Original Research Article

A PREY PREDATOR CONSERVATION MODEL FOR A FISHERY WITH A RESERVE AREA AND PREY REFUGE : A STUDY OFLAKE VICTORIA

Abstract

The loss of biodiversity in most fishery ecosystems worldwide has reached crisis levels driven by habitat loss, pollution, overfishing, invasive species, predation and climate change. This threatens sustainability of these fisheries. In Lake Victoria a major decline has been observed in haplochromine species due to predation by the invasive Nile perch and the species is now in danger of extinction. In this paper a prey-predator conservation model for a fishery with a reserve area and prey refuge has been formulated using a logistic nonlinear differential equations. The model incorporates Holling type II functional response of the predator towards the prey. This research is intended to study and analyze the effect of a reserve area on the stability of the prey-predator dynamic system of Nile perch (predator) and haplochromines (prey) in Lake Victoria. The lake ecosystem is divided into two parts, the reserve area and unreserved area. Equilibrium points have been determined and their local and global stability established by use of eigen value approach, Bendixon-Dulac criterion and Lyapunov function. The effect of the reserve area and prey refuge on the stability of the system has been determined by simulation in MATLAB. Results show that a reserve area makes the system stable. .

Keywords: Key words: Prey predator; conservation model; mathematical model; equilbrium points; stability analysis.

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1 Introduction

The increasing demand for food by human population exerts enormous pressure on natural resources causing overexploitation. For example most fisheries worldwide are facing overfishing. The situation is made worse by predators that compete for the same food resource. This has drawn concern of conservationists worldwide to try protect the ecosystem by use of scientific management approaches. Ecology is a study that deals with interactions of organisms with each other and the environment. An ecosystem consists of all species in a given area and the physical environment. Ecological interactions between different species occur in various ways, which include but not limited to competition and predation. Competition is where different species compete for scarce resources like food, space or environment. Competition may be between members of the same species (intraspecific) or members of the different species (interspecific). Intraspecific competition causes harm within the species. Predation is where one species (predator) feeds on another (prey). In this case the prey is converted into the predator through consumption, hence population growth of the predator is enhanced while that of the prey is reduced and its existence threatened. To understand the prey-predator dynamic interactions in the ecosystem several approaches have been used, one being mathematical modeling, models provide crucial insights into population dynamics of interacting species.

The earliest mathematical model proposed was the Lotka-Volterra model, since then many mathematical models have been developed to study the dynamics of ecological interactions. For instance prey-predator conservation models have been developed and used to study the factors that influence the growth and decline of natural resources like forests and fisheries. Conservation refers to preservation and protection of natural resources and the environment for sustainable future use. A prey-predator conservation model is a tool that ecologists use to understand the dynamics of the prey-predator interactions and how these interactions affect the population of the prey and predator species. Mathematical models use differential equations like the Lotka-Volterra equations to simulate the growth and the decline of the prey and the predator population, they incorporate factors like growth rate, predation rate, prey availability and harvesting rate. Prey-predator conservation models are critical in helping predict population trends of the prey and the predator in the ecosystem, and how the change in the population of the predator would affect the prey and vice versa. Similarly they help determine factors that affect the population dynamics such as prey availability, predation rate and harvesting rate, and other environmental factors. Conservationists then use this knowledge to develop conservation strategies such as predator control or prey enhancement, models also help assess the impact of these strategies. One of the strategies employed by conservationists is creation of a reserve area. A reserve area is a protected area set apart to conserve and protect wildlife usually with a goal of preserving endangered species and their habitat. A reserve area provides a safe haven for the prey to take refuge. Prey refuge is a strategy by the prey to evade the predator and reduce predation risk. The prey may take refuge in a protected reserve area or may develop certain natural adaptations like aggregating into schools where the prey takes refuge to avoid being targeted, since the attention of the predator is now spread to the whole group, this has been observed in fishes. In this study we include a reserve area and a prey refuge term in our model. A reserve area and prey refuge have a stabilizing effect on the model.

In this research we focus on a case study of Lake Victoria. We investigate the prey-predator dynamic system of the Nile perch (mbuta) and haplochromines (fulu). According to a research by Owiti et al. (2021) on Economic and Financial Impact Assessment (EFIA) of lake Victoria fishery in Kenya for management recommendations in (KMFRI) report found out that Lake Victoria makes up 66% of the fisheries in Kenya and contributed about 0.3% to the National GDP. However the contribution to the GDP by the fisheries sector has been dropping by averagely -3.16% per every 5 years. This research further reveals a decline in the biomass of commercially valued fishes, Nile perch, rastrineobola argentea (omena) and other fishes including haplochromines declined by 31%, 49% and 81% respectively. The decline in species begun in 1960s when Nile perch was introduced to date. In 1980s, Nile perch population had grown to 80% of the fish in the Lake and haplochromines species declined, a large quantity of haplochromines disappeared Kitchell et al. (1997) as cited in (Outa et al., 2020). The decline was largely attributed to predation by Nile perch. This is a confirmation that the fish resource in the lake is declining and there is an agent need for research to enrich the management practices that can help halt this trend, given the economic importance of the lake to the country and to the communities living around the lake. Sayer et al. (2018) in their research notes that Lake Victoria fishery is threatened by pollution, agriculture, overharvesting and predation by the invasive Nile perch. Nile perch is a ferocious predator exotic to the lake. It preys on other fishes but its favorite prey is the haplochromines cichlids species. Predation by the invasive Nile perch is driving most haplochromine species to the brink of extinction. The situation is so dire that the red line list index value for the native haplochromines cichlids dropped by 63% between 1960 and 2010 indicating increased risk of extinction. Sayer et al. (2018) in their IUCN research report 76% of the fish native to Lake Victoria is threatened with extinction. Haplochromines are important to the lake biodiversity since they feed on algae this prevents algae bloom in the lake. Growth of algae reduces oxygen in the water causing death of aquatic species. Haplochromines are also the favorite prey of the commercially valuable Nile perch. Therefore conservation of the haplochromines cichlids creates a prey reservoir available to the Nile perch for consumption, this prevents self-cannibalism hence conserving Nile perch. It is therefore crucially important that we understand the Nile perch-haplochromine dynamic interactions in the lake ecosystem to inform conservation strategies. Provide a factual background, clearly defined problem, proposed solution, a brief literature survey and the scope and justification of the work done, objective, etc.

