$k-$type Slant Helices for Modified Orthogonal Saban Frame on S_1^2 and H^2

Abstract

This article examines slant helices for modified orthogonal Saban frames on timelike and spacelike curves with geodesic curvature on S_1^2 and hyperbolic curves with geodesic curvature on H_0^2. In particular, it presents statements and proofs of novel theories for $k-$type slant helices.

1. Introduction

Surfaces and curves are two fundamental topics in differential geometry and curve theory. In particular not only the domain of geometers, but also physics and many engineering fields. For this reason, the field of curve theory has been examined from a new perspective. Frenet frames obtained from curves were constructed in Euclidean, Lorentzian, Galilean–Pseudo-Galilean spaces, and their geometric properties were examined [1-4]. The construction of Frenet frames was given for every dimension in these spaces. In order to add new theories to geometry. Furthermore, Frenet frames served as an important resource for studies in many fields such as electrical engineering, biology, and physics. The derivation of modified Frenet frames, equiform frames, quasi-frames, and Sabban frames from Frenet frames opened up many different avenues for geometers, offering the opportunity to examine new concepts [5-7]. The study of all these frames within Euclidean, Lorentzian, Galilean, and Pseudo-Galilean spaces has provided means to establish a tremendous theory. Furthermore, the study of the spatial forms of these frames in these spaces has enabled research to advance to much higher levels. Initially, Sasai defined a modified orthogonal framework based on curvature and torsion, starting from the Frenet framework in 3-dimensional Euclidean space [8-9]. Later, these frameworks were examined in terms of their spatial forms.

Firstly,the Saban frame was described by Koenderink [10] and then, Taşkoprü and Ali studied Smarandache curves according to the Saban frame in the Euclidean space [11, 12].

The Sabban framework and modified frameworks constructed in Euclidean and other spaces [13-14]. Also, it was studied in Minkowski space by Yakut et al. [15, 16].

In the theory of curves, one of the important areas on which many structures are built and new theories are constructed is helices. Helices have also enabled the expression and proof of strong theories in different spaces. The definition of the concept of slant helices has paved the way for the emergence of completely new ideas for differential geometers [17-19].

In recent years, in addition to slant helices, (k.m)-type slant helices have been defined by M. Yıldırım and M.Bektaş [20], adding new theories to differential geometry and providing a completely different perspective.

The development of the Saban framework across many areas has intruced a new perspective on curve theory together with the conspicuous results obtained from slant helices,has served as the roadmap for the construction of tis article.

The aim of this work is to express and prove new characterizations that provides the necessary and sufficient conditions for modified orthogonal Sabban frames to be slant helices in Minkowski space.

2. Basic Concepts

Let $<,>$ be symmetric non-degenerate (0,2) tensor with index one on R^3 defined by

$$<v,w>=-v_1w_1+v_2w_2+v_3w_3$$

for $v=(v_1,v_2,v_3)$ and $w=(w_1,w_2,w_3)$. By the standard isomophism between R^3 to T_pR^3, we extended $<,>$ to be a symetric non-degenerate (0,2) tensor field on R^3. The pair (R^3, $<,>$) is called a Minkowski (Lorentz) 3-space with index one. We denote it by E_1^3 and call $<,>$ a Minkowski metric with index one.

For $x \in E_1^3$, we put

$$S_1^2=\left\{ x=(x_1,x_2,x_3) \in E_1^3 \right| -x_1^2+x_2^2+x_3^2=1 \left.\right\}$$

and

$$H_1^3=\left\{ x=(x_1,x_2,x_3) \in E_1^3 \right| -x_1^2+x_2^2+x_3^2=-1 \left.\right\},$$

where $<,>$ is a Minkowski metric. In particular S_1^2 and H_0^2 are called a de Sitter space and Hyperbolic space (anti-de Sitter space) respectively. The H_0^2 space has two connected components: $H_{0,+}^2$ and $H_{0,-}^2$. In this work, H_0^2 is taken as $H_{0,+}^2$. That is, the upper leaf.

It follows that a tangent vector x to E_1^3 corresponds exactly one among the following Minkowski casusal characters

1. x is a spacelike vector if and only if $<x,x> \ >0$ or x =0.
2. x is a lightlike vector or a *null vector* if and only if $<x,x> \ =0$ and x $\neq 0$.
3. x is a timelike vector if and only if $<x,x> \ <0$.

