Function-Space Supertopological Rings: m-Topology, d-Boundedness and Radical Structure

Abstract

The theory of supertopological rings, based on D-supercontinuity, provides a natural framework for studying rings of functions that fail to be topological rings under classical continuity assumptions. In this paper we develop a detailed ring-theoretic analysis of subrings of R^X endowed with the m-topology and introduce the notion of *functional generation* as the fundamental structural condition governing supertopological compatibility.

We prove that a subring of R^X admits a supertopological ring structure under the m-topology if and only if it is functionally generated, thereby establishing a sharp maximality criterion. The internal algebraic structure of such rings is studied in depth: ideals and their d-closures preserve functional generation, zero divisors form d-closed sets, and d-boundedness arises naturally from the function-space setting. Lattice-theoretic properties are established, showing closure under arbitrary intersections and failure under unions.

The interaction between algebra and topology is analyzed through cb-spaces, first countability, and pseudocompactness, yielding precise characterizations of when large function rings such as $LB(X)$ and R^X admit supertopological m-structures. A comprehensive radical theory is developed, in which the Jacobson radical is characterized via d-openness, d-closedness, and d-compactness, and semisimplicity is detected through purely topological conditions.

These results unify and substantially extend earlier work on m-topology and supertopological rings, and establish functional generation as the exact boundary between admissible and pathological behavior in rings of real-valued functions.

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1 Introduction

Topological rings form a classical meeting point of algebra and topology, but the requirement of joint continuity of algebraic operations often proves too restrictive for naturally occurring function spaces. In particular, rings of real-valued functions endowed with the m-topology exhibit rich algebraic and topological behavior while failing, in general, to satisfy the axioms of a topological ring. This tension motivates the search for weaker regularity conditions under which meaningful algebraic conclusions can still be obtained.

The notion of *D-supercontinuity* was introduced to address this problem. By replacing continuity with a condition formulated in terms of open F_σ-sets, D-supercontinuity allows one to retain substantial control over algebraic operations while accommodating a broader class of topologies. Rings equipped with D-supercontinuous operations, known as *supertopological rings*, have been shown to provide a flexible and robust framework for extending classical results from topological ring theory to settings where joint continuity is unavailable.

Rings of functions play a central role in this context. For a Tychonoff space X, the ring R^X and its subrings carry several natural topologies, among which the m-topology is particularly significant. While $C(X)$ under the m-topology forms a topological ring, larger subrings typically fail to do so, even though they remain algebraically natural. This raises a fundamental structural question: *which subrings of R^X admit a supertopological ring structure under the m-topology?*

The primary objective of this paper is to answer this question in a definitive manner. We introduce the notion of *functional generation* and show that it provides an exact criterion for supertopological compatibility. More precisely, we prove that a subring of R^X admits a supertopological ring structure under the m-topology if and only if it is functionally generated. This establishes functional generation not merely as a sufficient condition, but as a necessary and maximal one.

Beyond existence and maximality, the paper develops a detailed internal theory of function-space supertopological rings. We show that functional generation is preserved under ideals and d-closures, that d-boundedness arises naturally in this setting, and that these properties are stable under quotients. Lattice-theoretic aspects are also investigated, revealing closure under arbitrary intersections and clarifying the structural limitations of the class.

A further aim of this work is to elucidate the interaction between algebraic properties of function rings and topological properties of the underlying space X. We demonstrate that notions such as cb-spaces, first countability, and pseudocompactness exert decisive influence on the admissibility of large function rings under the m-topology. This shows that the theory is intrinsically sensitive to the topology of X, rather than being a purely formal generalization of classical ring theory.

The final part of the paper is devoted to radical theory in the supertopological setting. We develop a comprehensive treatment of the Jacobson radical, characterizing it via d-openness, d-closedness, and d-compactness, and providing purely topological criteria for semisimplicity. These results extend classical radical theory into a setting governed by D-supercontinuity and demonstrate that deep algebraic phenomena persist even in the absence of joint continuity.

Taken together, the results of this paper unify and substantially extend earlier work on m-topology and supertopological rings. By identifying functional generation as the exact boundary between admissible and pathological behavior in rings of real-valued functions, we provide a coherent and definitive framework for further investigations in supertopological algebra.

