A PREY PREDATOR CONSERVATION MODEL FOR A FISHERY WITH A RESERVE AREA AND PREY REFUGE : A STUDY OF LAKE VICTORIA

Abstract

The loss of species in most fishery ecosystems worldwide has reached crisis levels driven by habitat loss, overfishing, invasive species predation and climate change. This threatens biodiversity and sustainability of these fisheries. In Lake Victoria a major decline has been observed in haplochromines due to predation by Nile perch and the species is now in danger of extinction haplochromines are important to the lake because the feed on algae preventing algal bloom. In this paper a prey-predator conservation model for a fishery with a reserve area and prey refuge has been formulated using a logistic nonlinear differential equations. The model incorporates Holling type II functional response of the predator towards the prey. In this research we study and analyze the stability of the prey-predator dynamic system of Nile perch (predator) and haplochromines (prey) in Lake Victoria. The lake ecosystem is divided into two parts, the reserve area and unreserved area. Equilibrium points have been determined and their local and global stability established by use of eigen value approach, Bendixon-Dulac criterion and lyapunov function. The effect of the reserve area and prey refuge on the stability of the system has been determined by simulation in MATLAB. Results show that a reserve area makes the system stable, and for certain values of migration rate $\sigma \leq 0.3$ the population of the predator can coexist with prey.

Keywords: Key words: Prey predator; conservation model; mathematical model; equilbrium points; stability analysis.

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1 Introduction

The increasing demand for food by human population exerts enormous pressure on natural resources causing overexploitation and depletion. Depletion of natural resources such as forests and fisheries results in loss of species and consequently loss of biodiversity. Biodiversity is the variability among living organisms. Biodiversity is the fabric that weaves the ecosystem together providing stability and resilience (Rathoure, 2024). The repercussions of loss of biodiversity are immense, it interferes with the functioning of the ecosystem and reduces the resilience of marine ecosystems (Duterte, 2025).

IUCN (International Union of Conservation and Nature) red list report highlights a severe crisis in fresh water fishery ecosystems, with about 24% of fresh water fishes and other invertebrates at high risk of extinction (Sayer et al., 2025). This is attributed to habitat loss, pollution, overfishing, climate change, invasive species and predation. This has drawn the concern of conservationists worldwide to try and protect the endangered species from extinction by use of scientific management approaches.

An ecosystem consists of all species in a given area and the physical environment. Ecological interactions between different species occur in various ways, which include but not limited to competition and predation. Competition is where different species compete for scarce resources like food, space or environment. Predation is where one species (predator) feeds on another (prey). In this case the prey is converted into the predator through consumption, hence population growth of the predator is enhanced while that of the prey is reduced and its existence threatened. In this research we focus on a case study of Lake Victoria fishery ecosystem. We investigate the prey-predator dynamic system of the Nile perch and haplochromines. (Sayer et al., 2018) in their research notes that Lake Victoria fishery is threatened by pollution, climate change, agriculture, overharvesting and predation by the invasive Nile perch. Climate change is causing heating up of water raising its temperature, this lowers oxygen in the water leading to algal bloom which affects aquatic life and causes fish stocks to decline (Rasolofoson et al., 2024). Nile perch preys on other fishes but its favorite prey is the haplochromines cichlids species. Predation by the invasive Nile perch is driving most haplochromines cichlids to the brink of extinction. The situation is so dire that the red list index value for the native haplochromines cichlids dropped by 63% between 1960 and 2010 indicating increased risk of extinction.(Sayer et al., 2018) in their IUCN research report 76% of the fish native to Lake Victoria basin is threatened with extinction. Recent researches show a progressive decline in fish species and biodiversity in Lake Victoria. It is estimated that out of over 500 haplochromine species in the Lake before the Nile perch was brought into the lake, about 200 of them are now extinct and the remaining are on the verge of extinction (Outa et al., 2020). The lake is home to many species not found anywhere else in the world and its biodiversity is still diminishing (Muthoka et al., 2024) Haplochromines are important to the lake biodiversity since they feed on algae this prevents algae bloom in the lake. Haplochromines are also the favorite prey of the commercially valuable Nile perch. Therefore conservation of the haplochromines cichlids creates a prey reservoir available to the Nile perch for consumption.

To understand the prey-predator dynamic interactions in the ecosystem several approaches have been used, one being mathematical modeling, models provide crucial insights into population dynamics of interacting species. The earliest mathematical model proposed was the Lotka-Volterra model, since then many models have been developed. For instance prey-predator conservation models have been developed and used to study the factors that influence the growth and decline of natural resources like forests and fisheries. Conservation refers to preservation and protection of natural resources and the environment for sustainable future use. Mathematical models use differential equations like the Lotka-Volterra equations to simulate the growth and the decline of the prey and the predator population, they incorporate factors like growth rate, predation rate, prey availability and harvesting rate. Prey-predator conservation models are critical in helping predict population trends of the prey and the predator in the ecosystem, similarly they help determine factors that affect the population dynamics such as prey availability, predation rate and harvesting rate, and other environmental factors. Conservationists then use this knowledge to develop conservation strategies such as predator control or prey enhancement. One of the strategies employed by conservationists is creation of a reserve area. A reserve area is a protected area set apart to conserve and protect wildlife usually with a goal of preserving endangered species and their habitat. A reserve area provides a safe haven for the prey to take refuge. Prey refuge is a strategy by the prey to evade the predator and reduce predation risk. The prey may take refuge in a protected reserve area or may develop certain natural adaptations like aggregating into schools where the prey takes refuge to avoid being targeted. In this study we include a reserve area and a prey refuge term in our model. A reserve area and prey refuge have a stabilizing effect on the model. In recent past mathematical models on sustainable exploitation of natural renewable resources, which includes conservation and optimum harvesting models have been formulated integrating various functional responses such as Holling type I, II, III and IV and modified Leslie Gower functional responses and harvesting.