2 Model Formulation

The theory of dynamical systems can be applied in mathematical modeling of ecosystems (Ramm & Hoang, 2013). Population dynamics describes the behavior of ecological systems as time changes. These systems consist of many species interacting in many complex ways (Jawad, 2018). Mathematical modeling has become an important method of studying and analyzing interactive dynamics of a growing population of competing species in an ecological system. Differential equations can be formulated to govern changes in population of interacting species with time. In this section we formulate a mathematical model to investigate the Prey-predator interactions of the Nile perch and haplochromine cichlids in Lake Victoria. The haplochromine cichlids species is the prey and the Nile perch is the predator. The fishery is subdivided into two regions, the fishing area and a protected reserve area where fishing is banned. The prey resides in both regions and migrates from the unreserved region to the reserved region randomly but the reverse is not permitted. The predator resides in the unreserved area and feeds on the prey in the unreserved area only. The predator fish has Holling type II functional response towards the prey in the unreserved area. Let $X(t)$ and $Y(t)$ represent the population densities of the haplochromines at any time t in the unreserved area and the reserved area respectively. Let $Z(t)$ represents the population density of the Nile perch at a time t. The predator and the prey are both harvested in the unreserved area. The growth of the prey in both zones is logistic in the absence of the predator, mx fraction of prey take refuge to evade predation, leaving $(1-m)x$ of the prey available to the predator for consumption.
We make the following assumptions for our model;

(i) The predator depends fully on the prey as its most preferred food and decays exponentially to extinction in the absence of the prey.

(ii) The prey has unlimited food supply.

(iii) The prey grows logistically in the absence of the predator and harvesting.

(iv) The predator is the only threat to the prey under the study. Other external factors like epidemics pollution affect both the prey and the predator equally

(v) Fishing effort is constant there is no redistribution of harvesting effort from the reserved area to the fishing zone.

The summary of variables and parameters used are described in the tables below

Table 1: Table of variables

Variable	Description of variables
X	Population densities of the haplochromines in the unreserved area
Y	Population densities of the haplochromines in the reserved area
Z	Population densities of the Nile perch in the unreserved area

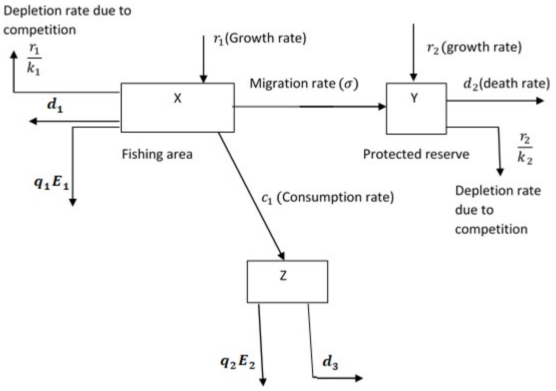

Figure 1: Flow diagram

Table 2: Table of parameters

Parameter	Parameter description
r_n	Intrinsic growth rate of the prey and the predator.
K_n	Carrying capacity of the environment.
b	Measure of the extent to which the environment provides protection to the prey.
c_1	Maximum value per capita reduction rate of prey due to consumption by the predator.
c_2	Rate of conversion of the predator into the prey.
d_n	Natural rate of death of the predator and the prey.
m	A fraction of the prey that takes refuge to avoid predation.
E_1	Harvesting effort dispensed towards harvesting of the prey.
E_2	Harvesting effort dispensed towards harvesting of the predator.
q_1	Catchability coefficients of the prey.
q_2	Catchability coefficients of the predator.
σ	Migration rate between the unreserved area to the reserved area.

Using the variables, parameters and the assumptions above the model equations are;

$$f_1(X,Y,Z) = \frac{dX}{dt} = r_1X\left(1-\frac{X}{K_1}\right) - \frac{c_1(1-m)XZ}{b+(1-m)X} - (\sigma + q_1E_1 + d_1)X \tag{2.1}$$

$$f_2(X,Y,Z) = \frac{dY}{dt} = r_2Y\left(1-\frac{Y}{K_2}\right) + \sigma X - d_2Y \tag{2.2}$$

$$f_3(X,Y,Z) = \frac{dZ}{dt} = r_3Z\left(1-\frac{Z}{K_3}\right) + \frac{c_2(1-m)XZ}{b+(1-m)X} - (q_2E_2 + d_3)Z \tag{2.3}$$

The ratios $\frac{(r_1X^2)}{K_1}$ and $\frac{(r_2Y^2)}{K_2}$ are the reduction rates of the prey due to interspecies competition. The ratio $\frac{c_1(1-m)XZ}{b+(1-m)X}$ is the decrease in prey population as a consequence of predation and the ratio $\frac{c_2(1-m)XZ}{b+(1-m)X}$ is the increase in predator population as result of eating the prey.