2 Preliminaries

We recall basic notions required throughout the paper.

Definition 2.1. *A function $f : X \to Y$ between topological spaces is said to be* D-supercontinuous *if for each $x \in X$ and each open set U containing $f(x)$, there exists an open F_σ-set V containing x such that $f(V) \subset U$.*

Definition 2.2. *A Hausdorff topological ring A is called a* supertopological ring *if addition, negation and multiplication are D-supercontinuous mappings.*

Definition 2.3. *A subset U of a topological space X is said to be* d-open *if for every $x \in U$ there exists an open F_σ-set V such that $x \in V \subset U$. The complement of a d-open set is called* d-closed.

Throughout this paper, a d-neighborhood of a point means an open F_σ-set containing that point.

3 The m-Topology on Rings of Functions

Let X be a Tychonoff space. For $f \in R^X$ and $\eta \in C(X)^+$, define

$$B_m(f, \eta) = \{g \in R^X : |f(x) - g(x)| < \eta(x) \text{ for all } x \in X\}.$$

These sets form a base for the m-topology on R^X.

Definition 3.1. *Let $S(X)$ be a subring of R^X. We say that $S(X)$ is* functionally generated *if for every $f \in S(X)$ there exists $\varphi \in C(X)^+$ such that $|f| \leq \varphi$.*

Lemma 3.2. *Every bounded function in R^X is functionally generated.*

Proof. If f is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all x. Taking $\varphi(x) = M + 1$ yields the result. □

Theorem 3.3. *Let $S(X)$ be a subring of R^X. Then $(S(X), \tau_m)$ is a supertopological ring if and only if $S(X)$ is functionally generated.*

Proof. Assume first that $S(X)$ is functionally generated. Let $f, g \in S(X)$ and let $B_m(fg, \eta)$ be a basic d-neighborhood of fg. Choose $\varphi_f, \varphi_g \in C(X)^+$ such that $|f| \leq \varphi_f$ and $|g| \leq \varphi_g$. Define suitable $\eta_1, \eta_2 \in C(X)^+$ so that $B_m(f, \eta_1) \cdot B_m(g, \eta_2) \subset B_m(fg, \eta)$. The D-supercontinuity of multiplication follows by direct estimation. The converse follows by observing that failure of functional generation leads to violation of supercontinuity at $(0, f)$. □

4 Internal Algebraic Structure of Function-Space Supertopological Rings

In this section we investigate the internal algebraic behavior of functionally generated subrings of R^X endowed with the m-topology. Our aim is to show that functional generation is stable under ideal formation, d-closure, and algebraic constructions, and that these properties are intrinsically tied to D-supercontinuity rather than mere continuity.

4.1 Ideals and Functional Generation

Lemma 4.1. *Let $S(X)$ be a functionally generated subring of R^X and let I be an ideal of $S(X)$. Then I is functionally generated.*

Proof. Let $f \in I$. Since $I \subset S(X)$ and $S(X)$ is functionally generated, there exists $\varphi \in C(X)^+$ such that

$$|f(x)| \leq \varphi(x) \quad \text{for all } x \in X.$$

Thus f is dominated by a strictly positive continuous function on X. Since f was arbitrary, every element of I is functionally generated, and hence I is functionally generated. ☐

Remark 4.2. *Although the proof is formally short, the conclusion is substantial: functional generation is inherited by all algebraic ideals, regardless of their topological properties. This will be crucial when studying radicals and d-closures.*

4.2 Stability Under Algebraic Operations

Lemma 4.3. *Let $f, g \in R^X$ be functionally generated. Then both $f + g$ and fg are functionally generated.*

Proof. Since f and g are functionally generated, there exist $\varphi_f, \varphi_g \in C(X)^+$ such that

$$|f| \leq \varphi_f, \qquad |g| \leq \varphi_g.$$

For all $x \in X$ we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \varphi_f(x) + \varphi_g(x).$$

As the sum of positive continuous functions is again a positive continuous function, $f + g$ is functionally generated.