For each point p $\in E_1^3$, the set of null vectors in the tangent space $T_pE_1^3$ is called the null-cone at the point p. The norm of a tangent vectors x to E_1^3 having any one of three causal characters is doneted and defined by $\|x\|=\sqrt{|<x,x>|}$. It is trivial to mention that a tangent vector x to E_1^3 is called a unit vector if and only if $\|x\|=1$, i.e., if and only if $<x,x> \ =\mp 1$. As usual, two tangent vectors x and y in to E_1^3 are called orthogonal if and only $<x,y> \ =0$.

Let $\alpha : I \to E_1^3$ be an arbitrary curve in E_1^3 and α' denote its velocity vector field. Then α may locally or globally have exactly one of the following casual characters of its velocity vectors $\alpha'(s)$ [21].

1. α is spacelike if and only if all its velocity vectors $\alpha'(s)$ are spacelike

(i.e. $<\alpha'(s),\alpha'(s)>>0$ or $\alpha'(s)=0$) for all possible s.

1. α is null (lightlike) if and only if all its velocity vectors $\alpha'(s)$ are null (lightlike)

(i.e. $<\alpha'(s), \alpha'(s)> = 0$ or $\alpha'(s) \neq 0$) for all possible s.

1. α is timelike if and only if all its velocity vectors $\alpha'(s)$ are timelike

(i.e. $<\alpha'(s), \alpha'(s)> < 0$) for all possible s.

3) The Modified Orthogonal Saban Frame With Geodesic Curvature On S_1^2 And H_0^2

Definition 3.1: Let α be a timelike (or spacelike) unit speed curve on S_1^2 (or H_0^2) and s be the arc length parameter of α and $\alpha'(s) = T(s)$. Assume that $T(s)$ is the unit tangent vector of $\alpha(s)$ and $\gamma(s) = \alpha(s) \bigwedge T(s)$, where $\alpha(s) \bigwedge T(s)$ is vector product of $\alpha(s)$ and $T(s)$. The relations between those vectors and the classical Saban frame (α, t, ζ) for timelike curve of $\eta(s)$ at non-zero points of κ_g are

$$\begin{cases} \alpha(s) = \alpha(s) \\ T(s) = \kappa_g(s) t(s) \\ \gamma(s) = \kappa_g(s) \zeta(s). \end{cases} \tag{3.1}$$

This frame $\{\alpha(s), T(s), \gamma(s)\}$ is called the modified orthogonal Saban frame of α on S_1^2 (or H_0^2) [14] .

Theorem 3.1: Let $\eta : I \subset R \longrightarrow S_1^2$ and $\{\eta(s), T(s), \gamma(s)\}$ be a unit speed regular timelike curve on S_1^2 and the modified Saban frame, respectively. Thus, the derivative formulas of modified orthogonal Saban frame for timelike curve of $\eta(s)$ is

$$\begin{bmatrix} \eta'(s) \\ T'(s) \\ \gamma'(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\kappa_g(s)} & 0 \\ \kappa_g(s) & \frac{\kappa_g'(s)}{\kappa_g(s)} & \kappa_g(s) \\ 0 & \kappa_g(s) & \frac{\kappa_g'(s)}{\kappa_g(s)} \end{bmatrix} = \begin{bmatrix} \eta(s) \\ T(s) \\ \gamma(s) \end{bmatrix} [14] . \tag{3.2}$$

Theorem 3.2: Let $\omega : I \subset R \longrightarrow S_1^2$ and $\{\omega(s), T(s), \gamma(s)\}$ be a unit speed regular spacelike curve on S_1^2 and the modified Saban frame for spacelike curve of $\omega(s)$, respectively. Thus, the derivative formulas of modified orthogonal Saban frame for spacelike curve of $\omega(s)$ is

$$\begin{bmatrix} \omega'(s) \\ T'(s) \\ \gamma'(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\kappa_g(s)} & 0 \\ -\kappa_g(s) & \frac{\kappa_g'(s)}{\kappa_g(s)} & -\kappa_g(s) \\ 0 & -\kappa_g(s) & \frac{\kappa_g'(s)}{\kappa_g(s)} \end{bmatrix} = \begin{bmatrix} \omega(s) \\ T(s) \\ \gamma(s) \end{bmatrix} [14] . \tag{3.3}$$