2.1 Analysis of the model

2.1.1 Positivity and Boundedness of Solutions

The analysis of the model is done in the feasible region Ω in which $(X(t), Y(t), Z(t)) \in \mathbb{R}^3_+$ we show that the solutions of the model are non-negative for all $t \geq 0$

Theorem 2.1. *The solutions* $(X(t), Y(t), Z(t)) \geq 0$ *for all* $t \geq 0$ *in* Ω

Proof. Taking into account equation (2.1)

$$\frac{dX}{dt} = r_1X\left(1 - \frac{X}{K_1}\right) - \frac{c_1(1-m)XZ}{b+(1-m)X} - (\sigma + q_1E_1 + d_1)X$$

If $X(t)$ is positive then this implies that

$$\frac{dX}{dt} \geq -(\sigma + q_1E_1 + d_1)X$$

Let $\mu = (\sigma + q_1E_1 + d_1) \implies \frac{dX}{dt} \geq -\mu X$
Separation of variables and using the initial condition gives

$$X(t) \geq X(0)e^{-(\sigma+q_1E_1+d_1)t}$$

Thus as $t \to \infty$, $X(t) \geq 0$ for all $t \geq 0$
Similarly for equations (2.2) and (2.3) we obtain

$$\begin{aligned} Y(t) &\geq Y(0)e^{-d_2t} \\ Z(t) &\geq Z(0)e^{-(q_2E_2+d_3)t} \end{aligned}$$

□

Therefore, for equations (2.1), (2.2), (2.3) as $t \to \infty$, $X(t) \geq 0, Y(t) \geq 0, Z(t) \geq 0$ for all $t \geq 0$
Next, we use the following theorem to show that the solutions are bounded

Theorem 2.2. *Assuming that* $c_1 \geq c_2$ *holds then the solutions of the system 2.1, 2.2, 2.3 that initiate in* $\Omega \subset \mathbb{R}^3_+$ *are bounded*

Proof. Considering $X(t), Y(t), Z(t)$ to be solutions to the model equations 2.1, 2.2, 2.3 with positive initial conditions and letting

$$\begin{aligned} U(t) &= X(t) + Y(t) + Z(t) \text{ then} \\ \frac{dU}{dt} &= \frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{dU}{dt} = {} & r_1X - \frac{r_1X^2}{K_1} - \frac{c_1(1-m)XZ}{b+(1-m)X} - \sigma X - (q_1E_1 + d_1)X + r_2Y - \frac{r_2Y^2}{K_2} \\ & +\sigma X - d_2Y + r_3Z\left(1 - \frac{Z}{K_3}\right) + \frac{c_2(1-m)XZ}{b+(1-m)X} - (q_2E_2 + d_3)Z \end{aligned}$$

But from above assumption of the theorem if $c_1 \geq c_2$ then we obtain

$$\begin{aligned}
\frac{dU}{dt} &\leq r_1X - \frac{r_1X^2}{K_1} - (q_1E_1 + d_1)X + r_2Y - \frac{r_2Y^2}{K_2} - d_2Y + r_3Z - r_3\frac{Z^2}{K_3} \\
&\quad -(q_2E_2 + d_3)Z \\
\frac{dU}{dt} + \xi U &\leq 2r_1X - \frac{r_1X^2}{K_1} + 2r_2Y - \frac{r_2Y^2}{K_2} + 2r_3Z - \frac{r_3Z^2}{K_3}
\end{aligned}$$

Where $\xi = \min\{r_1, r_2, r_3, d_1, d_2, d_3, q_1, q_2, E_1, E_2\}$ and let $\lim\limits_{X\to\infty} X = K_1$, $\lim\limits_{Y\to\infty} Y = K_2$ and $\lim\limits_{Z\to\infty} Z = K_3$ then

$$\begin{aligned}
\frac{dU}{dt} + \xi U &\leq r_1K_1 - \frac{r_1(X - K_1)^2}{K_1} + r_2K_2 - \frac{r_2(Y - K_2)^2}{K_2} + r_3K_3 - \frac{r_3(Z - K_3)^2}{K_3} \\
\frac{dU}{dt} + \xi U &\leq r_1K_1 + r_2K_2 + r_3K_3
\end{aligned}$$

Let $m = r_1K_1 + r_2K_2 + r_3K_3$. Therefore:

$$\begin{aligned}
\frac{dU}{dt} + \xi U &\leq m \\
\frac{dU}{dt} &\leq -\xi U + m
\end{aligned}$$

By use of Gronwall's differential inequality (Jawad, 2018), we obtain

$$0 \leq U(t) \leq \frac{m}{\xi}(1 - e^{-\xi t}) + U(0)e^{-\xi t}$$

Therefore as $t \to \infty$ we have

$$0 \leq \lim_{t\to\infty} \sup U(t) \leq \frac{m}{\xi}$$

□

Therefore $U(t)$ is bounded, and all solutions of the model equations 2.1, 2.2, 2.3 that initiate in $\Omega \subset \mathbb{R}^3_+$ are bounded

2.1.2 Equilibrium Points

We first scale down the number of parameters in the model equations 2.1, 2.2, 2.3 to simplify them. By letting

$$\begin{aligned}
\beta_1 &= r_1 - \sigma - q_1E_1 - d_1 \\
\beta_2 &= r_2 - d_2 \\
\beta_3 &= r_3 - d_3 - q_2E_2 \\
\omega_1 &= c_1(1 - m) \\
\omega_2 &= c_2(1 - m) \\
a &= \frac{b}{(1 - m)} \\
p_1 &= \frac{r_1}{K_1} \\
p_2 &= \frac{r_2}{K_2} \\
p_3 &= \frac{r_3}{K_3}
\end{aligned}$$