(Zhang et al., 2007) analyzed a prey predator model with a prey reserve area. The biological and bionomical equilibrium points of the system were determined. The stability analysis for equilibrium points was obtained. They concluded that irrespective of the presence or absence of the predator, the fishery can be sustained at a stable equilibrium.(Hai-Feng et al., 2012) studied a dynamic model for a fishery resource with a reserve area and taxation. They found out that increase in tax increased fish population in both areas, however fish population tend to equilibrium quickly in the reserve area than in the fishing area. (Mondal, 2014) analyzed a prey predator model with marine protected area (MPA) and harvesting. Results from this study showed that a protected marine reserve has a stabilizing impact on population dynamics. (Jawad, 2018) studied dynamics of a multispecies systems with prey refuge. He analyzed three models of four species involving a prey, a predator and a top predator. The models involved Holling type (I) functional response. He concluded that a reserve area may enable survival and stability of prey predator systems. (Raymond et al., 2019) investigated dynamics of a prey-predator fishery model with harvesting; a bionomical model using Holling type II functional response they examined the dynamics of prey-predator system of the Nile perch as the predator and tilapia and cichlids as the prey in Lake Victoria. They investigated the impact of harvesting, on the fish density of the system. The conditions for stability of equilibrium points both local and global were obtained. The results of the study revealed that cichlids and tilapia should not be overfished since they contribute to the growth of the Nile perch. (Khatun & Biswas, 2019) studied a mathematical model applied to renewable fishery management. The fishery had a protected area and fishing area, the existence of equilibrium points and their stability was determined. Results of the study revealed that, elevated death rate, fishing rate and high rate of consumption of the prey by the predator, within the unreserved region, leads to reduction of the species within the protected zone. Currently hybrid models incorporating ordinary differential equations and discrete equations for seasonal harvesting have been used to study prey predator dynamics. (Mu et al., 2025) investigated a hybrid control for the prey in a spatial prey predator model with cooperative hunting and fear effect time lag, they investigated how fear effect induced by cooperative hunting affects birth rate in the prey their results revealed that cooperative hunting and fear effect amplifies the spatial heterogeneous distribution of the prey and the predator and that a hybrid control scheme can mitigate negative effects of fear cooperative hunting. The models discussed above did not involve prey refuge aspect, which is a behavioral adaptation by the prey to evade the predator. In this research we incorporate prey refuge aspect in the model, we modify the predation term, taking into account a fraction m of the prey that takes refuge leaving (1m) of the prey available to the predator for consumption.

2 Model Formulation

The theory of dynamical systems can be applied in mathematical modeling of ecosystems (Ramm & Hoang, 2013). Population dynamics describes the behavior of ecological systems as time changes. These systems consist of many species interacting in many complex ways (Jawad, 2018).

Mathematical modeling has become an important method of studying and analyzing interactive dynamics of a growing population of competing species in an ecological system. Differential equations can be formulated to govern changes in population of interacting species with time. In this section we formulate a mathematical model to investigate the Prey-predator interactions of the Nile perch and haplochromine cichlids in Lake Victoria. The haplochromine cichlids species is the prey and the Nile perch is the predator. The fishery is subdivided into two regions, the fishing area and a

protected reserve area where fishing is banned. The prey resides in both regions and migrates from the unreserved region to the reserved region randomly but the reverse is not permitted. The predator resides in the unreserved area and feeds on the prey in the unreserved area only. The predator fish has Holling type II functional response towards the prey in the unreserved area. Let $X(t)$ and $Y(t)$ represent the population densities of the haplochromines at any time t in the unreserved area and the reserved area respectively. Let $Z(t)$ represents the population density of the Nile perch at a time t. The predator and the prey are both harvested in the unreserved area. The growth of the prey in both zones is logistic in the absence of the predator, mx fraction of prey take refuge to evade predation, leaving $(1-m)x$ of the prey available to the predator for consumption.
We make the following assumptions for our model;

- (i) The predator depends fully on the prey as its most preferred food and decays exponentially to extinction in the absence of the prey.
- (ii) The prey has unlimited food supply.
- (iii) The prey grows logistically in the absence of the predator and harvesting.
- (iv) The predator is the only threat to the prey under the study. Other external factors like epidemics pollution affect both the prey and the predator equally
- (v) Fishing effort is constant there is no redistribution of harvesting effort from the reserved area to the fishing zone.

The summary of variables and parameters used are described in the tables below

Table 1: Table of variables

Variable	Description of variables
X	Population densities of the haplochromines in the unreserved area
Y	Population densities of the haplochromines in the reserved area
Z	Population densities of the Nile perch in the unreserved area

Table 2: Table of parameters

Parameter	Parameter description
r_n	Intrinsic growth rate of the prey and the predator.
K_n	Carrying capacity of the environment.
b	Measure of the extent to which the environment provides protection to the prey.
c_1	Maximum value per capita reduction rate of prey due to consumption by the predator.
c_2	Rate of conversion of the predator into the prey.
d_n	Natural rate of death of the predator and the prey.
m	A fraction of the prey that takes refuge to avoid predation.
E_1	Harvesting effort dispensed towards harvesting of the prey.
E_2	Harvesting effort dispensed towards harvesting of the predator.
q_1	Catchability coefficients of the prey.
q_2	Catchability coefficients of the predator.
σ	Migration rate between the unreserved area to the reserved area.

