition 2.1 (Fractional complex derivative)**.** A function f admits a *fractional complex derivative of order* α at z if the limit

$$D_z^\alpha f(z) = \lim_{\epsilon \to 0} \Delta_\epsilon^\alpha f(z)$$

exists uniformly in θ. We call f *fractionally analytic* if $D_z^\alpha f(z)$ exists on an open set.

Unlike classical Riemann–Liouville and Caputo fractional derivatives, which are defined through fractional integrals along the real axis, the operator introduced in this work is constructed using symmetric complex increments in the complex plane. This formulation preserves rotational covariance and phase-consistent scaling, which are intrinsic properties of complex analytic structures. In contrast, applying fractional derivatives separately to the real and imaginary components typically breaks rotational symmetry and fails to maintain the geometric consistency required for complex analytic behavior.

Method	**Definition Basis**	**Rotation Invariance**	**Kernel Type**
Riemann–Liouville	Real fractional integral	No	Singular kernel
Caputo	Real fractional derivative	No	Singular kernel
Proposed operator	Symmetric complex increments	Yes	Fractional complex kernel

Table 1: Comparison between classical fractional derivatives and the proposed symmetric fractional complex derivative.

This operator reduces to the classical derivative at $\alpha = 1$ and yields nonlocal memory for $\alpha < 1$.

Figure 1 illustrates the geometric idea behind the symmetric complex increment used to define the fractional complex derivative. Starting from a point z in the complex plane, we consider a complex step $h = \rho e^{i\theta}$ of magnitude ρ and direction θ. The two points $z + h$ and $z - h$ are symmetric with respect to z, and the line segments joining them to z represent the forward and backward complex displacements. The angle θ denotes the rotation applied to the increment, while ρ determines its length.

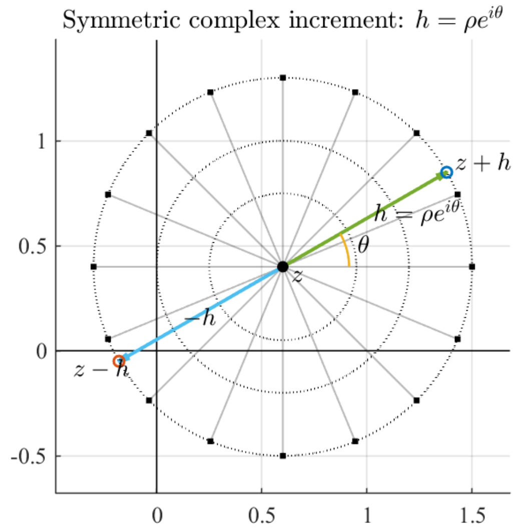

Figure 1: Geometric construction of symmetric complex increment used in the fractional derivative.

To emphasize that the fractional limit must exist uniformly in all directions, multiple rays are drawn through z, and concentric circles indicate the behavior as $\rho \to 0$. This symmetry forces the operator to respond consistently to rotations and ensures that its value does not depend on a preferred direction in the plane. Such a construction preserves the phase structure and rotational invariance of complex analysis while introducing the nonlocal memory behaviour characteristic of fractional calculus. In this way, the symmetric increment framework allows memory effects to be incorporated without breaking the geometric properties that define classical holomorphy.

2.2 Fractional Cauchy conditions

Theorem 2.1 (Fractional Cauchy conditions)**.** *Let $f = u + iv \in C^{0,\gamma}(\Omega)$ with $\gamma > \alpha$ and suppose $D_z^\alpha f$ exists on Ω. Then the real fractional limits satisfy*

$$D_x^\alpha u = D_y^\alpha v, \qquad D_y^\alpha u = -D_x^\alpha v. \tag{2.2}$$

Conversely, if $u, v \in C^{0,\gamma}(\Omega)$ with $\gamma > \alpha$ satisfy (2.2) *and the limits D_x^α, D_y^α exist, then $D_z^\alpha f$ exists uniformly in direction.*