The scaled down equations are:

$$\frac{dX}{dt} = \beta_1 X - p_1 X^2 - \frac{\omega_1 XZ}{(a+X)} \tag{2.5}$$

$$\frac{dY}{dt} = \beta_2 Y - p_2 Y^2 + \sigma X \tag{2.6}$$

$$\frac{dZ}{dt} = \beta_3 Z - p_3 Z^2 + \frac{\omega_2 XZ}{(a+X)} \tag{2.7}$$

Equating equations 2.5, 2.6, 2.7 to zero. We observe that for system of equations 2.5 2.6 2.7 three positive equilibria exist, the extinction equilibrium point $E_0(0,0,0)$, the predator extinction equilibrium point $E_1 = (X^*, Y^*, 0)$ and the co-existence equilibrium point $E_2 = (X^*, Y^*, Z^*)$. The existence of extinction equilibrium point is trivial we show the existence of the other equilibrium points.
For $E_1 = (X^*, Y^*, 0)$ we proceed as follows;
Considering equations 2.5 2.6 2.7 for $Z = 0$ we remain with two equations to solve

$$\begin{aligned} \beta_1 X - p_1 X^2 &= 0 \\ \beta_2 Y - p_2 Y^2 + \sigma X &= 0 \end{aligned}$$

The first equation gives

$$\begin{aligned} X(\beta_1 - p_1 X) &= 0 \\ \implies X &= 0 \text{ or} \\ \implies X^* &= \frac{\beta_1}{p_1} \end{aligned}$$

recalling the original parameters we have

$$X^* = \frac{K_1(r_1 - \sigma - q_1 E_1 - d_1)}{r_1} \tag{2.8}$$

Considering the second equation and using equation (2.8) we have

$$p_2 Y^2 - \beta_2 Y - \sigma \frac{\beta_1}{p_1} = 0$$

Thus

$$Y^* = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4p_2\sigma\frac{\beta_1}{p_1}}}{2p_2} \tag{2.9}$$

Therefore,

$$E_1(X^*, Y^*, 0) = \left(\frac{\beta_1}{p_1}, \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4p_2\sigma\frac{\beta_1}{p_1}}}{2p_2}, 0\right)$$

Thus recalling the original parameters, we observe that $E_1(X^*, Y^*, 0)$ exists if

$$r_1 > 0, r_2 > 0 \text{ and } r_1 > \sigma + q_1 E_1 + d_1, r_2 > d_2$$

For the equilibrium point $E_2(X^*, Y^*, Z^*)$ we proceed as follows, we set equations 2.5, 2.6, 2.7 to zero. Solving the third equation for non-zero Z we obtain

$$\beta_3 - p_3 Z + \frac{\omega_2 X}{(a+X)} = 0$$

Which gives Z^* as

$$Z^* = \frac{\beta_3(a+X) + \omega_2 X}{p_3(a+x)} \tag{2.10}$$

Solving the second equation, we obtain

$$\beta_2 Y - p_2 Y^2 + \sigma X \quad = \quad 0$$

That is,

$$Y^* = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4p_2\sigma X^*}}{2p_2} \tag{2.11}$$

Substituting for Z by Z^* in the first equation we obtain;

$$p_1p_3X^3 + p_3\beta_1X^2 - (p_1p_3a^2 + \beta_3\omega_1 + \omega_1\omega_2)X - (\omega_1\beta_3a + p_3\beta_1a^2) = 0$$

Therefore we obtain an equation of the form ;

$$aX^3 + bX^2 + cX + d = 0 \tag{2.12}$$

Where

$$\begin{aligned}
a &= p_1p_3 = \frac{r_1r_3}{K_1K_3} \\
b &= p_3\beta_1 = \frac{r_3}{K_3}(r_1 - (d_1 + q_1E_1 + \sigma)) \\
c &= p_1p_3a^2 + \beta_3\omega_1 + \omega_1\omega_2 = \frac{r_1r_3}{K_1K_3}\left(\frac{b}{1-m}\right) - c_1(1-m)(r_3 - (d_3 + q_2E_2)) \\
&\quad +c_1c_2(1-m)^2 \\
d &= -(\omega_1\beta_3a + p_3\beta_1a^2) = bc_1(r_3 - (d_3 + q_2E_2)) \\
&\quad +\frac{r_3}{K_3}(r_1 - (d_1 + q_1E_1 + \sigma))\left(\frac{b}{1-m}\right)^2
\end{aligned}$$

By use of Descartes rule of signs, equation (2.12) has a unique positive solution $X = X^*$, if the following inequalities hold;

$$\begin{aligned}
r_1 &> (d_1 + q_1E_1 + \sigma) \\
r_3 &> (d_3 + q_2E_2)
\end{aligned}$$

Therefore if the above inequalities hold the co-existence equilibrium $E_2(X^*, Y^*, Z^*)$ exists.