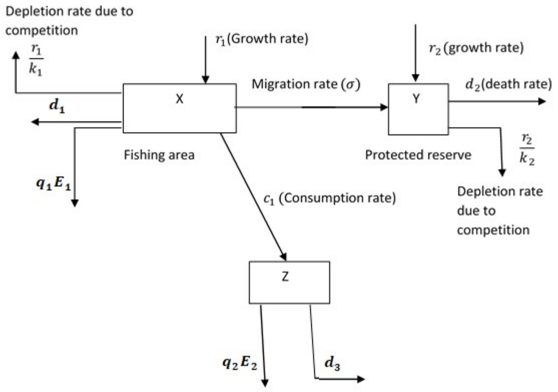

Figure 1: Flow diagram

Using the variables, parameters and the assumptions above the model equations are;

$$\begin{aligned} f_1(X,Y,Z) &= \frac{dX}{dt} = r_1X\left(1-\frac{X}{K_1}\right)-\frac{c_1(1-m)XZ}{b+(1-m)X}-(\sigma+q_1E_1+d_1)X \qquad (2.1)\\ f_2(X,Y,Z) &= \frac{dY}{dt} = r_2Y\left(1-\frac{Y}{K_2}\right)+\sigma X-d_2Y \qquad (2.2)\\ f_3(X,Y,Z) &= \frac{dZ}{dt} = r_3Z\left(1-\frac{Z}{K_3}\right)+\frac{c_2(1-m)XZ}{b+(1-m)X}-(q_2E_2+d_3)Z \qquad (2.3)\end{aligned}$$

The ratios $\frac{(r_1X^2)}{K_1}$ and $\frac{(r_2Y^2)}{K_2}$ are the reduction rates of the prey due to interspecies competition. The ratio $\frac{c_1(1-m)XZ}{b+(1-m)X}$ is the decrease in prey population as a consequence of predation and the ratio $\frac{c_2(1-m)XZ}{b+(1-m)X}$ is the increase in predator population as result of eating the prey.

2.1 Analysis of the model

2.1.1 Positivity and Boundedness of Solutions

The analysis of the model is done in the feasible region Ω in which $(X(t),Y(t),Z(t))\in\mathbb{R}_+^3$ we show that the solutions of the model are non-negative for all $t\geq 0$

Theorem 2.1. *The solutions* $(X(t),Y(t),Z(t))\geq 0$ *for all* $t\geq 0$ *in* Ω

Proof. Taking into account equation (2.1)

$$\frac{dX}{dt}=r_1X\left(1-\frac{X}{K_1}\right)-\frac{c_1(1-m)XZ}{b+(1-m)X}-(\sigma+q_1E_1+d_1)X$$

If $X(t)$ is positive then this implies that

$$\frac{dX}{dt}\geq-(\sigma+q_1E_1+d_1)X$$

Let $\mu=(\sigma+q_1E_1+d_1)\implies\frac{dX}{dt}\geq-\mu X$
Separation of variables and using the initial condition gives

$$X(t)\quad\geq\quad X(0)e^{-(\sigma+q_1E_1+d_1)t}$$

Thus as $t \to \infty$, $X(t) \geq 0$ for all $t \geq 0$
Similarly for equations (2.2) and (2.3) we obtain

$$\begin{aligned} Y(t) &\geq Y(0)e^{-d_2 t} \\ Z(t) &\geq Z(0)e^{-(q_2 E_2 + d_3)t} \end{aligned}$$

□

Therefore, for equations (2.1), (2.2), (2.3) as $t \to \infty$, $X(t) \geq 0, Y(t) \geq 0, Z(t) \geq 0$ for all $t \geq 0$. Hence in view of equations (2.1), (2.2), (2.3) all solutions are non-negative for $t \geq 0$. Next, we use the following theorem to show that the solutions are bounded

Theorem 2.2. *Assuming that $c_1 \geq c_2$ holds then the solutions of the system 2.1, 2.2, 2.3 that initiate in $\Omega \subset \mathbb{R}^3_+$ are bounded*

Proof. Considering $X(t), Y(t), Z(t)$ to be solutions to the model equations 2.1, 2.2, 2.3 with positive initial conditions and letting

$$\begin{aligned} U(t) &= X(t) + Y(t) + Z(t) \text{ then} \\ \frac{dU}{dt} &= \frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{dU}{dt} &= r_1 X - \frac{r_1 X^2}{K_1} - \frac{c_1(1-m)XZ}{b+(1-m)X} - \sigma X - (q_1 E_1 + d_1)X + r_2 Y - \frac{r_2 Y^2}{K_2} \\ &\quad + \sigma X - d_2 Y + r_3 Z\left(1 - \frac{Z}{K_3}\right) + \frac{c_2(1-m)XZ}{b+(1-m)X} - (q_2 E_2 + d_3)Z \end{aligned}$$

But from above assumption of the theorem if $c_1 \geq c_2$ then we obtain

$$\begin{aligned} \frac{dU}{dt} &\leq r_1 X - \frac{r_1 X^2}{K_1} - (q_1 E_1 + d_1)X + r_2 Y - \frac{r_2 Y^2}{K_2} - d_2 Y + r_3 Z - r_3 \frac{Z^2}{K_3} \\ &\quad - (q_2 E_2 + d_3)Z \\ \frac{dU}{dt} + \xi U &\leq 2r_1 X - \frac{r_1 X^2}{K_1} + 2r_2 Y - \frac{r_2 Y^2}{K_2} + 2r_3 Z - \frac{r_3 Z^2}{K_3} \end{aligned}$$

Where $\xi = \min\{r_1, r_2, r_3, d_1, d_2, d_3, q_1, q_2, E_1, E_2\}$ and
let $\lim\limits_{X \to \infty} X = K_1$, $\lim\limits_{Y \to \infty} Y = K_2$ and $\lim\limits_{Z \to \infty} Z = K_3$ then

$$\begin{aligned} \frac{dU}{dt} + \xi U &\leq r_1 K_1 - \frac{r_1(X - K_1)^2}{K_1} + r_2 K_2 - \frac{r_2(Y - K_2)^2}{K_2} + r_3 K_3 - \frac{r_3(Z - K_3)^2}{K_3} \\ \frac{dU}{dt} + \xi U &\leq r_1 K_1 + r_2 K_2 + r_3 K_3 \end{aligned}$$

Let $m = r_1 K_1 + r_2 K_2 + r_3 K_3$. Therefore:

$$\begin{aligned} \frac{dU}{dt} + \xi U &\leq m \\ \frac{dU}{dt} &\leq -\xi U + m \end{aligned}$$

