Non-conformability of the Moment Generating Function: Rayleigh Probability Function Case

Abstract

Non-existence has been highlighted in literature as the major limitation of the moment generating function (MGF) for some random variables (e.g. Log-Normal, Cauchy) due to divergence in their series. This study investigates (with focus on the Rayleigh probability function) the case where MGF exists but not conforming with the traditional method of deriving moments (μ_r' and or μ_r).

Keywords: Moment generating function, Rayleigh probability function, Non-existence, Log-Normal distribution, moments.
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1 Introduction

Let X be a random variable with probability density function $f(x)$ defined on space $\mathbb{R}$. If $\exists\ h > 0$ such that

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum\limits_{x\in\mathbb{R}} e^{tx} f(x); & \text{for discrete r.v} \\ \\ \int\limits_{x\in\mathbb{R}} e^{tx} f(x)dx; & \text{for continuous r.v} \end{cases} \tag{1}$$

exists for $-h < t < h$, then the $f(t)$ often denoted as $M(t)$ and defined by equation (1) is called the Moment generating function (mgf) of X (Bagui *et al.*, 2016; Dey *et al.*, 2014; Heymann, 2021; Inlow, 2010). If equation (1) exists, it is unique because if X and Y are two random variables and $M_X(t) = M_Y(t)$ then $F_X(x) = F_Y(y)$. As the name implies, equation (1) is a useful tool for determining moments as $M'(0) = \mu$, $M''(0) = \mu_2'$, $M'''(0) = \mu_3'$, $\cdots$, $M^r(0) = \mu_r'$. However, mgf does not exists for every random variable or every probability density function (Dey *et al.*, 2014). For example, mgf's of Cauchy and Log-Normal distributions are mathematically intractable. This constraint is common among class of distributions with heavy tail or specific growth pattern. Suppose $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$ are arbitrary constants, If $Y \sim N(\mu, \sigma^2)$ and $X = \exp(y)$ then the function

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right]; & x > 0 \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

is called the Log-Normal probability function with mgf as

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right] dx. \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left[te^u - \frac{(u-\mu)^2}{2\sigma^2}\right] du; \quad \text{for } x = e^u \end{aligned} \tag{3}$$

For $t > 0$, the term $\exp(te^u)$ grows exceedingly fast as $u \to \infty$ due to the double-exponential growth, specifically $\exp(te^u) \approx e^{e^u}$. This increment is considerably faster than the decay provided by the Gaussian term $\exp\left[-\frac{(u-\mu)^2}{2\sigma^2}\right]$. Consequently, as $u \to \infty$, the integral diverges, leading to the conclusion that the MGF for the log-normal distribution does not exist for $t > 0$ (Heymann, 2021; Tellambura and Senaratne, 2010).

This study asserts that apart from the non-existence limitation of mgf in some cases, non-conformability (with focus on the Rayleigh probability function) is another major deficiency. That is, mgf can be mathematically tractable but not adhering with results of other techniques for deriving moments.

2 Preliminaries

Definition 2.1 *Suppose the scale parameter $\theta > 0$, the function*

$$f(x;\theta) = \begin{cases} \frac{2x}{\theta^2} e^{-\left(\frac{x}{\theta}\right)^2}, & \text{for } x > 0 \\ 0, & \text{elsewhere} \end{cases} \quad (4)$$

named after the British Nobel prize winner in Physics, Lord Baron John William Strutt Rayleigh (1842–1919), is called Rayleigh probability function.

Theorem 2.1 *Suppose (X, Y) has the standard bivariate normal distribution, the polar coordinate distance $R = X^2 + Y^2$ has the Rayleigh distribution with scale parameter $\sqrt{2}$ (Dey et al., 2014; Shala and Merovci, 2024).*

Proof: Given two independent coordinates x and y from normal distributions with zero mean and the same variance σ^2 the distance

$$z = \sqrt{x^2 + y^2} \quad (5)$$

is distributed according to the Rayleigh function. The x and y may e.g. be regarded as the velocity components of a particle moving in a plane. To realise this we first write

$$w = \frac{z^2}{\sigma^2} = \frac{x^2 + y^2}{\sigma^2} = \frac{x^2}{\sigma^2} + \frac{y^2}{\sigma^2} \quad (6)$$

From (6),

$$\frac{dw}{dz} = \frac{2z}{\sigma^2} \quad (7)$$

Since $\frac{x}{\sigma}$ and $\frac{y}{\sigma}$ are distributed as standard normal variables the sum of their squares has the χ_2^2, i.e.