Theorem 3.3: Let $\psi : I \subset R \longrightarrow H_0^2$ and $\{ \psi(s), T(s), \gamma(s)\}$ be a unit speed regular hyperbolic curve on H_0^2 and the modified Saban frame on H_0^2, respectively. Hence, the derivative formulas of modified orthogonal Saban frame is

$$\begin{bmatrix} \psi'(s) \\ T'(s) \\ \gamma'(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\kappa_g(s)} & 0 \\ \kappa_g(s) & \frac{\kappa_g'(s)}{\kappa_g(s)} & \kappa_g(s) \\ 0 & -\kappa_g(s) & \frac{\kappa_g'(s)}{\kappa_g(s)} \end{bmatrix} = \begin{bmatrix} \psi(s) \\ T(s) \\ \gamma(s) \end{bmatrix} [14] \ . \tag{3.4}$$

4) k-type Slant Helices for The Modified Orthogonal Saban Frame With Geodesic Curvature On S_1^2 And H_0^2

Definition 4.1 Let γ (s) be an unit-speed regular curve on E_1^3 with Frenet frame $\{V_1(s), V_2(s), V_3(s)\}$. γ (s) be an unit-speed regular curve said to be a k-type slant helix if there exists a non-zero fixed direction vector U such that $< V_k(s), U >= C$, where C is a real constant and $k \in \{1,\ 2,\ 3\}$ [22].

Theorem 4.1: Let $\eta : I \subset R \longrightarrow S_1^2$ be an unit-speed regular timelike curve on S_1^2 and .$\{\eta(s), T(s), \gamma(s)\}$ be Modified Orthogonal Saban frame of $\eta(s)$. If the curve $\eta(s)$ is a 1-type slant helix, it is also a 3-type slant helix.

Proof: Let's assume that the timelike curve $\eta(s)$ in S_1^2, given by the Modified orthogonal Saban frame $\{\eta(s), T(s), \gamma(s)\}$. If $\eta(s)$ is a 1-type slant helix, then there is a c_1 constant, such that

$$\langle \eta(s), u \rangle = c_{1.} \tag{4.1}$$

Now, if we take the derivative of Equation (4.1) and using equation (3.2), we may write

$$\langle \eta'(s), u \rangle = 0$$

$$\frac{1}{\kappa_g(s)} \langle T(s), u \rangle = 0$$

$\kappa_g(s) \neq 0$ and

$$\langle T(s), u \rangle = 0. \tag{4.2}$$

If differentiate of (4.2) and using (3.2), we get

$$\kappa_g(s) \langle \eta(s), u \rangle + \frac{\kappa_g'(s)}{\kappa_g(s)} \langle T(s), u \rangle + \kappa_g(s) \langle \gamma(s), u \rangle = 0$$

$$\kappa_g(s) c_1 + \kappa_g(s) \langle \gamma(s), u \rangle = 0$$

$$\langle \gamma(s), u \rangle = -c_1.$$

Thus, $\eta(s)$ is a 3-type slant helix.

Theorem 4.2: Let $\eta : I \subset R \longrightarrow S_1^2$ be an unit-speed regular timelike curve on S_1^2 and $\{\eta(s), T(s), \gamma(s)\}$ be Modified Orthogonal Saban frame of $\eta(s)$. If the curve $\eta(s)$ is a 2-type slant helix and $\kappa_g'(s) = \kappa_g^2(s)$, it is also a 1-type slant helix and 3-type slant helix.

Proof: Let's assume that the timelike curve $\eta(s)$ in S_1^2, given by with the Modified orthogonal Saban frame $\{\eta(s), T(s), \gamma(s)\}$.If $\eta(s)$is a 1-type slant helix, then there is a c_2 constant, such that

$$\langle T(s), u \rangle = c_{2.} \tag{4.3}$$

Now, if we differentive of Equation (4.3) and using equation (3.2), we may write

$$\kappa_g(s)\langle \eta(s), \mathrm{u}\rangle + \frac{\kappa_g'(s)}{\kappa_g(s)}\langle T(s), u\rangle + \kappa_g(s)\langle \gamma(s), u\rangle = 0$$

$$\kappa_g(s)(\langle \eta(s), \mathrm{u}\rangle + \langle \gamma(s), u\rangle) = -\frac{\kappa_g'(s)}{\kappa_g(s)}c_2$$

$$\langle \eta(s), \mathrm{u}\rangle + \langle \gamma(s), u\rangle = -\frac{\kappa_g'(s)}{\kappa_g^2(s)}c_2$$

Thus, From the hypotesis $\eta(s)$ is a 1-type and 3-type slant helix.