By use of Gronwall's differential inequality (Jawad, 2018), we obtain

$$0 \leq U(t) \leq \frac{m}{\xi}(1 - e^{-\xi t}) + U(0)e^{-\xi t}$$

Therefore as $t \to \infty$ we have

$$0 \leq \lim_{t \to \infty} \sup U(t) \leq \frac{m}{\xi}$$

□

Therefore $U(t)$ is bounded, and all solutions to the system of model equations, (2.1), (2.2), (2.3) that initiate in $\mathbb{R}^3_+$ are attracted to the region

$$\Omega = \{(x+y+z) \in \mathbb{R}^3_+ : U = x+y+z \leq \frac{m}{\xi}\}$$

hence solutions are bounded. Therefore from this proof because $\lim_{t\to\infty} \sup U(t)$ is independent of initial conditions then the system is uniformly bounded. In addition since c_1 is the capture rate of the prey and c_2 is the conversion rate of the prey into the predator then it is reasonable assume that $c_1 \geq c_2$.

2.1.2 Equilibrium Points

We first scale down the number of parameters in the model equations 2.1, 2.2, 2.3 to simplify them. By letting

$$\begin{aligned}
\beta_1 &= r_1 - \sigma - q_1 E_1 - d_1 \\
\beta_2 &= r_2 - d_2 \\
\beta_3 &= r_3 - d_3 - q_2 E_2 \\
\omega_1 &= c_1(1-m) \\
\omega_2 &= c_2(1-m) \\
a &= \frac{b}{(1-m)} \\
p_1 &= \frac{r_1}{K_1} \\
p_2 &= \frac{r_2}{K_2} \\
p_3 &= \frac{r_3}{K_3}
\end{aligned}$$

The scaled down equations are:

$$\frac{dX}{dt} = \beta_1 X - p_1 X^2 - \frac{\omega_1 XZ}{(a+X)} \tag{2.5}$$

$$\frac{dY}{dt} = \beta_2 Y - p_2 Y^2 + \sigma X \tag{2.6}$$

$$\frac{dZ}{dt} = \beta_3 Z - p_3 Z^2 + \frac{\omega_2 XZ}{(a+X)} \tag{2.7}$$

Equating equations 2.5, 2.6, 2.7 to zero. We observe that for system of equations 2.5 2.6 2.7 three positive equilibria exist, the extinction equilibrium point $E_0(0,0,0)$, the predator extinction equilibrium point $E_1 = (X^*, Y^*, 0)$ and the co-existence equilibrium point $E_2 = (X^*, Y^*, Z^*)$. The existence of extinction equilibrium point is trivial we show the existence of the other equilibrium points.
For $E_1 = (X^*, Y^*, 0)$ we proceed as follows;
Considering equations 2.5 2.6 2.7 for $Z = 0$ we remain with two equations to solve

$$\begin{aligned}
\beta_1 X - p_1 X^2 &= 0 \\
\beta_2 Y - p_2 Y^2 + \sigma X &= 0
\end{aligned}$$

The first equation gives

$$\begin{aligned}
X(\beta_1 - p_1 X) &= 0 \\
\implies X &= 0 \text{ or} \\
\implies X^* &= \frac{\beta_1}{p_1}
\end{aligned}$$

recalling the original parameters we have

$$X^* = \frac{K_1(r_1 - \sigma - q_1E_1 - d_1)}{r_1} \tag{2.8}$$

Considering the second equation and using equation (2.8) we have

$$p_2Y^2 - \beta_2Y - \sigma\frac{\beta_1}{p_1} \quad = \quad 0$$

Thus

$$Y^* = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4p_2\sigma\frac{\beta_1}{p_1}}}{2p_2} \tag{2.9}$$

Therefore,

$$E_1(X^*, Y^*, 0) = \left(\frac{\beta_1}{p_1}, \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4p_2\sigma\frac{\beta_1}{p_1}}}{2p_2}, 0\right)$$

Thus recalling the original parameters, we observe that $E_1(X^*, Y^*, 0)$ exists if

$r_1 > 0, r_2 > 0$ and $r_1 > \sigma + q_1E_1 + d_1, r_2 > d_2$

For the equilibrium point $E_2(X^*, Y^*, Z^*)$ we proceed as follows, we set equations 2.5, 2.6, 2.7 to zero. Solving the third equation for non-zero Z we obtain

$$\beta_3 - p_3Z + \frac{\omega_2X}{(a + X)} \quad = \quad 0$$

Which gives Z^* as

$$Z^* = \frac{\beta_3(a + X) + \omega_2X}{p_3(a + x)} \tag{2.10}$$

Solving the second equation, we obtain

$$\beta_2Y - p_2Y^2 + \sigma X \quad = \quad 0$$

That is,

$$Y^* = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4p_2\sigma X^*}}{2p_2} \tag{2.11}$$

Substituting for Z by Z^* in the first equation we obtain;

$$p_1p_3X^3 + p_3\beta_1X^2 - (p_1p_3a^2 + \beta_3\omega_1 + \omega_1\omega_2)X - (\omega_1\beta_3a + p_3\beta_1a^2) = 0$$

Therefore we obtain an equation of the form ;

$$aX^3 + bX^2 + cX + d = 0 \tag{2.12}$$

Where

$$\begin{aligned}
a &= p_1p_3 = \frac{r_1r_3}{K_1K_3} \\
b &= p_3\beta_1 = \frac{r_3}{K_3}(r_1 - (d_1 + q_1E_1 + \sigma)) \\
c &= p_1p_3a^2 + \beta_3\omega_1 + \omega_1\omega_2 = \frac{r_1r_3}{K_1K_3}\left(\frac{b}{1-m}\right) - c_1(1-m)(r_3 - (d_3 + q_2E_2)) \\
&\quad + c_1c_2(1-m)^2 \\
d &= -(\omega_1\beta_3a + p_3\beta_1a^2) = bc_1(r_3 - (d_3 + q_2E_2)) \\
&\quad + \frac{r_3}{K_3}(r_1 - (d_1 + q_1E_1 + \sigma))\left(\frac{b}{1-m}\right)^2
\end{aligned}$$

By use of Descartes rule of signs, equation (2.12) has a unique positive solution $X = X^*$, if the following inequalities hold;

$$\begin{aligned} r_1 &> (d_1 + q_1 E_1 + \sigma) \\ r_3 &> (d_3 + q_2 E_2) \end{aligned}$$

Therefore if the above inequalities hold the co-existence equilibrium $E_2(X^*, Y^*, Z^*)$ exists.