2.2 Local stability of the equilibrium points

The local stability of each equilibrium point is found by determining eigen values of the Jacobian matrix associated with the equilibrium point. For an equilibrium point to be stable eigen values of the Jacobian matrix must have negative real parts. The Jacobian matrix is found as follows;

$$J(E_i) = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{bmatrix}$$

Considering the system of equations 2.5, 2.6, 2.7

$$\begin{aligned}
f_1(X, Y, Z) &= \beta_1X - p_1X^2 - \frac{\omega_1XZ}{a+X} \\
f_2(X, Y, Z) &= \beta_2y - p_2Y^2 + \sigma X \\
f_3(X, Y, Z) &= \beta_3Z - p_3Z^2 + \frac{\omega_2XZ}{a+X}
\end{aligned}$$

This gives;

$$J(E_i) = \begin{bmatrix} \beta_1 - 2p_1X - \frac{a\omega_1 Z}{(a+X)^2} & 0 & \frac{-a\omega_1 X}{a+X} \\ \sigma & \beta_2 - 2p_2Y & 0 \\ \frac{a\omega_2 Z}{(a+X)^2} & 0 & \beta_3 - 2p_3Z + \frac{\omega_2 X}{a+X} \end{bmatrix}$$

2.2.1 Local stability

For extinction equilibrium point $E_0(0,0,0)$ the Jacobian matrix is given by

$$J(E_0) = \begin{bmatrix} \beta_1 & 0 & 0 \\ \sigma & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

whose eigenvalues, after recalling the original paramaters, are

$$\begin{aligned} \lambda_1 &= r_1 - (\sigma + q_1E_1 + d_1) \\ \lambda_2 &= r_2 - d_2 \\ \lambda_3 &= r_3 - (d_3 + q_2E_2) \end{aligned}$$

Since $r_2 > d_2$ the eigen value λ_2 is positive therefore $E_0(0,0,0)$ is unstable.

For local stability of predator extinction equilibrium point $E_1(X^*, Y^*, 0)$ the Jacobian at $E_1(X^*, Y^*, 0)$ is;

$$J(E_1) = \begin{bmatrix} \beta_1 - 2p_1X & 0 & \frac{-a\omega_1 X}{a+X} \\ \sigma & \beta_2 - 2p_2Y & 0 \\ 0 & 0 & \beta_3 + \frac{\omega_2 X}{a+X} \end{bmatrix}$$

Let

$$\begin{aligned} A_{11} &= \beta_1 - 2p_1X = r_1 - (\sigma + q_1E_1 + d_1) - 2\frac{r_1}{K_1}X \\ A_{13} &= \frac{-a\omega_1 X}{a+X} = \frac{c_2(1-m)X}{\frac{b}{1-m} + X} \\ A_{21} &= \sigma \\ A_{22} &= \beta_2 - 2p_2Y = r_2 - d_2 - 2\frac{r_2}{K_2}Y \\ A_{33} &= \beta_3 + \frac{\omega_2 X}{a+X} = r_3 - (d_3 + q_2E_2) + \frac{c_2(1-m)X}{\frac{b}{1-m} + X} \end{aligned}$$

Therefore

$$J(E_1) = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

$$J(E_1) - \lambda = \begin{bmatrix} A_{11} - \lambda & 0 & A_{13} \\ A_{21} & A_{22} - \lambda & 0 \\ 0 & 0 & A_{33} - \lambda \end{bmatrix}$$

The characteristics equation is given by $|J(E_1) - \lambda| = 0$. Thus we have

$$\begin{aligned}(A_{11} - \lambda)[(A_{22} - \lambda)(A_{33} - \lambda)] &= 0 \\ (A_{11} - \lambda)[A_{22}A_{33} - A_{22}\lambda - A_{33}\lambda + \lambda^2] &= 0\end{aligned}$$

Simplifying this we obtain the characteristic equation;

$$\lambda^3 - (A_{11} + A_{22} + A_{33})\lambda^2 + (A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33})\lambda - (A_{11}A_{22}A_{33}) = 0$$

This equation is of the form;

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + c = 0 \tag{2.13}$$

According to Routh-Hurwitz criterion the eigen values(λ) are negative iff, $a_1 > 0$, $c > 0$, and $a_1a_2 > c$, hence the equilibrium point $E_1(X^*, Y^*, 0)$ is locally asymptotically stable otherwise it is unstable. Finally, the local stability of co-existence equilibrium point $E_2(X^*, Y^*, Z^*)$ is determined by the Jacobian matrix

$$J(E_3) = \begin{bmatrix} \beta_1 - 2p_1X^* - \frac{a\omega_1 Z^*}{(a+X^*)^2} & 0 & \frac{-a\omega_1 X^*}{a+X^*} \\ \sigma & \beta_2 - 2p_2Y^* & 0 \\ \frac{a\omega_2 Z^*}{(a+X^*)^2} & 0 & \beta_3 - 2p_3Z^* + \frac{\omega_2 X^*}{a+X^*} \end{bmatrix}$$

In a similar manner as we did to the equilibrium point E_2 we have the characteristics equation given by

$$(A_{11} - \lambda)[(A_{22} - \lambda)(A_{33} - \lambda)] + A_{13}[A_{31}(A_{22} - \lambda)] = 0$$

On simplifying we obtain

$$\begin{aligned}&\lambda^3 - (A_{11} + A_{22} + A_{33})\lambda^2 - (A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33} + A_{31})\lambda \\ &-(A_{11}A_{22}A_{33} + A_{13}A_{22}A_{31}) = 0\end{aligned}$$

This is a characteristic equation of the form;

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + c = 0 \tag{2.14}$$

By Routh-Hurwitz criterion the eigen values(λ) are negative iff, $a_1 > 0$, $c > 0$, and $a_1a_2 > c$, hence the equilibrium point $E_2(X^*, Y^*, Z^*)$ locally asymptotically stable otherwise it is unstable.

