A NOTE ON FRACTIONAL SEMIGROUP THEORY AND FRACTIONAL EVOLUTION EQUATIONS

Abstract. This paper considers fractional semigroup theory and fractional evolution equations. First and foremost, we present some fractional semigroups and prove the semigroup property associated with Riemann-Liouville fractional integral operator which is not necessarily true for the fractional differential operator. Consequently, we utilize the fractional β-semigroup of operators to solve the fractional abstract Cauchy problem.

Keywords: Fractional β-semigroup, fractional Cauchy problem, Riemann-Liouville operator.

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1. INTRODUCTION

Fractional differential equations have recently received an increasing attention in research, see for example [9], [10], [12], [13], [17], [18], [19], [20] and [21]. Indeed, fractional differential equation is that aspect of mathematics, precisely mathematical analysis that considers the application of derivatives and integrals in the fractional order sense [4]. With this in mind, it is worthy to understand the fundamental properties of fractional integral and differential operators, especially the classical order differential operators.

Numerous definitions apply to fractional differential equations, categorized into two types; local (singular) kernels and non-singular kernels. The Riemann-Liouville and Caputo fractional derivatives, falling under the singular kernel, utilize power law kernels [6]. In contrast, the Atangana-Baleanu and Caputo-Fabrizio fractional derivatives, belonging to the non-singular kernel category, employ the Mittag-Leffler function and exponential decay function as their kernels.

The importance of semigroup theory cannot be over emphasized, see for example [5], [8], [14] etc. Semigroup theory can be used as a tool to solve a class of problems referred to as evolution equations. These are the types of equations which are evident in engineering, finance, biology, physics etc. These classes of problems are usually described for a differential equation by an initial value problem which can be ordinary or partial. One classic example of a vector-valued differential equation is the abstract Cauchy problem, typically represented as

$$\begin{cases} {}^{d}/_{dt}\, u(t) = Au(t), & t \geq 0 \\ u(0) = v \in \mathbb{C}^n \end{cases} \tag{1.1}$$

and $A = (a_{ij})$ is a square matrix such that $a_{ij} \in \mathbb{C}$ for $i, j = 1,2,\dots,n$. The solution to (1.1) is

$$u(t) = e^{tA}v, \quad t \geq 0,$$

where e^{tA} is referred to as the fundamental matrix of (1.1) and it is equal to 1 whenever $t = 0$.

More so, $e^{tA} = \sum_{l=0}^{\infty} \frac{t^l}{l!} A^l, \; t \in \mathbb{R}$.

It is important to note that A and e^{tA} are regarded as linear operators, $A \in \mathcal{L}(X)$, $e^{tA} \in \mathcal{L}(X)$, where $X = \mathbb{C}$, utilizing any of their equivalent norms.

In addition, for each $x \in X$, $A : D(A) \subseteq X \to X$ is a linear operator while the unknown function is $u : [0, \infty) \to X$.

Recent research trends in existing literature demonstrate that fractional- order calculus captures non-localities and the memory effects in physical processes, yielding more reliable results for mathematical systems. For the sake of these merits, the initial value problem or abstract Cauchy problem can be represented as

$$\begin{cases} {}_0^cD_t^{\beta}\, u(t) = Au(t), & t > 0, \quad \beta \in (0,1] \\ u(0) = u_0 \end{cases} \tag{1.2}$$

where ${}_0^cD_t^{\beta}$ denotes the Caputo fractional derivative operator which has lower limit of zero and upper limit of t. Using Laplace transform or Homotopy perturbation method, the Caputo fractional order $\beta > 0$ has the following exact solution

$$u(t) = E_{\beta,\,1}(At^{\beta}),$$

where the quality $E_{\beta,\,1}(.)$ represents the Mittag-Leffler function of parameter defined as

$$E_{\beta,\,1}(At^{\beta}) = \sum_{l=0}^{\infty} \frac{(At)^l}{\Gamma(\beta l + 1)} \quad , \qquad Re(\beta) > 0.$$

It is important to note that $E_{\beta,\,1}(At^{\beta})$ can be seen as linear operators and $E_{\beta,\,1}(At^{\beta}) \in \mathcal{L}(X)$ where $X \in \mathbb{C}$.