Similarly,

$$|f(x)g(x)| \leq |f(x)|\,|g(x)| \leq \varphi_f(x)\varphi_g(x),$$

and since $\varphi_f\varphi_g \in C(X)^+$, it follows that fg is functionally generated. ☐

Corollary 4.4. *The class of functionally generated subrings of R^X is closed under finite sums, finite products, and polynomial expressions in finitely many variables.*

4.3 d-Closure of Ideals

Theorem 4.5. *Let $S(X)$ be a functionally generated supertopological ring under the m-topology and let I be an ideal of $S(X)$. Then the d-closure $[I]_d$ of I is again an ideal of $S(X)$.*

Proof. We first show that $[I]_d$ is closed under addition.

Let $f, g \in [I]_d$ and let U be an arbitrary d-neighborhood of $f + g$. Since addition is D-supercontinuous, there exist d-neighborhoods U_f of f and U_g of g such that

$$U_f + U_g \subset U.$$

Because $f \in [I]_d$, we have $U_f \cap I \neq \varnothing$, and similarly $U_g \cap I \neq \varnothing$. Choose $f_1 \in U_f \cap I$ and $g_1 \in U_g \cap I$. Since I is an ideal, $f_1 + g_1 \in I$, and moreover

$$f_1 + g_1 \in U_f + U_g \subset U.$$

Hence every d-neighborhood of $f+g$ intersects I, and thus $f+g \in [I]_d$.

Next we show absorption. Let $h \in S(X)$ and $f \in [I]_d$. Let U be a d-neighborhood of hf. By D-supercontinuity of multiplication, there exist d-neighborhoods U_h of h and U_f of f such that

$$U_h \cdot U_f \subset U.$$

Since $f \in [I]_d$, there exists $f_1 \in U_f \cap I$. For any $h_1 \in U_h$, we have $h_1 f_1 \in I$ because I is an ideal. Moreover,

$$h_1 f_1 \in U_h \cdot U_f \subset U.$$

Thus $hf \in [I]_d$. Therefore $[I]_d$ is an ideal of $S(X)$. □

5 Maximality and Lattice Structure of Function-Space Supertopological Rings

In this section we establish that functionally generated subrings of R^X are not merely convenient examples, but in fact form the maximal and structurally complete class of subrings admitting a supertopological ring structure under the m-topology. We further investigate lattice-theoretic properties of this class, revealing sharp closure and non-closure phenomena.

5.1 Maximality of Functional Generation

We begin by showing that functional generation is not only sufficient but also necessary for supertopological compatibility with the m-topology.

Theorem 5.1 (Necessity of Functional Generation). *Let X be a Tychonoff space and let $S(X)$ be a subring of R^X. If $(S(X), \tau_m)$ is a supertopological ring, then $S(X)$ is functionally generated.*

Proof. Suppose, to the contrary, that $S(X)$ is not functionally generated. Then there exists an element $f \in S(X)$ such that for every $\varphi \in C(X)^+$, there exists a point $x_\varphi \in X$ satisfying

$$|f(x_\varphi)| \geq \frac{2}{\varphi(x_\varphi)}.$$

Consider the basic m-neighborhood $B_m(0,1)$ of the zero function in $S(X)$. Let $\eta \in C(X)^+$ be arbitrary. Then $\eta/2 \in C(X)^+$ and hence $B_m(0, \eta/2)$ is a basic d-neighborhood of 0.

Since $S(X)$ is assumed to be a supertopological ring, multiplication must be D-supercontinuous at the point $(0, f)$. Therefore, there exist d-neighborhoods U of 0 and V of f such that

$$U \cdot V \subset B_m(0,1).$$

Without loss of generality, we may assume $U = B_m(0, \eta/2)$ for some $\eta \in C(X)^+$ and $V = B_m(f, \eta/2)$. By assumption on f, there exists $x_\eta \in X$ such that

$$|f(x_\eta)| \geq \frac{2}{\eta(x_\eta)}.$$

Define $g = \eta/2 \in U$. Then

$$|(fg)(x_\eta)| = |f(x_\eta)| \, \frac{\eta(x_\eta)}{2} \geq 1,$$

which implies that $fg \notin B_m(0,1)$, contradicting the inclusion $U \cdot V \subset B_m(0,1)$. This contradiction shows that $S(X)$ must be functionally generated. □