2.3 Global stability

We prove the global stability of $E_1(X^*, Y^*, 0)$ and $E_2(X^*, Y^*, Z^*)$ as follows; According to Dubey and Upadhyay (2004) as cited in Akugizibwe (2010) we prove global stability of E_1 by using the Bendixon-Dulac criterion that gives conditions for non-existence of periodic orbits. Considering the system of equations 2.5, 2.6, 2.7 and using $E_1(X^*, Y^*, 0)$ we obtain;

$$\begin{aligned}f_1(X, Y) &= \frac{dX}{dt} = \beta_1 X - p_1 X^2 \\ f_2(X, Y) &= \frac{dY}{dt} = \beta_2 Y - p_2 Y^2 + \sigma X\end{aligned}$$

We now show that there are no closed orbits in the positive X, Y plane. We first choose a Dulac function, according to the work of (Osuna & Vargas-De-León, 2015) in their article : Construction of Dulac functions for mathematical models in population biology, they established that the Dulac function for Lotka Voltera systems is of the form $\frac{1}{X_1X_2}$ in this case we choose the Dulac function D as $\frac{1}{XY}$. Let $F(X, Y) = f_1(X, Y,) + f_2(X, Y)$

$$
\begin{aligned}
DF(X,Y) &= \frac{1}{XY}(\beta_1 X - p_1 X^2) + \frac{1}{XY}(\beta_2 Y - p_1 Y^2 + \sigma X) \\
&= \frac{X}{XY}(\beta_1 - p_1 X) + \frac{Y}{XY}(\beta_2 - p_1 Y) + \frac{1}{XY}(\sigma X) \\
div(DF) &= \frac{d}{dX}\left(\frac{\beta_1}{Y} - \frac{p_1 X}{Y}\right) + \frac{d}{dY}\left(\frac{\beta_2}{X} - \frac{p_2 Y}{X}\right) + \frac{d}{dY}(\frac{\sigma}{Y}) \\
div(DF) &= -\frac{p_1}{Y} - \frac{p_2}{X} - \frac{\sigma}{Y}
\end{aligned}
$$

Where

$$
\begin{aligned}
p_1 &= \frac{r_1}{K_1} \\
p_2 &= \frac{r_2}{K_2}
\end{aligned}
$$

$div(DF) \neq 0$ and $div(DF)$ has the same negative sign everywhere in the X, Y plane if $\sigma > 0$, which implies that ,

$$
\begin{aligned}
\frac{dX}{dt} &= f_1(X,Y) \\
\frac{dY}{dt} &= f_2(X,Y)
\end{aligned}
$$

has no closed orbits in the X, Y plane hence no periodic solutions in the entire region $\Omega \subset \mathbb{R}^3_+$ hence the equilibrium point $E_1(X^*, Y^*, 0)$ is globally stable if
$r_1 > 0, r_2 > 0, \sigma > 0$ for all $t > 0$, otherwise it is globally unstable. To wind up on stability, we determine the global stability of $E_2(X^*, Y^*, Z^*)$ using a suitable lyapunov function.

Theorem 2.3. *Assuming the co-existence equilibrium point $E_2(X^*, Y^*, Z^*)$ is locally asymptotically stable in $\Omega \subset \mathbb{R}^3_+$ then it is globally stable in $\mathbb{R}^3_+$*

Proof. Consider the positive definite Lyapunov function at $E_2(X^*, Y^*, Z^*)$

$$
\begin{aligned}
V(X,Y,Z) &= \left(X - X^* - X^* \ln \frac{X}{X^*}\right) + \left(Y - Y^* - Y^* \ln \frac{Y}{Y^*}\right) \\
&\quad + \left(Z - Z^* - Z^* \ln \frac{Z}{Z^*}\right)
\end{aligned}
$$

Differentiating $V(X,Y,Z)$ with respect to time t, and using equations 2.5, 2.6, 2.7we obtain

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{X - X^*}{X}\left(X(\beta_1 - p_1 X - \frac{\omega_1 Z}{a + X})\right) + \frac{Y - Y^*}{Y}\left(Y(\beta_2 - p_2 Y) + \sigma X\right) \\
&\quad + \frac{Z - Z^*}{Z}\left(Z(\beta_3 - p_3 Z + \frac{\omega_2 X}{a + X})\right) \\
\frac{dV}{dt} &= (X - X^*)\left[\left(-p_1 X - \frac{\omega_1 Z}{a + X}\right) - \left(-p_1 X^* - \frac{\omega_1 Z^*}{a + X^*}\right)\right] + \\
&\quad (Y - Y^*)\left[-p_2 Y - (-p_2 Y^*) + \frac{Y - Y^*}{Y}(\sigma X - \sigma X^*)\right] + \\
&\quad (Z - Z^*)\left[\left(-p_3 Z + \frac{\omega_2 X}{a + X}\right) - \left(-p_3 Z^* + \frac{\omega_2 X^*}{a + X^*}\right)\right]
\end{aligned}
$$

which simplifies to,

$$
\begin{aligned}
\frac{dV}{dt} &= -p_1(X - X^*)^2 - \frac{(X - X^*)(Z - Z^*)(\omega_1 - \omega_2)}{(a + X)(a + X^*)} - P_2(Y - Y^*)^2 \\
&\quad -\sigma(Y - Y^*)\left(\frac{X^* Y - X Y^*}{Y Y^*}\right)
\end{aligned}
$$

□

Therefore $\frac{dV}{dt} < 0$, which is negative definite, therefore, $E_2(X^*, Y^*, Z^*)$ is globally asymptotically stable.