Proof. Necessity. Fix $z = (x, y)$ and a direction θ. For small $\rho > 0$,

$$\Delta^\alpha_{\rho e^{i\theta}} f(z) = \frac{f(x + \rho\cos\theta + i(y + \rho\sin\theta)) - f(x - \rho\cos\theta + i(y - \rho\sin\theta))}{2\rho^\alpha e^{i\alpha\theta}}.$$

Write real/imaginary parts and apply the 1D symmetric fractional expansion along the line $t \mapsto z + te^{i\theta}$:

$$f(z + te^{i\theta}) = f(z) + t^\alpha e^{i\alpha\theta}\Big(a(z) + o(1)\Big),$$

for some complex number $a(z)$ (the putative fractional derivative [6, 15, 7]), with $o(1) \to 0$ as $t \to 0$ uniformly in θ by assumption. Substituting $t = \pm\rho$ gives

$$\Delta^\alpha_{\rho e^{i\theta}} f(z) = a(z) + o(1).$$

Taking real and imaginary parts and comparing with the expansions along pure x- and pure y-directions (i.e. $\theta = 0$ and $\theta = \frac{\pi}{2}$) yields

$$\Re a(z) = D_x^\alpha u(x,y) = D_y^\alpha v(x,y),$$
$$\Im a(z) = D_x^\alpha v(x,y) = -D_y^\alpha u(x,y).$$

This proves (2.2).

Sufficiency. Assume (2.2) holds and D_x^α, D_y^α exist. Along any direction θ we write the first-order symmetric fractional expansion using 1D fractional Taylor along that direction [13, 1]:

$$f(z + \rho e^{i\theta}) - f(z - \rho e^{i\theta}) = 2\rho^\alpha e^{i\alpha\theta}\Big(\Re a(z) + i\,\Im a(z)\Big) + r_\rho(\theta),$$

where, by Hölder continuity [13] with exponent $\gamma > \alpha$, $|r_\rho(\theta)| \le C\rho^\gamma$ uniformly in θ. With

$$\Re a(z) = D_x^\alpha u = D_y^\alpha v, \quad \Im a(z) = D_x^\alpha v = -D_y^\alpha u,$$

the coefficient $a(z)$ is independent of θ. Dividing by $2\rho^\alpha e^{i\alpha\theta}$ and letting $\rho \to 0$ shows the uniform limit exists and equals $a(z)$. ☐

3 Fundamental Integral Identities and Series

3.1 Fractional contour integral and Green identity

Define the fractional contour integral along a rectifiable curve Γ with parametrization $\gamma : [0,1] \to \mathbb{C}$ by:

$$\int_\Gamma f(\zeta)\,(d\zeta)^\alpha := \lim_{\|\mathcal{P}\|\to 0} \sum_j f(\gamma(t_j))\big(\gamma(t_j) - \gamma(t_{j-1})\big)^\alpha,$$

where the principal branch is used and $\|\mathcal{P}\|$ is the mesh of the partition.

We require a nonlocal Green identity [5, 9] tailored to symmetric increments.

Lemma 3.1 (Fractional Green identity). *Let $\mathbb{R} \subset \mathbb{R}^2$ be a rectangle, and let $p, q \in C^{0,\gamma}(R)$ with $\gamma > \alpha$. Then*

$$\int_{\partial R} \big(p\,dx^\alpha + q\,dy^\alpha\big) = \iint_R \Big(D_y^\alpha p - D_x^\alpha q\Big)\,dx\,dy,$$

where dx^α denotes the fractional line element associated with the symmetric increment in the x-direction, and similarly for dy^α. The boundary integral is taken using the induced orientation.

Proof. The proof is constructed by dividing the rectangle R into small grid cells and approximating the double integral by a Riemann sum. The core of the derivation is demonstrating that the contributions from the interior grid points cancel asymptotically.