$$g(w) = \frac{1}{2^{\frac{2}{2}}\Gamma\left(\frac{2}{2}\right)} w^{\frac{2}{2}-1} e^{-\frac{w}{2}} = \frac{1}{2} e^{-\frac{w}{2}} \quad (8)$$

Using (7) as the Jacobian of the transformation,

$$f(z) = g\left(w = \frac{z^2}{\sigma^2}\right) \left|\frac{dw}{dz}\right| = \frac{1}{2} e^{-\frac{z^2}{2\sigma^2}} \frac{2z}{\sigma^2} = \frac{2z}{(\sqrt{2}\sigma)^2} e^{-\left(\frac{z}{\sqrt{2}\sigma}\right)^2} \quad (9)$$

which is famously recognise as the Rayleigh probability density function.

3 Results

The r^{th}-moment of $X \sim Rayleigh(\theta)$ is

$$\begin{aligned} E(X^r) = \int_{\mathbb{R}} x^r f(x)dx &= \frac{2}{\theta^2}\int_0^\infty x^{r+1}e^{-\left(\frac{x}{\theta}\right)^2}dx \\ &= \frac{2}{\theta^2}\int_0^\infty \left(\theta\, u^{\frac{1}{2}}\right)^{r+1} e^{-u}\frac{\theta\, u^{-\frac{1}{2}}}{2}du; \quad \text{by letting } u = \left(\frac{x}{\theta}\right)^2 \\ &= \frac{r\theta^r\Gamma\left(\frac{r}{2}\right)}{2} \end{aligned} \tag{10}$$

Putting $r = 1, 2, 3, \cdots$ in (10),

$$\begin{aligned} &\mu = \frac{\theta\sqrt{\pi}}{2}, \quad \mu_2' = \theta^2, \quad \mu_3' = \frac{3\theta^3\sqrt{\pi}}{4}, \quad \mu_4' = 2\theta^4, \quad \mu_5' = \frac{15\theta^5\sqrt{\pi}}{8}, \quad \mu_6' = 6\theta^6, \\ &\mu_7' = \frac{105\theta^7\sqrt{\pi}}{16}, \quad \mu_8' = 24\theta^8, \quad \mu_9' = \frac{945\theta^9\sqrt{\pi}}{32}, \quad \mu_{10}' = 120\theta^{10}, \quad \mu_{11}' = \frac{10395\theta^{11}\sqrt{\pi}}{64}, \\ &\mu_{12}' = 720\theta^{12}, \quad \mu_{13}' = \frac{135135\theta^{13}\sqrt{\pi}}{128}, \quad \mu_{14}' = 5040\theta^{14}, \quad \mu_{15}' = \frac{2027025\theta^{15}\sqrt{\pi}}{256}, \quad \cdots \end{aligned} \tag{11}$$

Consequently,

$$\sigma_X^2 = \mu_2' - \mu^2 = \theta^2 - \left[\frac{\theta\sqrt{\pi}}{2}\right]^2 = \theta^2\left(1 - \frac{\pi}{4}\right). \tag{12}$$

Using (1) and (4), mgf of Rayleigh random variable is

$$\begin{aligned} M_X(t) &= \frac{2}{\theta^2}\int_0^\infty xe^{-\frac{1}{\theta^2}\left[x^2-\theta^2tx\right]}dx \\ &= \frac{2}{\theta^2}\int_0^\infty xe^{-\frac{1}{\theta^2}\left[\left(x-\frac{1}{2}\theta^2t\right)^2-\frac{1}{4}\theta^4t^2\right]}dx \\ &= \frac{2}{\theta^2}e^{\frac{1}{4}\theta^2t^2}\int_0^\infty xe^{-\left(\frac{x-\frac{1}{2}\theta^2t}{\theta}\right)^2}dx \end{aligned} \tag{13}$$

Let $u = \frac{x-\frac{1}{2}\theta^2t}{\theta}$,

$$\begin{aligned} M_X(t) &= \frac{2}{\theta^2}e^{\frac{1}{4}\theta^2t^2}\int_0^\infty \left(u\theta + \frac{1}{2}\theta^2t\right)e^{-u^2}\theta\,du \\ &= \frac{2}{\theta}e^{\frac{1}{4}\theta^2t^2}\left[\theta\int_0^\infty ue^{-u^2}du + \frac{\theta^2t}{2}\int_0^\infty e^{-u^2}du\right] \\ &= \frac{2}{\theta}e^{\frac{1}{4}\theta^2t^2}\left[\frac{\theta}{2} + \frac{\theta^2t\sqrt{\pi}}{4}\right] \\ &= e^{\frac{1}{4}\theta^2t^2}\left[1 + \frac{\theta\sqrt{\pi}t}{2}\right]. \end{aligned} \tag{14}$$