Theorem 4.3: Let $\eta : I \subset R \longrightarrow S_1^2$ be an unit-speed regular timelike curve on S_1^2 and $\{\eta(s), T(s), \gamma(s)\}$ be Modified Orthogonal Saban frame of $\eta(s)$. If the curve $\eta(s)$ is a 3-type slant helix and $\kappa_g(s) = as + b$ $(a, b \in R)$, it is also a 2-type slant helix

Proof: Let's assume that the timelike curve $\eta(s)$ in S_1^2, given by the Modified orthogonal Saban frame $\{\eta(s), T(s), \gamma(s)\}$. If $\eta(s)$ is a 3-type slant helix, then there is a c_3 constant, such that

$$\langle \gamma(s), u\rangle = c_3 \tag{4.4}$$

Now, if we take the derivative of Equation (4.4) and using equation (3.2), we may write

$$\kappa_g(s)\langle T(s), u\rangle + \frac{\kappa_g'(s)}{\kappa_g(s)}\langle \gamma(s), u\rangle = 0$$

$$\kappa_g(s)\langle T(s), u\rangle + \frac{\kappa_g'(s)}{\kappa_g(s)}c_3 = 0$$

$$\langle T(s), u\rangle = -c_3\kappa_g'(s).$$

Thus,from the hypotesis $\eta(s)$ is a 2-type slant helix.

Using the proof methods of these theorems, the following theorems can be proven easily.

Theorem 4.4: Let $\omega : I \subset R \longrightarrow S_1^2$ be an unit-speed regular spacelike curve on S_1^2 and $\{\omega(s), T(s), \gamma(s)\}$ be Modified orthogonal Saban frame of $\omega(s)$. If the space curve $\omega(s)$ is a 1-type slant helix, it is also a 3-type slant helix

Theorem 4.5: Let $\omega : I \subset R \longrightarrow S_1^2$ be an unit-speed regular spacelike curve on S_1^2 and $\{\omega(s), T(s), \gamma(s)\}$ be Modified orthogonal Saban frame of $\omega(s)$. If the spacelike curve $\omega(s)$ is a 2-type slant helix and $\kappa_g'(s) = \kappa_g^2(s)$, it is also a 1-type slant helix and 3-type slant helix

Theorem 4.6: Let $\omega : I \subset R \longrightarrow S_1^2$ be an unit-speed regular spacelike curve on S_1^2 and $\{\omega(s), T(s), \gamma(s)\}$ be Modified orthogonal Saban frame of $\omega(s)$. If the spacelike curve $\omega(s)$ a 3-type slant helix and $\kappa_g(s) = as + b$ $(a, b \in R)$, it is also a 2-type slant helix

Theorem 4.7: Let $\psi : I \subset R \longrightarrow H_0^2$ be an unit-speed regular curve on H_0^2 and $\{\psi(s), T(s), \gamma(s)\}$ be Modified orthogonal Saban frame of $\psi(s)$. If the space curve $\psi(s)$ is a 1-type slant helix, it is also a 3-type slant helix

Theorem 4.8: Let $\psi : I \subset R \longrightarrow H_0^2$ be an unit-speed regular curve on H_0^2 and $\{\psi(s), T(s), \gamma(s)\}$ be Modified orthogonal Saban frame of $\psi(s)$. If the regular curve $\psi(s)$ is a 2-type slant helix and $\kappa_g'(s) = \kappa_g^2(s)$, then a 1-type and 3-type slant helix

Theorem 4.9: : Let $\psi : I \subset R \longrightarrow H_0^2$ be an unit-speed regular curve on H_0^2 and $\{\psi(s), T(s), \gamma(s)\}$ be Modified orthogonal Saban frame of $\psi(s)$. If the spacelike curve $\psi(s)$ is a 3-type slant helix and $\kappa_g(s) = as + b$ $(a, b \in R)$, it is also a 2-type slant helix

5. Conclusion

The modified Saban collisions constructed for timelike and spacelike curves in S_1^2 and regular curves in H_0^2 represent a significant area of differential geometry, which has been the focus of considerable research in recent years. Furthermore, an additional significant domain pertains to slant helices, which find extensive application in numerous disciplines by both physicists and geometers. This study, which examines these two topics, covers a very broad field but is still a gold mine waiting to be explored in the literature, with the potential for many more works to emerge.

This organised study examines the characterisations of the following types of slant helices: 1st, 2nd and 3rd-type slant helices for timelike curves in S_1^2, 1st, 2nd and 3rd-type slant helices for spacelike curves in S_1^2, and 1st, 2nd and 3rd-type slant helices for regular curves in H_0^2. These characterisations are defined for modified orthogonal Saban frames. The definition of three-type slant helices has been established for regular curves in H_0^2.

Consequently, the slant helices defined for modified orthogonal Saban frames have the potential to generate novel research opportunities for mathematicians and physicists.

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