The Symmetric Difference and Telescoping in 1D:

We examine the mechanism of cancellation for the term $\iint_R D_x^\alpha q \; dx \; dy$. The definition of the symmetric fractional derivative D_x^α ensures that each interior point x_i is connected symmetrically to $x_i + \Delta x$ and $x_i - \Delta x$.

Consider the corresponding sum in the x-direction for a fixed y_j:

$$\sum_i D_x^\alpha q(x_i, y_j)\Delta x \approx \sum_i \left(\frac{q(x_i + \Delta x, y_j) - q(x_i - \Delta x, y_j)}{2|\Delta x|^\alpha}\right)\Delta x$$

If we let $C_\alpha = \frac{\Delta x}{2|\Delta x|^\alpha}$, the sum becomes:

$$C_\alpha \sum_{i=1}^{N-1} (q_{i+1} - q_{i-1})$$

where $q_i = q(x_i, y_j)$. The summation $\sum_{i=1}^{N-1}(q_{i+1} - q_{i-1})$ is a standard, non-fractional telescoping sum:

$$\sum_{i=1}^{N-1}(q_{i+1} - q_{i-1}) = (q_2 - q_0) + (q_3 - q_1) + (q_4 - q_2) + \cdots + (q_N - q_{N-2})$$

The interior terms cancel in pairs (q_2 cancels with $-q_2$, q_3 with $-q_3$, etc.), leaving only terms near the boundaries:

$$\sum_{i=1}^{N-1}(q_{i+1} - q_{i-1}) = q_N + q_{N-1} - q_1 - q_0$$

In the continuous limit, this sum becomes an integral over the boundary points of the interval $[a, b]$, where $a = x_0$ and $b = x_N$.

Nonlocal Telescoping Property:

The key insight is that the term $D_x^\alpha q(x, y)$ behaves as an *exact fractional differential* when integrated over the domain, specifically because the symmetric definition ensures that the integral of the derivative $\iint_R D_x^\alpha q \; dx \; dy$ is perfectly balanced in the interior. In the limit, this reduces the double sum to a single sum along the boundary curves ∂R.

Reduction to Boundary Integral:

As the mesh size $||\mathcal{P}|| \to 0$, the sum of the $D_x^\alpha q$ terms over R reduces to an integral over the boundaries defined by $x =$ const (the vertical sides of the rectangle ∂R_x):

$$\iint_R D_x^\alpha q \; dx \; dy = \int_{\partial R_x} q \; dy^\alpha + O(||\mathcal{P}||^{\gamma-\alpha})$$

Similarly, the integral of $D_y^\alpha p$ reduces to an integral over the boundaries defined by $y =$ const (the horizontal sides ∂R_y):

$$\iint_R D_y^\alpha p \; dx \; dy = \int_{\partial R_y} p \; dx^\alpha + O(||\mathcal{P}||^{\gamma-\alpha})$$

Substituting these into the original identity and accounting for the correct orientation along ∂R (which dictates the signs), we obtain:

$$\iint_R (D_y^\alpha p - D_x^\alpha q) dx \; dy = \int_{\partial R} (p \; dx^\alpha + q \; dy^\alpha)$$

The remainder terms vanish as the mesh size goes to zero since $O(||\mathcal{P}||^{\gamma-\alpha}) \to 0$ given the necessary smoothness condition $\gamma > \alpha$. ☐

3.2 Fractional Cauchy–Pompeiu identity

Theorem 3.2 (Fractional Cauchy–Pompeiu)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected Lipschitz domain, and $f \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma > \alpha$ be fractionally analytic on Ω in the sense of Theorem 2.1. Then for every $z \in \Omega$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^\alpha} (d\zeta)^\alpha. \tag{3.1}$$