Differentiate (14) successively using product rule of Calculus,

$$M'_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{\theta\sqrt{\pi}}{2} + \frac{\theta^2 t}{2} + \frac{\theta^3\sqrt{\pi}t^2}{4}\right]$$

$$M''_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{\theta^2}{2} + \frac{3\theta^3\sqrt{\pi}t}{4} + \frac{\theta^4 t^2}{4} + \frac{\theta^5\sqrt{\pi}t^3}{8}\right]$$

$$M'''_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{3\theta^3\sqrt{\pi}}{4} + \frac{3\theta^4 t}{4} + \frac{6\theta^5\sqrt{\pi}t^2}{8} + \frac{\theta^6 t^3}{8} + \frac{\theta^7\sqrt{\pi}t^4}{16}\right]$$

$$M^{iv}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{3\theta^4}{4} + \frac{15\theta^5\sqrt{\pi}t}{8} + \frac{6\theta^6 t^2}{8} + \frac{10\theta^7\sqrt{\pi}t^3}{16} + \frac{\theta^8 t^4}{16} + \frac{\theta^9\sqrt{\pi}t^5}{32}\right]$$

$$M^{v}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{15\theta^5\sqrt{\pi}}{8} + \frac{15\theta^6 t}{8} + \frac{45\theta^7\sqrt{\pi}t^2}{16} + \frac{10\theta^8 t^3}{16} + \frac{15\theta^9\sqrt{\pi}t^4}{32} + \frac{\theta^{10} t^5}{32} + \frac{\theta^{11}\sqrt{\pi}t^6}{64}\right]$$

$$M^{vi}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{15\theta^6}{8} + \frac{105\theta^7\sqrt{\pi}t}{16} + \frac{45\theta^8 t^2}{16} + \frac{105\theta^9\sqrt{\pi}t^3}{32} + \frac{15\theta^{10} t^4}{32} + \frac{21\theta^{11}\sqrt{\pi}t^5}{64} + \frac{\theta^{12} t^6}{64} + \frac{\theta^{13}\sqrt{\pi}t^7}{128}\right]$$

$$M^{vii}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{105\theta^7\sqrt{\pi}}{16} + \frac{105\theta^8 t}{16} + \frac{420\theta^9\sqrt{\pi}t^2}{32} + \frac{105\theta^{10} t^3}{32} + \frac{210\theta^{11}\sqrt{\pi}t^4}{64} + \frac{21\theta^{12} t^5}{64} + \frac{28\theta^{13}\sqrt{\pi}t^6}{128} + \frac{\theta^{14} t^7}{128} + \frac{\theta^{15}\sqrt{\pi}t^8}{256}\right]$$

$$M^{viii}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{105\theta^8}{16} + \frac{945\theta^9\sqrt{\pi}t}{32} + \frac{420\theta^{10} t^2}{32} + \frac{1260\theta^{11}\sqrt{\pi}t^3}{64} + \frac{210\theta^{12} t^4}{64} + \frac{378\theta^{13}\sqrt{\pi}t^5}{128} + \frac{28\theta^{14} t^6}{128} + \frac{36\theta^{15}\sqrt{\pi}t^7}{256} + \frac{\theta^{16} t^8}{256} + \frac{\theta^{17}\sqrt{\pi}t^9}{512}\right]$$

$$M^{ix}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{945\theta^9\sqrt{\pi}}{32} + \frac{945\theta^{10} t}{32} + \frac{4725\theta^{11}\sqrt{\pi}t^2}{64} + \frac{1260\theta^{12} t^3}{64} + \frac{3150\theta^{13}\sqrt{\pi}t^4}{128} + \frac{378\theta^{14} t^5}{128} + \frac{630\theta^{15}\sqrt{\pi}t^6}{256} + \frac{36\theta^{16} t^7}{256} + \frac{45\theta^{17}\sqrt{\pi}t^8}{512} + \frac{\theta^{18} t^9}{512} + \frac{\theta^{19}\sqrt{\pi}t^{10}}{1024}\right]$$