To explore further details about semigroups of operators and abstract Cauchy problems, refer to [5] and [14]. For information on the inverse form of the abstract Cauchy problem, refer to [7] and [15].

The investigation of fractional semigroups parallels that of fractional powers initially explored by Bochner in [2]. In [3], the analysis centered on the problem of fractional powers of closed operators and the semigroups they produce. Additionally, Popescu in [16] explored the fractional Cauchy problem linked with a Feller semigroup.

The physical motivation behind studying fractional semigroup theory lies in its ability to describe dynamical processes with memory effects and long-range interactions more accurately. Traditional semigroup theory deals with linear operators that generate evolution equations with time. However, when dealing with systems exhibiting fractional dynamics, traditional semigroup theory may not suffice.

Fractional semigroup theory extends the classical theory to handle fractional-order operators, allowing for the study of systems with non-local and non-Markovian dynamics. This is crucial in understanding complex physical phenomena where memory effects play a significant role, such as in the behavior of complex fluids, diffusion in porous media, or the dynamics of biological systems. By incorporating fractional operators, fractional semigroup theory provides a more comprehensive framework for modeling and analyzing these systems, leading to deeper insights and more accurate predictions.

This paper aims to provide a foundational characterization of fractional semigroup theory and fractional evolution equations.

This paper has four sections. In section 1, we gave a brief overview of the topic under study while some known definitions and results were presented in section 2. In section 3, we characterized some fractional semigroup theory as well as their properties. Finally in section 4, we presented the concept of fractional evolution equations.

2. PRELIMINARIES

In this section, we review essential definitions and established results crucial for this paper. For any notation and terminologies not explicitly covered, readers are encouraged to consult references [1], [5], [6], [11] and [14].

Definition 2.1. Let S be a set and $*$ be a binary operation on S. The pair $(S,*)$ is a semigroup if $\forall\, a,b,c \in S, \quad a*(b*c) = (a*b)*c$

Example 2.2. $(\mathbb{Z},+), (\mathbb{R},+), (\mathbb{Z},\ *), \ (\mathbb{R},*)$ etc.

Definition 2.3. Let X be a Banach space over $\mathbb{C}$ with norm $\|\,.\,\|$ and let $\mathcal{L}(X)$ be the set of all linear bounded (continuous) operators $T : D(T) = X \to X$. $\mathcal{L}(X)$ is also a Banach space with respect to the operator norm,

$$\|T\| = \sup\{\|Tx\| \ : x \in X, \ \|x\| \le 1\}.$$

Definition 2.4. We call a semigroup a one parameter family $\{T(t)\,; t \ge 0\} \subset \mathcal{L}(X)$ if:

i) $T(t+s) = T(t)T(s), \quad t,s > 0$

ii) $T(0) = I$

where I is the identity operator acting on X.

Definition 2.5. We call a semigroup of class C_0 (otherwise known as strongly continuous semigroup) a one parameter semigroup family $\{T(t)\,; t \ge 0\}$ of linear bounded operators if

i) $T(t+s) = T(t)T(s), \quad t,s > 0$

ii) $T(0) = I$

iii) $\lim_{t \to 0^+} \| T(t)x - x \| = 0, \ \ \forall \, x \in X.$

Definition 2.6. A uniformly continuous semigroup is referred to as a one parameter family $\{T(t) \, ; t \geq 0\}$ of bounded linear operators from X to X if

i) $T(t+s) = T(t)T(s), \quad t, s > 0$

ii) $T(0) = I$

iii) $\lim_{t \to 0^+} \| T(t)x - I \| = 0$

Remark 2.7. It is worth noting that uniformly continuous semigroups are contained within the set of strongly continuous semigroups.

Definition 2.8. We say that $A : D(A) \subset X \to X$ is the infinitesimal generator of the semigroup $\{T(t) : t \geq 0\}$ if it satisfies

$$Ax = \lim_{h \to 0^+} \frac{1}{h} (T(h)x - x), \ \ \forall \, x \in X,$$

where $D(A)$ is the set of all $x \in X$ such that the above limit exists.