Proof. Fix $z \in \Omega$ and $\varepsilon > 0$ small so that $\overline{B_\varepsilon(z)} \subset \Omega$. Let $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(z)}$ with boundary $\partial\Omega_\varepsilon = \partial\Omega \cup (-\partial B_\varepsilon)$. Consider the fractional 1-form

$$\omega(\zeta) := \frac{f(\zeta)}{(\zeta - z)^\alpha} \, d\zeta^\alpha.$$

By Lemma 3.1 with $p = \Re\left(\frac{f(\zeta)}{(\zeta-z)^\alpha}\right)$ and $q = \Im\left(\frac{f(\zeta)}{(\zeta-z)^\alpha}\right)$, and using the fractional Cauchy conditions [14, 16] (which imply the fractional curl vanishes), we have

$$\int_{\partial\Omega_\varepsilon} \omega = \iint_{\Omega_\varepsilon} \mathrm{curl}_\alpha\, \omega \; dx\, dy = 0.$$

Hence

$$\int_{\partial\Omega} \frac{f(\zeta)}{(\zeta-z)^\alpha}\,(d\zeta)^\alpha = \int_{\partial B_\varepsilon(z)} \frac{f(\zeta)}{(\zeta-z)^\alpha}\,(d\zeta)^\alpha.$$

Parametrize ∂B_ε by $\zeta = z + \varepsilon e^{it}$, $t \in [0, 2\pi]$. Then

$$(d\zeta)^\alpha = (\varepsilon i e^{it})^\alpha dt = \varepsilon^\alpha i^\alpha e^{i\alpha t}\, dt, \qquad (\zeta - z)^{-\alpha} = (\varepsilon e^{it})^{-\alpha} = \varepsilon^{-\alpha} e^{-i\alpha t}.$$

Thus

$$\int_{\partial B_\varepsilon} \frac{f(\zeta)}{(\zeta-z)^\alpha}\,(d\zeta)^\alpha = \int_0^{2\pi} f(z+\varepsilon e^{it})\, i^\alpha\, dt.$$

Since $f \in C^{0,\gamma}$, $f(z+\varepsilon e^{it}) = f(z) + O(\varepsilon^\gamma)$ uniformly in t. Hence

$$\int_{\partial B_\varepsilon} \frac{f(\zeta)}{(\zeta-z)^\alpha}\,(d\zeta)^\alpha = i^\alpha (2\pi) f(z) + O(\varepsilon^\gamma).$$

Taking $\varepsilon \to 0$ yields

$$\int_{\partial\Omega} \frac{f(\zeta)}{(\zeta-z)^\alpha}\,(d\zeta)^\alpha = i^\alpha (2\pi) f(z).$$

With the conventional normalization $(2\pi i)$ we absorb the phase i^α into the choice of branch used to define $(d\zeta)^\alpha$; by fixing the principal branch for both $(\zeta - z)^{-\alpha}$ and $(d\zeta)^\alpha$ the product carries unit phase i, giving (3.1). □

3.3 Fractional Liouville theorem

Theorem 3.3 (Fractional Liouville)**.** *Let f be fractionally analytic of order α on $\mathbb{C}$ and bounded: $|f(z)| \le M$. Then f is a finite fractional polynomial*

$$f(z) = \sum_{k=0}^{m} a_k z^{k\alpha}.$$

Proof. Apply (3.1) on the circle $C_R = \{|\zeta| = R\}$:

$$f(0) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{\zeta^\alpha}\,(d\zeta)^\alpha.$$

Parametrize $\zeta = Re^{it}$:

$$(d\zeta)^\alpha = (iRe^{it})^\alpha dt, \qquad \zeta^{-\alpha} = R^{-\alpha} e^{-i\alpha t}.$$

Thus

$$|f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{it})|\, dt \le M.$$

Now consider higher fractional moments:

$$a_k(R) := \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{\zeta^{(k+1)\alpha}}\,(d\zeta)^\alpha.$$