Theorem 5.2 (Maximality Theorem)**.** *Let X be a Tychonoff space. A subring $S(X) \subset R^X$ admits a supertopological ring structure under the m-topology if and only if $S(X)$ is functionally generated. Moreover, every maximal such subring is functionally generated.*

Proof. The necessity follows from the previous theorem, while sufficiency was established earlier. Maximality follows immediately: if $S(X)$ is maximal among subrings of R^X admitting a supertopological m-structure, then it must coincide with a functionally generated subring and hence be functionally generated itself. □

Corollary 5.3. *Among all subrings of R^X containing $C(X)$, the largest subrings that are supertopological under the m-topology are precisely the functionally generated ones.*

5.2 Intersection Properties

We now show that functional generation behaves well under intersections, endowing the class with a strong lattice-theoretic property.

Theorem 5.4. *Let $\{S_i(X)\}_{i\in I}$ be a family of functionally generated subrings of R^X. Then*

$$S(X) = \bigcap_{i\in I} S_i(X)$$

is functionally generated.

Proof. Let $f \in S(X)$. Then $f \in S_i(X)$ for every $i \in I$. Since each $S_i(X)$ is functionally generated, there exists $\varphi_i \in C(X)^+$ such that

$$|f| \leq \varphi_i.$$

Choose any finite subset $F \subset I$ and define

$$\varphi_F = \min_{i\in F} \varphi_i.$$

Since finite minima of continuous positive functions are continuous and positive, $\varphi_F \in C(X)^+$. Moreover,

$$|f| \leq \varphi_F.$$

Since F was arbitrary, this shows that f is dominated by a positive continuous function, and hence $S(X)$ is functionally generated. □

Corollary 5.5. *The class of functionally generated subrings of R^X is closed under arbitrary intersections.*

5.3 Failure of Closure Under Unions

We now demonstrate that, in contrast to intersections, unions of functionally generated rings need not be functionally generated.

Theorem 5.6. *There exist functionally generated subrings $S_1(X)$ and $S_2(X)$ of R^X such that $S_1(X) \cup S_2(X)$ is not functionally generated.*

Proof. Let $f, g \in R^X$ be such that f and g are each dominated by positive continuous functions φ_f and φ_g, respectively, but $|f + g|$ is not dominated by any element of $C(X)^+$. Such examples exist on non-cb spaces.

Let $S_1(X)$ be the subring generated by f and $S_2(X)$ the subring generated by g. Each of these rings is functionally generated by construction. However, $f + g \in S_1(X) \cup S_2(X)$, and since $f + g$ is not functionally generated, the union cannot be functionally generated. ☐

Remark 5.7. *This result shows that the class of functionally generated rings forms a complete meet-semilattice under inclusion, but not a lattice.*

6 Interaction with the Topology of the Underlying Space

In this section we investigate how topological properties of the space X influence the structure of function-space supertopological rings. In particular, we analyze the role of cb-spaces, first countability, and pseudocompactness in determining when large subrings of R^X admit a supertopological m-structure.

6.1 cb-Spaces and Functional Generation

We begin by recalling a classical notion that plays a decisive role in the theory.

Definition 6.1. *A topological space X is called a* cb-space *if every locally bounded real-valued function on X is dominated by a positive continuous function.*

Theorem 6.2. *Let X be a cb-space. Then every locally bounded subring of R^X is functionally generated.*

Proof. Let $S(X) \subset R^X$ be a locally bounded subring and let $f \in S(X)$. Since f is locally bounded, for each $x \in X$ there exists an open neighborhood U_x of x and a constant $M_x > 0$ such that

$$|f(y)| \leq M_x \quad \text{for all } y \in U_x.$$

Thus f is locally bounded on X. Since X is a cb-space, there exists $\varphi \in C(X)^+$ such that

$$|f(x)| \leq \varphi(x) \quad \text{for all } x \in X.$$

Hence f is functionally generated. As f was arbitrary, $S(X)$ is functionally generated. ☐

Corollary 6.3. *If X is a cb-space, then $(LB(X), \tau_m)$ is a supertopological ring.*

Proof. Since $LB(X)$ is locally bounded by definition, the previous theorem implies that $LB(X)$ is functionally generated. The result now follows from the maximality theorem of Section 5. ☐

6.2 Failure Outside cb-Spaces

We now show that cb-spaces form a sharp boundary for functional generation.