2.2 Local stability of the equilibrium points

The local stability of each equilibrium point is found by determining eigen values of the Jacobian matrix associated with the equilibrium point. For an equilibrium point to be stable eigen values of the Jacobian matrix must have negative real parts. The Jacobian matrix is found as follows;

$$J(E_i) = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{bmatrix}$$

Considering the system of equations 2.5, 2.6, 2.7

$$\begin{aligned} f_1(X,Y,Z) &= \beta_1 X - p_1 X^2 - \frac{\omega_1 XZ}{a+X} \\ f_2(X,Y,Z) &= \beta_2 y - p_2 Y^2 + \sigma X \\ f_3(X,Y,Z) &= \beta_3 Z - p_3 Z^2 + \frac{\omega_2 XZ}{a+X} \end{aligned}$$

This gives;

$$J(E_i) = \begin{bmatrix} \beta_1 - 2p_1 X - \frac{a\omega_1 Z}{(a+X)^2} & 0 & \frac{-a\omega_1 X}{a+X} \\ \sigma & \beta_2 - 2p_2 Y & 0 \\ \frac{a\omega_2 Z}{(a+X)^2} & 0 & \beta_3 - 2p_3 Z + \frac{\omega_2 X}{a+X} \end{bmatrix}$$

2.2.1 Local stability

For extinction equilibrium point $E_0(0,0,0)$ the Jacobian matrix is given by

$$J(E_0) = \begin{bmatrix} \beta_1 & 0 & 0 \\ \sigma & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

whose eigenvalues, after recalling the original paramaters, are

$$\begin{aligned} \lambda_1 &= r_1 - (\sigma + q_1 E_1 + d_1) \\ \lambda_2 &= r_2 - d_2 \\ \lambda_3 &= r_3 - (d_3 + q_2 E_2) \end{aligned}$$

Since $r_2 > d_2$ the eigen value λ_2 is positive therefore $E_0(0,0,0)$ is unstable.

For local stability of predator extinction equilibrium point $E_1(X^*, Y^*, 0)$ the Jacobian at $E_1(X^*, Y^*, 0)$ is;

$$J(E_1) = \begin{bmatrix} \beta_1 - 2p_1 X & 0 & \frac{-a\omega_1 X}{a+X} \\ \sigma & \beta_2 - 2p_2 Y & 0 \\ 0 & 0 & \beta_3 + \frac{\omega_2 X}{a+X} \end{bmatrix}$$

Let

$$\begin{aligned}
A_{11} &= \beta_1 - 2p_1X = r_1 - (\sigma + q_1E_1 + d_1) - 2\frac{r_1}{K_1}X \\
A_{13} &= \frac{-a\omega_1X}{a+X} = \frac{c_2(1-m)X}{\frac{b}{1-m}+X} \\
A_{21} &= \sigma \\
A_{22} &= \beta_2 - 2p_2Y = r_2 - d_2 - 2\frac{r_2}{K_2}Y \\
A_{33} &= \beta_3 + \frac{\omega_2X}{a+X} = r_3 - (d_3 + q_2E_2) + \frac{c_2(1-m)X}{\frac{b}{1-m}+X}
\end{aligned}$$

Therefore

$$\begin{aligned}
J(E_1) &= \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \\
J(E_1) - \lambda &= \begin{bmatrix} A_{11}-\lambda & 0 & A_{13} \\ A_{21} & A_{22}-\lambda & 0 \\ 0 & 0 & A_{33}-\lambda \end{bmatrix}
\end{aligned}$$

The characteristics equation is given by $|J(E_1) - \lambda| = 0$. Thus we have

$$\begin{aligned}
(A_{11}-\lambda)[(A_{22}-\lambda)(A_{33}-\lambda)] &= 0 \\
(A_{11}-\lambda)[A_{22}A_{33} - A_{22}\lambda - A_{33}\lambda + \lambda^2] &= 0
\end{aligned}$$

Simplifying this we obtain the characteristic equation;

$$\lambda^3 - (A_{11}+A_{22}+A_{33})\lambda^2 + (A_{11}A_{22}+A_{11}A_{33}+A_{22}A_{33})\lambda - (A_{11}A_{22}A_{33}) = 0$$

This equation is of the form;

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + c = 0 \tag{2.13}$$

According to Routh-Hurwitz criterion the eigen values(λ) are negative iff, $a_1 > 0, c > 0$, and $a_1a_2 > c$, hence the equilibrium point $E_1(X^*, Y^*, 0)$ is locally asymptotically stable otherwise it is unstable. Finally, the local stability of co-existence equilibrium point $E_2(X^*, Y^*, Z^*)$ is determined by the Jacobian matrix

$$J(E_3) = \begin{bmatrix} \beta_1 - 2p_1X^* - \frac{a\omega_1Z^*}{(a+X^*)^2} & 0 & \frac{-a\omega_1X^*}{a+X^*} \\ \sigma & \beta_2 - 2p_2Y^* & 0 \\ \frac{a\omega_2Z^*}{(a+X^*)^2} & 0 & \beta_3 - 2p_3Z^* + \frac{\omega_2X^*}{a+X^*} \end{bmatrix}$$

In a similar manner as we did to the equilibrium point E_2 we have the characteristics equation given by

$$(A_{11}-\lambda)[(A_{22}-\lambda)(A_{33}-\lambda)] + A_{13}[A_{31}(A_{22}-\lambda)] = 0$$

On simplifying we obtain

$$\begin{aligned}
&\lambda^3 - (A_{11}+A_{22}+A_{33})\lambda^2 - (A_{11}A_{22}+A_{11}A_{33}+A_{22}A_{33}+A_{31})\lambda \\
&-(A_{11}A_{22}A_{33} + A_{13}A_{22}A_{31}) = 0
\end{aligned}$$

This is a characteristic equation of the form;

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + c = 0 \tag{2.14}$$

By Routh-Hurwitz criterion the eigen values(λ) are negative iff, $a_1 > 0$, $c > 0$, and $a_1a_2 > c$, hence the equilibrium point $E_2(X^*, Y^*, Z^*)$ locally asymptotically stable otherwise it is unstable.