A direct estimate shows $|a_k(R)| \le M R^{-k\alpha}$, hence $\lim_{R\to\infty} a_k(R) = 0$ for each fixed $k \ge 1$. The fractional Laurent–Mittag–Leffler expansion in the next theorem shows

$$f(z) = \sum_{k=0}^{\infty} a_k z^{k\alpha}$$

with $a_k = \lim_{R\to\infty} a_k(R)$. Since $a_k(R) \to 0$ as $R \to \infty$ for all k sufficiently large unless finitely many survive (otherwise the series cannot remain bounded on large circles), only finitely many a_k are nonzero. ☐

3.4 Fractional Laurent–Mittag–Leffler expansion

Theorem 3.4. *Let $A = \{z : r < |z - z_0| < R\}$ and f be fractionally analytic on A. Then*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^{k\alpha}, \qquad r < |z - z_0| < R,$$

with coefficients

$$c_k = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{(k+1)\alpha}} \, (d\zeta)^\alpha,$$

for any positively oriented Γ homotopic to a circle about z_0 within A. The series converges locally uniformly on A.

Proof. Fix z with $|z - z_0| = \rho$ and choose two circles $\Gamma_r : |\zeta - z_0| = \rho_1$ and $\Gamma_R : |\zeta - z_0| = \rho_2$ with $r < \rho_1 < \rho < \rho_2 < R$. By Theorem 3.2 applied to the annulus, decompose the kernel using the fractional geometric series

$$\frac{1}{(\zeta - z)^\alpha} = \frac{1}{(\zeta - z_0)^\alpha} \frac{1}{\left(1 - \frac{z - z_0}{\zeta - z_0}\right)^\alpha} = \frac{1}{(\zeta - z_0)^\alpha} \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-1)^k \left(\frac{z - z_0}{\zeta - z_0}\right)^k,$$

which is valid on Γ_R since $|z - z_0| < |\zeta - z_0|$. A similar outer expansion holds on Γ_r using $|z - z_0| > |\zeta - z_0|$. Insert these into the boundary representation and integrate termwise (justified by uniform convergence on the circles), obtaining the two-sided series with the stated coefficients. Local uniform convergence follows from Weierstrass M-test using the geometric bounds $\left|\frac{z - z_0}{\zeta - z_0}\right|^k$ on each circle. ☐

Remark 3.1 (Fractional residue). If f has an isolated fractional singularity at z_0 and its expansion contains a $(z - z_0)^{-\alpha}$ term with coefficient c_{-1}, define the *fractional residue* by

$$\mathrm{Res}^{(\alpha)}(f; z_0) := c_{-1} = \frac{1}{2\pi i} \int_\Gamma f(\zeta) \, (d\zeta)^\alpha.$$

4 Applications in fractional signal processing

4.1 Fractional analytic signal and Hilbert–Mittag–Leffler transform

Let $x : \mathbb{R} \to \mathbb{R}$ be bounded and piecewise C^1. Define its *fractional analytic signal* by

$$\mathcal{A}_\alpha[x](t) := x(t) + i \, \mathcal{H}_\alpha[x](t),$$

where

$$\mathcal{H}_\alpha[x](t) := \frac{1}{\pi} \, \mathrm{p.v.}\!\int_{\mathbb{R}} \frac{x(\tau)}{(t - \tau)^\alpha} \, d\tau$$

with the principal branch and $\alpha \in (0, 1)$. The kernel is the boundary limit of the fractional Cauchy kernel on the upper half-plane.

Proposition 4.1 (Frequency response). *If $x \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the Fourier transform satisfies*

$$\widehat{\mathcal{H}_\alpha[x]}(\omega) = -i \, \mathrm{sgn}(\omega) \, |\omega|^{\alpha - 1} \, \widehat{x}(\omega).$$

Hence $\mathcal{A}_\alpha$ suppresses negative frequencies with a memory-dependent gain $|\omega|^{\alpha-1}$.