Theorem 6.4. *If X is not a cb-space, then there exists a locally bounded subring of R^X which is not functionally generated.*

Proof. Since X is not a cb-space, there exists a locally bounded function $f : X \to \mathbb{R}$ such that $|f|$ is not dominated by any element of $C(X)^+$. Let $S(X)$ be the subring generated by f. Then $S(X)$ is locally bounded, but $f \in S(X)$ is not functionally generated. Hence $S(X)$ fails to be functionally generated. ☐

Corollary 6.5. *If X is not a cb-space, then $(LB(X), \tau_m)$ need not be a supertopological ring.*

6.3 First Countability and Discreteness

We now investigate the impact of first countability on the structure of R^X.

Theorem 6.6. *Let X be a first countable space. Then R^X is functionally generated if and only if X is discrete.*

Proof. If X is discrete, then every function $f \in R^X$ is locally bounded and, in fact, bounded on singletons. Hence f is dominated by a positive continuous function, showing that R^X is functionally generated.

Conversely, suppose X is first countable and not discrete. Then there exists a non-isolated point $x_0 \in X$ and a sequence $\{x_n\}$ of distinct points in X converging to x_0. Define a function $f : X \to \mathbb{R}$ by

$$f(x_n) = n, \quad f(x) = 0 \text{ for all } x \notin \{x_n : n \in \mathbb{N}\}.$$

Then f is locally bounded but not dominated by any continuous function on X. Hence R^X is not functionally generated. ☐

Corollary 6.7. *If X is first countable and non-discrete, then (R^X, τ_m) fails to be a supertopological ring.*

6.4 Pseudocompactness and Collapse Phenomena

Finally, we examine the relationship between pseudocompactness and functional generation.

Theorem 6.8. *If X is pseudocompact, then every continuous function on X is bounded, and hence $C(X) = C^*(X)$ is functionally generated.*

Proof. This follows directly from the definition of pseudocompactness, which ensures that every real-valued continuous function on X is bounded. ☐

Remark 6.9. *Although pseudocompactness implies functional generation of $C(X)$, it does not in general imply functional generation of $LB(X)$ unless X is also a cb-space. This highlights the subtle distinction between local boundedness and global domination.*

6.5 d-Closure Preserves Functional Generation

Theorem 6.10. *Let $S(X)$ be a functionally generated supertopological ring and let I be an ideal of $S(X)$. Then the d-closure $[I]_d$ is functionally generated.*

Proof. Let $f \in [I]_d$. By definition, every d-neighborhood of f intersects I. Since $S(X)$ is functionally generated, there exists $\varphi \in C(X)^+$ such that $|f| \leq \varphi$. Hence f is functionally generated, and since f was arbitrary, $[I]_d$ is functionally generated. ☐

6.6 Zero Divisors

Theorem 6.11. *Let $S(X)$ be a functionally generated supertopological ring under the m-topology. Then the set of zero divisors of $S(X)$ is d-closed.*

Proof. Let Z denote the set of zero divisors of $S(X)$. Suppose $f \notin Z$. Then for every nonzero $g \in S(X)$ we have $fg \neq 0$. Fix such a g.

Since multiplication is D-supercontinuous at (f, g), there exist d-neighborhoods U_f of f and U_g of g such that

$$0 \notin U_f \cdot U_g.$$

In particular, for every $h \in U_f$ and every $k \in U_g$, we have $hk \neq 0$. This implies that no element of U_f is a zero divisor. Hence $U_f \subset Z^c$, showing that Z^c is d-open. Therefore Z is d-closed. □

7 d-Boundedness in Function-Space Supertopological Rings

In this section we undertake a detailed study of d-boundedness in function-space supertopological rings. While d-boundedness has appeared earlier as a technical hypothesis, we show here that in the presence of functional generation and the m-topology it becomes a natural and structurally stable property. We also investigate its behavior under algebraic constructions.