$$M^{x}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{945\theta^{10}}{32} + \frac{10395\theta^{11}\sqrt{\pi}t}{64} + \frac{4725\theta^{12} t^2}{64} + \frac{17325\theta^{13}\sqrt{\pi}t^3}{128} + \frac{3150\theta^{14} t^4}{128} + \frac{6930\theta^{15}\sqrt{\pi}t^5}{256} + \frac{630\theta^{16} t^6}{256} + \frac{990\theta^{17}\sqrt{\pi}t^7}{512} + \frac{45\theta^{18} t^8}{512} + \frac{55\theta^{19}\sqrt{\pi}t^9}{1024} + \frac{\theta^{20} t^{10}}{1024} + \frac{\theta^{21}\sqrt{\pi}t^{11}}{2048}\right]$$

$$M^{xi}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{10395\theta^{11}\sqrt{\pi}}{64} + \frac{10395\theta^{12} t}{64} + \frac{62370\theta^{13}\sqrt{\pi}t^2}{128} + \frac{17325\theta^{14} t^3}{128} + \frac{51975\theta^{15}\sqrt{\pi}t^4}{256} + \frac{6930\theta^{16} t^5}{256} + \frac{13860\theta^{17}\sqrt{\pi}t^6}{512} + \frac{990\theta^{18} t^7}{512} + \frac{1485\theta^{19}\sqrt{\pi}t^8}{1024} + \frac{55\theta^{20} t^9}{1024} + \frac{66\theta^{21}\sqrt{\pi}t^{10}}{2048} + \frac{\theta^{22} t^{11}}{2048} + \frac{\theta^{23} t^{12}}{4096}\right]$$

$$M^{xii}_X(t) = e^{\frac{1}{4}\theta^2 t^2}\left[\frac{10395\theta^{12}}{64} + \frac{135135\theta^{13}\sqrt{\pi}t}{128} + \frac{530145\theta^{14} t^2}{128} + \frac{270270\theta^{15}\sqrt{\pi}t^3}{256} + \frac{51975\theta^{16} t^4}{256} + \frac{135135\theta^{17}\sqrt{\pi}t^5}{512} + \frac{13860\theta^{18} t^6}{512} + \frac{25740\theta^{19}\sqrt{\pi}t^7}{1024} + \frac{1485\theta^{20} t^8}{1024} + \frac{2145\theta^{21}\sqrt{\pi}t^9}{2048} + \frac{66\theta^{22} t^{10}}{2048} + \frac{78\theta^{23} t^{11}}{4096} + \frac{\theta^{24} t^{12}}{4096} + \frac{\theta^{25} t^{13}}{8192}\right]$$

and so on. Set $t = 0$ in all the derivatives to get:

$$M'_X(0) = \frac{\theta\sqrt{\pi}}{2},\ M''_X(0) = \frac{\theta^2}{2},\ M'''_X(0) = \frac{3\theta^3\sqrt{\pi}}{4},\ M^{iv}_X(0) = \frac{3\theta^4}{4},\ M^{v}_X(0) = \frac{15\theta^5\sqrt{\pi}}{8},$$
$$M^{vi}_X(0) = \frac{15\theta^6 t}{8},\ M^{vii}_X(0) = \frac{105\theta^7\sqrt{\pi}}{16},\ M^{viii}_X(0) = \frac{105\theta^8}{16},\ M^{ix}_X(0) = \frac{945\theta^9\sqrt{\pi}}{32},$$
$$M^{x}_X(0) = \frac{945\theta^{10}}{32},\ M^{xi}_X(0) = \frac{10395\theta^{11}\sqrt{\pi}}{64},\ M^{xii}_X(0) = \frac{10395\theta^{12}}{64},\ M^{xiii}_X(0) = \frac{135135\theta^{13}\sqrt{\pi}}{128},\ \cdots \tag{15}$$

As a result,

$$Var(X) = M''_X(0) - [M'_X(0)]^2 = \frac{\theta^2}{2} - \left[\frac{\theta\sqrt{\pi}}{2}\right]^2 = \theta^2\left(\frac{1}{2} - \frac{\pi}{4}\right). \tag{16}$$

The assumption that mgf (if exist) is an alternative approach for obtaining μ'_r (raw moments) of a r.v. X is sometimes unrealistic as established in this study. Despite (14) satisfies all principal features of mgf such as: uniformly continuous and infinitely differentiable, $M(0) = 1$, $|M_X(t)| \leq 1$, in most cases, its results in (15) and (16) are noncompliant with the orthodox moment method presented in (11) and (12), respectively.

Conclusion

MGF of Rayleigh probability function exists, unique, converge absolutely and infinitely differentiable, this study establishes its non-commensuration with the conventional moment method. Henceforth, researchers and educationists should double-check for agreement betwixt the two techniques.

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