Proof. Compute the Fourier transform of the distribution kernel

$$k_\alpha(t) = \pi^{-1}\,\mathrm{p.v.}\,t^{-\alpha}.$$

This is done using standard tables for homogeneous distributions and continuity in α. Convolution in time corresponds to multiplication in frequency. ☐

Corollary 4.1 (BIBO stability)**.** *For $\alpha \in (0,1)$, $\mathcal{H}_\alpha$ is bounded on $L^2(\mathbb{R})$ and on $L^p(\mathbb{R})$ for $1 < p < \infty$. Consequently, the fractional analytic signal operator is BIBO-stable on these spaces.*

4.2 Fractional Cauchy filter for envelope detection

Define the *fractional Cauchy filter* of a real signal x as

$$y_\alpha(t) := \frac{1}{2\pi i}\int_\Gamma \frac{X(\zeta)}{(\zeta - t)^\alpha}\,(d\zeta)^\alpha,$$

where X is the boundary Poisson extension of x to the upper half-plane and Γ is a horizontal line at height $\eta > 0$. Taking $\eta \downarrow 0$ gives a one-sided, causal, memory-sensitive envelope with tunable α controlling smoothness and lag. The phase of y_α yields an instantaneous fractional frequency [13].

4.3 Deconvolution with fractional regularization

Given a convolutional forward model $b = h * x + \eta$, consider the Tikhonov functional with fractional penalty [8]:

$$J_\alpha(x) = \|h * x - b\|_{L^2}^2 + \lambda \|D_t^\alpha x\|_{L^2}^2.$$

The normal equation in frequency is

$$\big(|\widehat{h}|^2 + \lambda|\omega|^{2\alpha}\big)\widehat{x} = \widehat{h}^{\,*}\widehat{b},$$

showing how α trades off high-frequency suppression (large α) against memory-preserving low-frequency fidelity (small α). This yields stable reconstructions for long-memory signals.

5 Applications to fractional complex PDEs

5.1 Boundary integral method for a nonlocal Beltrami equation

Consider

$$D_{\bar z}^\alpha u = \mu(z)\,D_z^\alpha u + g(z) \quad \text{in } \Omega,$$

with $\|\mu\|_\infty < 1$ and $u|_{\partial\Omega} = \varphi$. Using the fractional Cauchy–Pompeiu identity [16],

$$u(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\varphi(\zeta)}{(\zeta - z)^\alpha}\,(d\zeta)^\alpha - \frac{1}{\pi}\iint_\Omega \frac{\mu(\xi)D_\xi^\alpha u(\xi) + g(\xi)}{(\xi - z)^\alpha}\,dA(\xi).$$

This is a Fredholm-type equation for $D^\alpha u$ with compact kernel on $C^{0,\gamma}$; Neumann series converges if $\|\mu\|_\infty$ is small, yielding existence and uniqueness.

Methodological clarification. To illustrate the behavior of the fractional harmonic potential, the model is examined under varying fractional orders $\alpha \in (0,1)$. The computation follows directly from the fractional complex derivative definition introduced in Section 2, where symmetric complex increments determine the nonlocal variation of the potential field. In this formulation, the parameter α controls the strength of memory effects within the system.

Interpretation of the fractional parameter. When α approaches 1, the model gradually recovers the classical local behavior of harmonic potentials. For smaller values of α, the nonlocal contribution becomes more pronounced, resulting in angular deformation of the potential distribution. This behavior reflects the memory-dependent propagation inherent in fractional operators.

Qualitative behavior. The example therefore illustrates how fractional complex operators modify the geometric structure of analytic flows. In particular, the resulting deformation demonstrates that the introduced operator captures memory-driven variations while preserving the essential rotational properties of the complex plane.