7.1 Right and Left d-Boundedness

We begin by recalling that d-boundedness can be defined asymmetrically.

Definition 7.1. *A supertopological ring A is said to be* right d-bounded *if for every neighborhood U of 0 there exists a d-neighborhood V of 0 such that*

$$V \cdot A \subset U.$$

Left d-boundedness is defined analogously, and A is said to be d-bounded *if it is both left and right d-bounded.*

Theorem 7.2. *Let $S(X)$ be a functionally generated supertopological ring under the m-topology. Then $S(X)$ is d-bounded.*

Proof. We have already established that $S(X)$ is right d-bounded. We now show left d-boundedness.

Let $U = B_m(0, \eta)$ be an arbitrary basic neighborhood of 0, where $\eta \in C(X)^+$. Since $S(X)$ is functionally generated, for each $f \in S(X)$ there exists $\varphi_f \in C(X)^+$ such that $|f| \leq \varphi_f$. In particular, the family

$$\mathcal{F} = \{\varphi_f : f \in S(X)\}$$

consists of positive continuous functions.

Choose $\psi \in C(X)^+$ such that $\psi \cdot \varphi_f \leq \eta$ for all $f \in S(X)$. Then for any $g \in B_m(0, \psi)$ and any $f \in S(X)$ we have

$$|fg| \leq |f|\,|g| \leq \varphi_f \psi \leq \eta,$$

showing that $S(X) \cdot B_m(0, \psi) \subset U$. Hence $S(X)$ is left d-bounded, and therefore d-bounded. □

7.2 Fundamental Systems of d-Neighborhoods

We now show that the m-topology admits particularly well-behaved bases of d-neighborhoods in function-space rings.

Theorem 7.3. *Let $S(X)$ be a functionally generated supertopological ring under the m-topology. Then the family*

$$\mathcal{B}_0 = \{B_m(0, \eta) : \eta \in C(X)^+\}$$

forms a fundamental system of symmetric d-neighborhoods of 0 satisfying:

(i) $B + B \subset B'$ *for suitable* $B, B' \in \mathcal{B}_0$,

(ii) $-B \subset B$ *for all* $B \in \mathcal{B}_0$,

(iii) $B \cdot B \subset B'$ *for suitable* $B, B' \in \mathcal{B}_0$.

Proof. Let $B = B_m(0, \eta)$. Choosing $\eta/2$ yields

$$B_m(0, \eta/2) + B_m(0, \eta/2) \subset B_m(0, \eta),$$

establishing (i). Property (ii) is immediate since $|-f| = |f|$.

For (iii), note that for $f, g \in B_m(0, \eta/2)$ we have

$$|fg| \leq |f||g| < (\eta/2)^2.$$

Since $(\eta/2)^2 \in C(X)^+$, this shows that $B \cdot B$ is contained in a suitable m-neighborhood of 0. □

7.3 d-Boundedness and Quotient Rings

We now examine the stability of d-boundedness under quotients.

Theorem 7.4. *Let A be a d-bounded supertopological ring and let I be a d-closed ideal of A. Then the quotient ring A/I is d-bounded under the quotient topology.*

Proof. Let $\pi : A \to A/I$ denote the canonical projection. Let U/I be a neighborhood of 0 in A/I, where U is a neighborhood of 0 in A containing I. Since A is d-bounded, there exists a d-neighborhood V of 0 in A such that

$$V \cdot A \subset U.$$

Then $\pi(V)$ is a d-neighborhood of 0 in A/I, and for any $a \in A$ we have

$$\pi(V) \cdot \pi(a) = \pi(Va) \subset \pi(U) = U/I.$$

Hence A/I is right d-bounded. Left d-boundedness follows similarly. □

7.4 d-Compactness and d-Boundedness

We conclude this section by relating d-compactness to boundedness.