2.3 Global stability

We prove the global stability of $E_1(X^*, Y^*, 0)$ and $E_2(X^*, Y^*, Z^*)$ as follows; According to Dubey and Upadhyay (2004) as cited in Akugizibwe (2010) we prove global stability of E_1 by using the Bendixon-Dulac criterion that gives conditions for non-existence of periodic orbits. Considering the system of equations 2.5, 2.6, 2.7 and using $E_1(X^*, Y^*, 0)$ we obtain;

$$\begin{aligned} f_1(X,Y) &= \frac{dX}{dt} = \beta_1 X - p_1 X^2 \\ f_2(X,Y) &= \frac{dY}{dt} = \beta_2 Y - p_2 Y^2 + \sigma X \end{aligned}$$

We now show that there are no closed orbits in the positive X, Y plane. We first choose a Dulac function, according to the work of (Osuna & Vargas-De-León, 2015) in their article : Construction of Dulac functions for mathematical models in population biology, they established that the Dulac function for Lotka Voltera systems is of the form $\frac{1}{X_1 X_2}$ in this case we choose the Dulac function D as $\frac{1}{XY}$. Let $F(X,Y) = f_1(X,Y,) + f_2(X,Y)$

$$\begin{aligned} DF(X,Y) &= \frac{1}{XY}(\beta_1 X - p_1 X^2) + \frac{1}{XY}(\beta_2 Y - p_1 Y^2 + \sigma X) \\ &= \frac{X}{XY}(\beta_1 - p_1 X) + \frac{Y}{XY}(\beta_2 - p_1 Y) + \frac{1}{XY}(\sigma X) \\ div(DF) &= \frac{d}{dX}\left(\frac{\beta_1}{Y} - \frac{p_1 X}{Y}\right) + \frac{d}{dY}\left(\frac{\beta_2}{X} - \frac{p_2 Y}{X}\right) + \frac{d}{dY}(\frac{\sigma}{Y}) \\ div(DF) &= -\frac{p_1}{Y} - \frac{p_2}{X} - \frac{\sigma}{Y} \end{aligned}$$

Where

$$\begin{aligned} p_1 &= \frac{r_1}{K_1} \\ p_2 &= \frac{r_2}{K_2} \end{aligned}$$

$div(DF) \neq 0$ and $div(DF)$ has the same negative sign everywhere in the X, Y plane if $\sigma > 0$, which implies that ,

$$\begin{aligned} \frac{dX}{dt} &= f_1(X,Y) \\ \frac{dY}{dt} &= f_2(X,Y) \end{aligned}$$

has no closed orbits in the X, Y plane hence no periodic solutions in the entire region $\Omega \subset \mathbb{R}^3_+$ hence the equilibrium point $E_1(X^*, Y^*, 0)$ is globally stable if
$r_1 > 0, r_2 > 0, \sigma > 0$ for all $t > 0$, otherwise it is globally unstable. To wind up on stability, we determine the global stability of $E_2(X^*, Y^*, Z^*)$ using a suitable lyapunov function.

Theorem 2.3. *Assuming the co-existence equilibrium point $E_2(X^*, Y^*, Z^*)$ is locally asymptotically stable in $\Omega \subset \mathbb{R}^3_+$ then it is globally stable in $\mathbb{R}^3_+$*

Proof. Consider the positive definite Lyapunov function at $E_2(X^*, Y^*, Z^*)$

$$\begin{aligned} V(X,Y,Z) &= \left(X - X^* - X^* \ln\frac{X}{X^*}\right) + \left(Y - Y^* - Y^* \ln\frac{Y}{Y^*}\right) \\ &\quad + \left(Z - Z^* - Z^* \ln\frac{Z}{Z^*}\right) \end{aligned}$$

Differentiating $V(X,Y,Z)$ with respect to time t, and using equations 2.5, 2.6, 2.7we obtain

$$\begin{aligned}
\frac{dV}{dt} &= \frac{X-X^*}{X}\left(X(\beta_1 - p_1X - \frac{\omega_1 Z}{a+X})\right) + \frac{Y-Y^*}{Y}\left(Y(\beta_2 - p_2Y) + \sigma X\right) \\
&\quad + \frac{Z-Z^*}{Z}\left(Z(\beta_3 - p_3Z + \frac{\omega_2 X}{a+X})\right) \\
\frac{dV}{dt} &= (X-X^*)\left[\left(-p_1X - \frac{\omega_1 Z}{a+X}\right) - \left(-p_1X^* - \frac{\omega_1 Z^*}{a+X^*}\right)\right] + \\
&\quad (Y-Y^*)\left[-p_2Y - (-p_2Y^*) + \frac{Y-Y^*}{Y}(\sigma X - \sigma X^*)\right] + \\
&\quad (Z-Z^*)\left[\left(-p_3Z + \frac{\omega_2 X}{a+X}\right) - \left(-p_3Z^* + \frac{\omega_2 X^*}{a+X^*}\right)\right]
\end{aligned}$$

which simplifies to,

$$\begin{aligned}
\frac{dV}{dt} &= -p_1(X-X^*)^2 - \frac{(X-X^*)(Z-Z^*)(\omega_1-\omega_2)}{(a+X)(a+X^*)} - P_2(Y-Y^*)^2 \\
&\quad -\sigma(Y-Y^*)\left(\frac{X^*Y - XY^*}{YY^*}\right)
\end{aligned}$$

□

Therefore $\frac{dV}{dt} < 0$, which is negative definite, therefore, $E_2(X^*, Y^*, Z^*)$ is globally asymptotically stable.
Since the co-existence equilibrium is locally stable under certain conditions and globally stable then both the prey and the predator can co-exist at equilibrium point with the creation of a reserve area.