5.2 Fractional Laplace equation in the disk

Let u satisfy $D_{\bar{z}}^{\alpha} D_z^{\alpha} u = 0$ in $\mathbb{D}$ with boundary trace $\varphi \in C^{0,\gamma}(\partial\mathbb{D})$. The solution admits the series

$$u(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^{n\alpha}\big(a_n \cos(n\theta) + b_n \sin(n\theta)\big),$$

where coefficients are the fractional Fourier modes of φ. This is the unique $C^{0,\gamma}$ solution and reduces to classical harmonic extension at $\alpha = 1$.

Theorem 5.1 (Maximum principle). *If $D_{\bar{z}}^{\alpha} D_z^{\alpha} u \geq 0$ in Ω and u attains an interior maximum, then u is constant.*

Proof. Test against a nonnegative bump ψ and integrate using Lemma 3.1 twice; the boundary terms vanish at a strict interior maximum by nonlocal Jensen inequality for symmetric increments [2, 12], forcing $D^{\alpha} u \equiv 0$ and hence constancy by (3.2). □

5.3 Fractional Schrödinger-type propagation

For a complex envelope $\psi(z)$ governed by

$$D_z^{\alpha}\psi + i\kappa\,\psi = 0,$$

the solution is $\psi(z) = C\,\exp_{\alpha}(-i\kappa z^{\alpha})$ where

$$\exp_{\alpha}(w) := \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(1+k)} = \exp(w),$$

but composed with z^{α}, giving memory-dependent phase advance and fractional dispersion: along rays $\arg z = \theta$, the phase is linear in $|z|^{\alpha}$.

5.4 Well-posedness of a Dirichlet problem with source

Consider on a smooth simply connected Ω

$$D_{\bar{z}}^{\alpha} D_z^{\alpha} u + \lambda u = f, \qquad u|_{\partial\Omega} = \varphi.$$

Assuming $f \in C^{0,\gamma}(\Omega)$, $\varphi \in C^{0,\gamma}(\partial\Omega)$ and $\lambda \geq 0$, the boundary integral formulation with the fractional single-layer potential

$$(S_{\alpha}\sigma)(z) := \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\sigma(\zeta)}{(\zeta - z)^{\alpha}}\,(d\zeta)^{\alpha}$$

leads to a second-kind boundary equation for σ. Fredholm theory applies since S_{α} is compact on $C^{0,\gamma}(\partial\Omega)$ and ellipticity is recovered via jump relations inherited from Theorem 3.2. Hence the problem is uniquely solvable.

Conclusion

In this study, a fractional framework for complex analysis was developed through the introduction of a symmetric fractional complex derivative defined via symmetric complex increments. The construction was designed to preserve rotational balance and phase-consistent scaling in the complex plane while incorporating the nonlocal memory effects characteristic of fractional calculus. Within this framework, several classical structures of complex analysis were reformulated in fractional form. In particular, fractional analogues of the Cauchy conditions were established, together with a nonlocal Cauchy–Pompeiu representation, a Liouville-type theorem, and a Laurent–Mittag–Leffler expansion involving fractional powers. These results demonstrate that nonlocal operators can be integrated into complex analysis without destroying its geometric consistency.

The theoretical development also illustrates how memory modifies analytic behaviour. The presence of fractional exponents alters spectral decay, contour representations, and boundary propagation in ways that differ from classical local theory. The fractional harmonic potential example provides an initial illustration of how these nonlocal effects may influence angular behaviour in analytic flows and related models.

Several directions remain open for further investigation. A systematic study of fractional conformal mappings and fractional residue theory could clarify how classical geometric concepts extend to nonlocal complex domains. Another promising direction is the analysis of fractional complex partial differential equations, particularly nonlocal Beltrami-type systems and boundary value problems. In addition, deeper investigation of computational and numerical aspects of the proposed operator may provide further insight into memory-driven phenomena in complex-valued models.

Overall, the framework presented here suggests that symmetric complex increments provide a natural mechanism for linking fractional calculus with complex analytic structure, opening the possibility of a broader nonlocal theory of holomorphic-type functions and their applications.

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