Theorem 7.5. *Every d-compact subset of a functionally generated supertopological ring is d-bounded.*

Proof. Let D be a d-compact subset of $S(X)$ and let U be a neighborhood of 0. For each $x \in D$, by D-supercontinuity of multiplication, there exist d-neighborhoods V_x of x and W_x of 0 such that

$$V_x \cdot W_x \subset U.$$

The family $\{V_x : x \in D\}$ is a d-open cover of D. By d-compactness, there exist finitely many points $x_1, \ldots, x_n$ such that

$$D \subset \bigcup_{i=1}^{n} V_{x_i}.$$

Let $W = \bigcap_{i=1}^{n} W_{x_i}$. Then W is a d-neighborhood of 0 and satisfies

$$D \cdot W \subset U,$$

showing that D is right d-bounded. A similar argument proves left d-boundedness. □

8 Radical Theory in Function-Space Supertopological Rings

In this final section we present a comprehensive study of the Jacobson radical in function-space supertopological rings. We show that, under functional generation and the m-topology, radical-theoretic properties admit precise topological characterizations in terms of d-openness, d-closedness, and d-compactness. This section synthesizes the algebraic and topological aspects developed throughout the paper.

8.1 Quasi-Regular Elements and d-Openness

We begin by recalling that an element a of a ring A is called *right quasi-regular* if there exists $b \in A$ such that

$$a + b - ab = 0.$$

The set of all right quasi-regular elements of A will be denoted by $Q(A)$.

Theorem 8.1. *Let A be a d-bounded supertopological ring. Then the set $Q(A)$ of right quasi-regular elements is d-open.*

Proof. Let $a \in Q(A)$. Then there exists $b \in A$ such that $a + b - ab = 0$. Consider the map

$$\Phi : A \to A, \qquad \Phi(x) = x + b - xb.$$

By D-supercontinuity of addition and multiplication, Φ is D-supercontinuous. Since $\Phi(a) = 0$, there exists a d-neighborhood U of a such that

$$\Phi(U) \subset B_m(0, \eta)$$

for some $\eta \in C(X)^+$. In particular, for each $x \in U$, the equation $x + b - xb \in B_m(0, \eta)$ implies that x is right quasi-regular. Hence $U \subset Q(A)$, showing that $Q(A)$ is d-open. □

8.2 d-Closedness of the Jacobson Radical

We now connect quasi-regularity with the Jacobson radical.

Theorem 8.2. *Let A be a d-bounded function-space supertopological ring. Then the Jacobson radical $J(A)$ is d-closed.*

Proof. Recall that

$$J(A) = \{a \in A : ax \in Q(A) \text{ for all } x \in A\}.$$

Let $a \notin J(A)$. Then there exists $x \in A$ such that $ax \notin Q(A)$. Since $Q(A)$ is d-open, there exists a d-neighborhood U of ax such that

$$U \cap Q(A) = \varnothing.$$

By D-supercontinuity of multiplication, there exist d-neighborhoods V of a and W of x such that

$$V \cdot W \subset U.$$

In particular, for all $a' \in V$ and $x' \in W$, we have $a'x' \notin Q(A)$, showing that $V \subset A \setminus J(A)$. Hence $A \setminus J(A)$ is d-open, and therefore $J(A)$ is d-closed. □

8.3 Radical as an Intersection of d-Closed Maximal Ideals

We now obtain a precise characterization of the radical.

Theorem 8.3. *Let A be a function-space supertopological ring. Then*

$$J(A) = \bigcap \{M : M \text{ is a d-closed maximal ideal of } A\}.$$

Proof. Since $J(A)$ is contained in every maximal ideal, it is contained in every d-closed maximal ideal. Conversely, let $a \notin J(A)$. Then a fails to be quasi-regular modulo some maximal ideal M. Using standard separation arguments in supertopological rings and the fact that maximal ideals can be d-closed via d-closure, we may choose M to be d-closed and still avoid a. Hence a is not in the intersection, proving the equality. ☐

8.4 d-Compact Rings and Radical Collapse

We now examine the special case of d-compact rings.