3 Results and Discussion

In this section we present numerical simulations for our model. The parameters values used in simulation are as shown in the table below

Table 3: Parameter values of the model

Parameter	Value	Source
r_1	2.3	(Mpele et al., 2014)
r_2	3.0	estimated
r_3	4.8	(Mpele et al., 2014)
K_1	200	(Mpele et al., 2014)
K_2	300	estimated
K_3	600	(Mpele et al., 2014)
σ	0.2	estimated
b	20	estimated
c_1	0.05	estimated
c_2	0.02	estimated
d_1	0.3	(Mayengo et al., 2014)
d_2	0.25	estimated
d_3	0.2	(Mayengo et al., 2014)
m	0.3	estimated
E_1	0.6	estimated
E_2	1.5	(Mayengo et al., 2014)
q_1	0.14	(Raymond et al., 2019)
q_2	0.125	(Raymond et al., 2019)

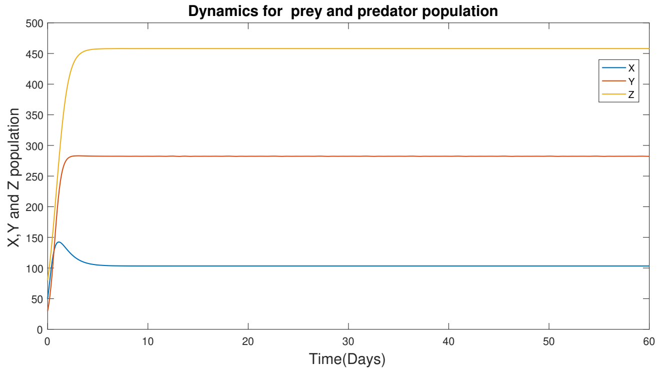

Figure 2: Prey and predator population with time for parameters given above

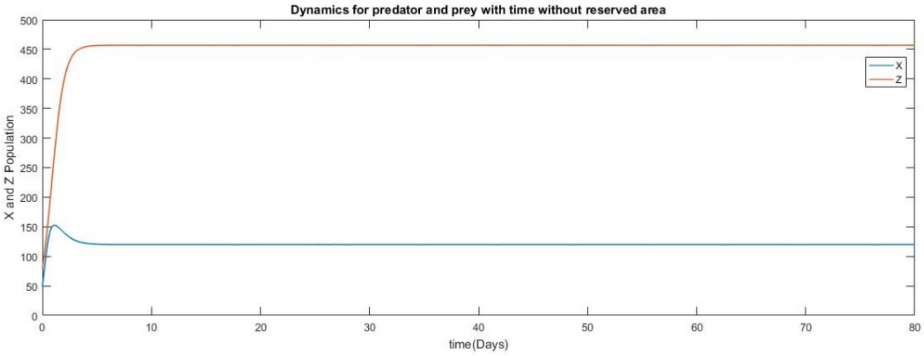

Figure 3: The population of the haplochromines (prey) and the Nile perch with time

From figure 2 and figure 3 the population of both the prey and the predator converge to their equilibrium and attains stability then grows at constant rate. However the prey population in the unreserved area oscillates briefly and becomes stable at a much lower population hence takes less time to reach equilibrium point compared to the prey in figure 3 without a reserve area. This shows that a reserve area hastens attainment of stability. Figure 2 also shows that haplochromines and the Nile perch can co-exist at equilibrium point for the initial parameter values in presence of a reserve area.

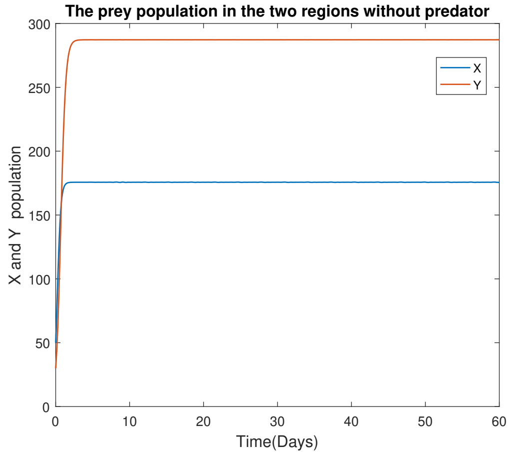

Figure 4: The population of X and Y with time

Figure 4 shows that the population of the prey in both areas converges to equilibrium and becomes stable, then grows at constant rate. It shows that the prey in the two regions can co-exist if $\sigma \leq 0.3$ as seen in figure 5. It also shows that the predator free equilibrium is stable.Without a reserve area predator free equilibrium cannot be stable

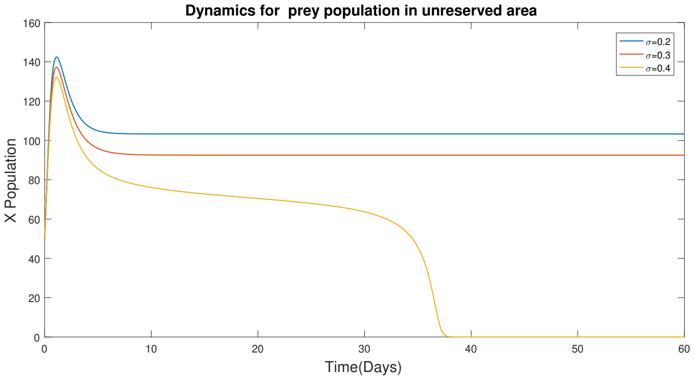

Figure 5: The prey in the unreserved area for various values of σ

Figure 5 shows that the prey population in the unreserved area decreases as migration rate (σ) increases. However the population oscillates briefly before approaching equilibrium and becoming stable for lower values of migration rate(σ). When $\sigma = 0.4$ the prey population goes to extinction after about 38 days. When more prey migrates it increases the decay rate beyond the growth rate of the prey and this pushes the prey to extinction. The prey population in the unreserved area exists if $\sigma \leq 0.3$, therefore the haplocromine(prey) and the Nile perch(predator) can co-exist if $\sigma \leq 0.3$.