3 Results and Discussion

In this section we present numerical simulations for our model. The parameters values used in simulation are as shown in the table below

Table 3: Parameter values of the model

Parameter	Value	Source
r_1	2.3	(Mpele et al., 2014)
r_2	3.0	estimated
r_3	4.8	(Mpele et al., 2014)
K_1	200	(Mpele et al., 2014)
K_2	300	estimated
K_3	600	(Mpele et al., 2014)
σ	0.2	estimated
b	20	estimated
c_1	0.05	estimated
c_2	0.02	estimated
d_1	0.3	(Mayengo et al., 2014)
d_2	0.25	estimated
d_3	0.2	(Mayengo et al., 2014)
m	0.3	estimated
E_1	0.6	estimated
E_2	1.5	(Mayengo et al., 2014)
q_1	0.14	(Raymond et al., 2019)
q_2	0.125	(Raymond et al., 2019)

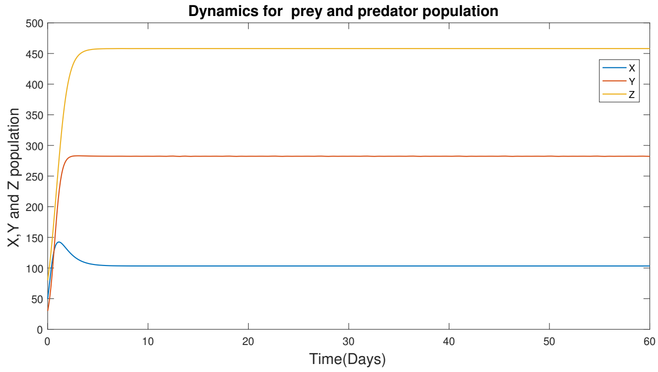

Figure 2: Prey and predator population with time for parameters given above

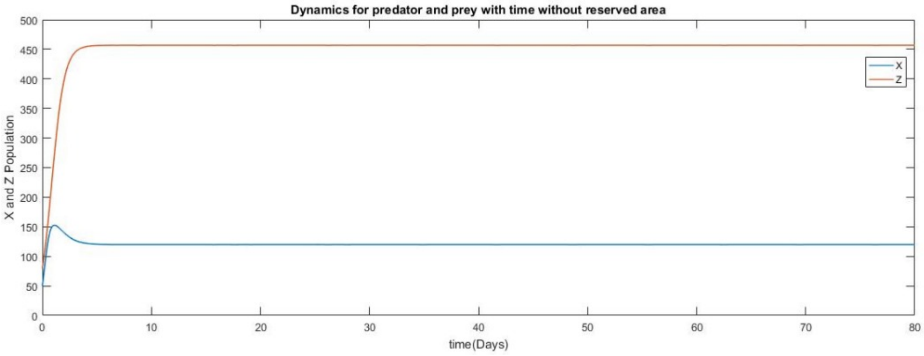

Figure 3: The population of the haplochromines (prey) and the Nile perch with time

From figure 2 and figure 3 the population of both the prey and the predator converge to their equilibrium and attains stability then grows at constant rate. However the prey population in the unreserved area oscillates briefly and becomes stable at a much lower population hence takes less time to reach equilibrium point compared to the prey in figure 3 without a reserve area. This shows that a reserve area hastens attainment of stability. Figure 2 also shows that haplochromines and the Nile perch can co-exist at equilibrium point for the initial parameter values in presence of a reserve area.

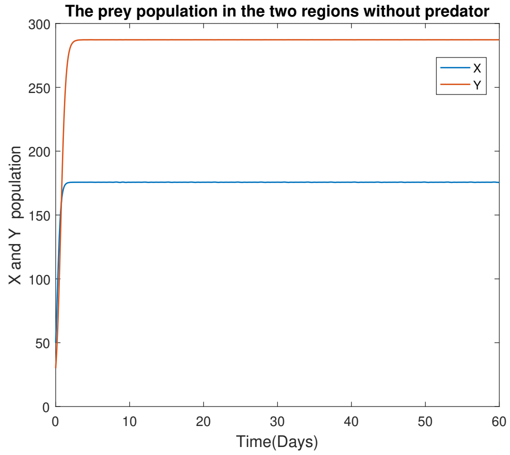

Figure 4: The population of X and Y with time

Figure 4 shows that the population of the prey in both areas converges to equilibrium and becomes stable, then grows at constant rate. It shows that the prey in the two regions can co-exist if $\sigma \leq 0.3$ as seen in figure 5. It also shows that the predator free equilibrium is stable.Without a reserve area predator free equilibrium cannot be stable

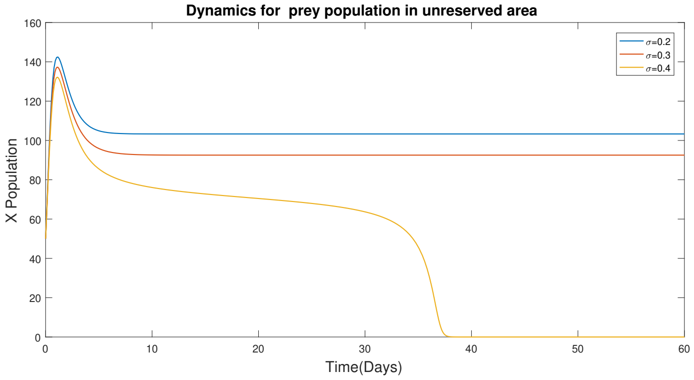

Figure 5: The prey in the unreserved area for various values of σ

Figure 5 shows that the prey population in the unreserved area decreases as migration rate (σ) increases. However the population oscillates briefly before approaching equilibrium and becoming stable for lower values of migration rate(σ). When $\sigma = 0.4$ the prey population goes to extinction after about 38 days. When more prey migrates it increases the decay rate beyond the growth rate of the prey and this pushes the prey to extinction. The prey population in the unreserved area exists if $\sigma \leq 0.3$, therefore the haplocromine(prey) and the Nile perch(predator) can co-exist if $\sigma \leq 0.3$.