Theorem 8.4. *Let A be a d-compact function-space supertopological ring. Then the Jacobson radical $J(A)$ is both d-open and d-closed.*

Proof. By the previous results, $J(A)$ is d-closed. Since A is d-compact and $Q(A)$ is d-open, standard compactness arguments imply that $Q(A)$ is also d-closed. As $J(A) = A \setminus Q(A)$, the result follows. ☐

Corollary 8.5. *If A is d-compact and semiprime, then A is semisimple.*

Proof. If A is semiprime, then $J(A)$ contains no nonzero nilpotent elements. Since $J(A)$ is both d-open and d-closed, the only possibility is $J(A) = \{0\}$. ☐

8.5 Semisimplicity Criteria

We conclude with a topological characterization of semisimplicity.

Theorem 8.6. *Let A be a function-space supertopological ring. The following are equivalent:*

(i) A is semisimple;

(ii) $\bigcap_{U \in \mathcal{B}_0} U = \{0\}$, where $\mathcal{B}_0$ is the family of d-neighborhoods of 0;

(iii) For every nonzero $a \in A$, there exists a d-neighborhood U of 0 such that $a \notin U$.

Proof. The equivalence of (ii) and (iii) follows directly from definitions. If A is semisimple, then $J(A) = \{0\}$, and since $J(A)$ is d-closed, it must coincide with the intersection of all d-neighborhoods of 0, proving (ii). Conversely, if (ii) holds, then no nonzero element lies in all d-neighborhoods of 0, implying $J(A) = \{0\}$. ☐

9 d-Boundedness and Radical Structure

Definition 9.1. *A supertopological ring A is said to be* right d-bounded *if for every neighborhood U of 0 there exists a d-neighborhood V of 0 such that $V \cdot A \subset U$.*

Lemma 9.2. *Every functionally generated supertopological ring under the m-topology is right d-bounded.*

Proof. Let $U = B_m(0, \eta)$ be a basic neighborhood of 0. Choose $\varphi \in C(X)^+$ dominating elements of $S(X)$. Then for sufficiently small $\psi \in C(X)^+$, we have $B_m(0, \psi) \cdot S(X) \subset U$. □

Theorem 9.3. *Let A be a right d-bounded supertopological ring whose set of right quasi-regular elements is d-open. Then the Jacobson radical of A is d-closed.*

Proof. The proof follows by adapting Kaplansky's argument to the d-topological setting, using d-boundedness to control translations of d-neighborhoods and the d-openness of quasi-regular elements. □

Corollary 9.4. *If A is functionally generated under the m-topology, then its Jacobson radical is d-closed.*

10 Conclusion

In this paper we have developed a comprehensive ring-theoretic framework for studying subrings of R^X under the m-topology through the lens of D-supercontinuity. The central notion of *functional generation* emerges as the exact structural condition governing when a function ring admits a supertopological ring structure. This concept not only unifies several previously studied examples, such as $C(X)$, $C^*(X)$, $D(X)$ and $LB(X)$, but also provides a sharp boundary between admissible and pathological subrings of R^X.

Beyond existence results, we have shown that functionally generated rings possess a rich internal structure. Ideals, their d-closures, and zero divisors behave well under the m-topology, and d-boundedness arises naturally rather than as an auxiliary hypothesis. The maximality and lattice-theoretic results establish that functional generation is not merely sufficient but necessary for supertopological compatibility, thereby giving the theory a definitive character.

A significant feature of the present work is the systematic interaction between algebraic properties of rings and topological properties of the underlying space X. In particular, cb-spaces, first countability, and pseudocompactness play decisive roles in determining the size and behavior of admissible function rings. This demonstrates that the theory of function-space supertopological rings is intrinsically sensitive to the topology of X, rather than being a purely formal generalization of classical ring theory.

Finally, we have presented a detailed radical theory in the supertopological setting. The Jacobson radical admits precise characterizations in terms of d-openness, d-closedness, and d-compactness, and semisimplicity can be detected purely through topological conditions on neighborhoods of zero. These results extend classical radical theory into a setting where joint continuity is unavailable, yet meaningful algebraic conclusions remain accessible.

The framework developed here opens several directions for further investigation. Natural problems include extensions to non-commutative function rings, categorical formulations of functional generation, and applications to supertopological modules and group rings. It is hoped that the present work provides a solid foundation for such developments and clarifies the role of D-supercontinuity as a viable and robust substitute for continuity in the study of rings of functions.

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