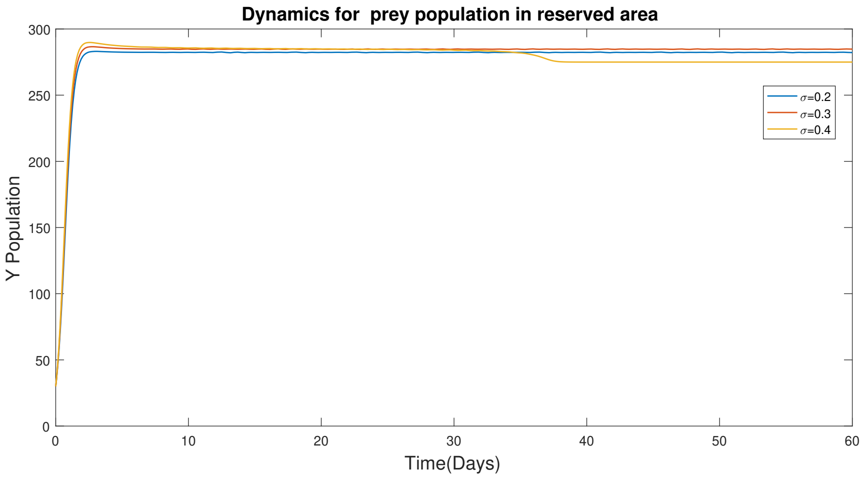

Figure 6: The population of haplochromines in the reserve area at various migration rates (σ)

Figure 6 shows that population of the prey rises sharply then slightly decreases before approaching equilibrium and attaining stability. At equilibrium the population grows at a constant rate as migration rate (σ) of the prey increases, with a very a small decrease after about 38 days when $\sigma = 0.4$. This slight decrease coincides with the decrease to extinction of the prey population in the unreserved area because after about 38 days there were zero migrating fishes. The haplocromine(prey) population in the reserved area takes shorter time to attain stability and is more stable compared to the prey in the unreserved area.

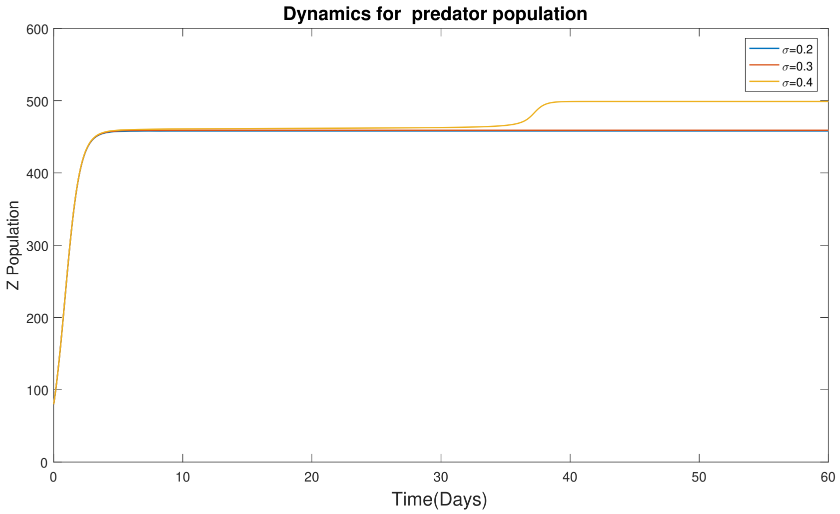

Figure 7: The population of Nile perch in the unreserved area for various migration rates(σ)

Figure 7 shows that the population of the predator rises steeply before converging to equilibrium and attaining stability. The population eventually grows at a constant rate. It can also be observed that the population of the predator increases slightly with increasing values of migration rate (σ) of the prey. When $\sigma = 0.4$ and after about 38 days the population of the predator increases and attains new equilibrium before becoming stable. This increase coincided with the decrease to extinction of the haplochromine population in the unreserved area. The population of the predator increases after about 38 days as a result of conversion of the prey into the predator. It can be observed that for certain values of migration rate $\sigma \leq 0.3$ the population of the predator and can co-exist with prey

4 CONCLUSIONS

In this research we have formulated and analyzed a two species prey-predator model with reserve area and prey refuge for Nile perch and haplochromines. The following results were obtained from the analysis

a Stability analysis of equilibrium points established that the extinction equilibrium is unstable, the predator free equilibrium, and the coexistence equilibrium exist and are both locally stable. The predator free equilibrium is globally stable when $r_1 > 0$, $r_2 > 0$, $\sigma > 0$ for all $t > 0$, this implies that a reserve area is necessary for the predator free equilibrium to be globally stable. The analytic results also show that the coexistence equilibrium is Lyapunov stable.

b Numerical simulations show that the coexistence equilibrium is stable when the migration rate $\sigma \leq 0.3$.

We conclude from these results that the prey and the predator in the unreserved area can coexist with the prey in reserved area at a stable equilibrium hence creation of a reserve area has a stabilizing effect on the prey-predator dynamic system and can be used to conserve the endangered haplochromines.

We therefore recommend to the fishery managers, as an enhancement of the Kenyan government policy spelt out in the wildlife conservation and management act of 2013 regulation 2016, that creation of conservation reserve areas should be extended beyond coastal line to inland fresh water fisheries like Lake Victoria that are threatened with loss of biodiversity in order to protect endangered species from extinction.

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