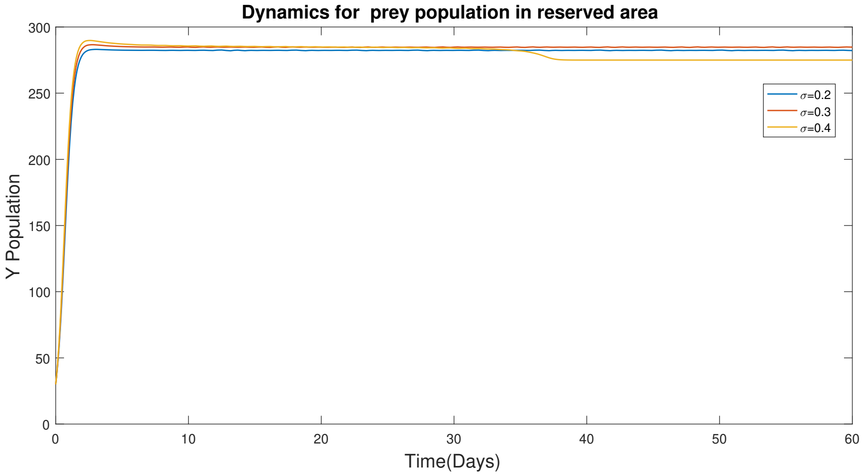

Figure 6: The population of haplochromines in the reserve area at various migration rates (σ)

Figure 6 shows that population of the prey rises sharply then slightly decreases before approaching equilibrium and attaining stability. At equilibrium the population grows at a constant rate as migration rate (σ) of the prey increases, with a very a small decrease after about 38 days when $\sigma = 0.4$. This slight decrease coincides with the decrease to extinction of the prey population in the unreserved area because after about 38 days there were zero migrating fishes. The haplocromine(prey) population in the reserved area takes shorter time to attain stability and is more stable compared to the prey in the unreserved area.

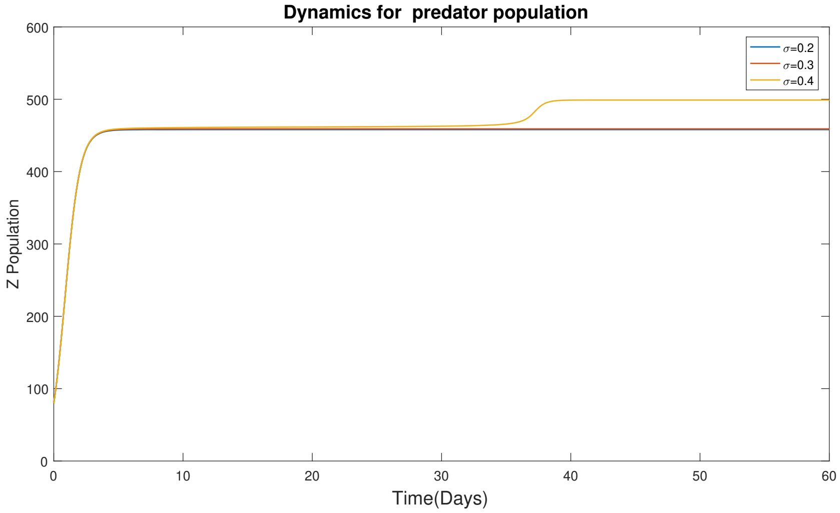

Figure 7: The population of Nile perch in the unreserved area for various migration rates(σ)

Figure 7 shows that the population of the predator rises steeply before converging to equilibrium and attaining stability. The population eventually grows at a constant rate. It can also be observed that the population of the predator increases slightly with increasing values of migration rate (σ) of the prey. When $\sigma = 0.4$ and after about 38 days the population of the predator increases and attains new equilibrium before becoming stable. This increase coincided with the decrease to extinction of the haplochromine population in the unreserved area. The population of the predator increases after about 38 days as a result of conversion of the prey into the predator. It can be observed that for certain values of migration rate $\sigma \leq 0.3$ the population of the predator and can co-exist with prey

4 CONCLUSIONS

In this research we have formulated and analyzed a two species prey-predator model with reserve area and prey refuge for Nile perch and haplochromines. The following results were obtained from the analysis

a Stability analysis of equilibrium points established that the extinction equilibrium is unstable, the predator free equilibrium, and the coexistence equilibrium exist are both locally stable. The predator free equilibrium is globally stable when $r_1 > 0$, $r_2 > 0$, $\sigma > 0$ for all $t > 0$, this implies that a reserve area is necessary for the predator free equilibrium to be globally stable. The analytic results also show that the coexistence equilibrium is Lyapunov stable.

b Numerical simulations show that the coexistence equilibrium is stable when the migration rate $\sigma \leq 0.3$.

We conclude from these results that the prey and the predator in the unreserved area can coexist with the prey in reserved area at a stable equilibrium hence creation of a reserve area has a stabilizing effect on the prey-predator dynamic system and can be used to conserve the endangered haplochromines.

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