Definition 2.9. [6]. The Riemann-Liouville fractional integral of the function $h \in C' \, ([0, T], \ \mathbb{R}^+)$ of order $\beta > 0$ is given by

$${}_0J_t^{\beta} \, h(t) = \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta - 1} \, h(s) \, ds, \ t > 0, \ \beta \in \mathbb{N}.$$

Definition 2.10. [4]. The Riemann-Liouville fractional derivative of the function $h \in C' \, ([0, T], \ \mathbb{R}^+)$ and $p - 1 < \beta \leq p, \ p \in \mathbb{R}^+$ is given by

$${}_0^{Rl}D_t^{\beta} \, h(t) = D^p \, {}_0J_t^{p-\beta} \, h(t) = D^p \left[\tfrac{1}{\Gamma(p-\beta)} \int_0^t \ \tfrac{h(s)}{(t-s)^{\beta+1-p}} \ ds \right],$$

Here $D^p \equiv \frac{d^p}{dt^p}$ and if $h(t) = t^{\alpha}$, then ${}_0^{Rl}D_t^{\beta} \, t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}, \ \beta > 0, \ \alpha > -1, \ t > 0$

which is the fractional derivative of power function.

Definition 2.11. [4]. The Caputo fractional derivative of the function $h \in C' \, ([0, T], \ \mathbb{R}^+)$ and $p - 1 < \beta < p$, where $p \in \mathbb{R}^+$ is given by

$${}_0^cD_t^{\beta} \, h(t) = \ {}_0J_t^{p-\beta} \, D^p h(t) = \frac{1}{\Gamma(p-\beta)} \int_0^t \frac{h^{(p)}(s)}{(t-s)^{\beta+1-p}} \, ds$$

For the purpose of clarity, the Riemann-Liouville and Caputo fractional derivatives are not equal i.e.

$$D^p \, {}_0J_t^{p-\beta} \, h(t) \neq \ {}_0J_t^{p-\beta} \, D^p h(t).$$

In fact with the aid of Definition 2.10 we have that

$$ {}_{0}^{c}D_{t}^{\beta}\, h(t) = {}_{0}^{R}D_{t}^{\beta}\, h(t) - \sum_{l=0}^{p-1} \frac{t^{l-\beta}}{\Gamma(l-\beta+1)} h^{(k)}(0) $$

$$ = {}_{0}^{R}D_{t}^{\beta} \left(h(t) - \sum_{l=0}^{p-1} \frac{t^{l}}{l!} h^{(k)}(0) \right) $$

where $p-1 < \beta < p,\ p \in \mathbb{N}$.

Definition 2.12. Let X be a Banach space and $\beta \in (0, a]$ for any $a > 0$. A family $\{T(t)\,; t \geq 0\} \subset \mathcal{L}(X)$ is called a fractional β-semigroup of operators if:

i) $T(t+s)^{\frac{1}{\beta}} = T(t)^{\frac{1}{\beta}} T(s)^{\frac{1}{\beta}}$ for all $t, s \in [0, \infty)$

ii) $T(0) = I,$ where I is the same with Definition 2.4

Obviously, if $\beta = 1$, then we return to Definition 2.4

Example 2.13. [1]. Suppose we let $X = C[0, \infty)$ to be the space of real valued function that is continuous on $[0, \infty)$. Now define $T(t)$ on X by the rule that $T(t)x = e^{2\sqrt{x}}x$. Then $\{T(t)\,; t \geq 0\}$ is a $\frac{1}{2}$ - semigroup since for $t, s \geq o, x \in X$, we have that

$$ \begin{aligned} T(t+s)^2 x = e^{2\sqrt{(t+s)^2}} x &= e^{2(t+s)} x \\ &= e^{2t}(e^{2s}x) \\ &= T(s^2)(e^{2t}x) \\ &= T(s^2)T(t^2)x. \end{aligned} $$

It can be easily shown that $T(0) = I$.

Definition 2.14. A fractional β-semigroup $T(t)$ is called a C_0-semigroup if $\lim_{t \to 0^+} \| T(t)x - x \| = 0,\ \forall\, x \in X$.

Definition 2.15. The linear operator $A : D(A) \subset X \to X$ is referred to as the β −infinitesimal generator of the semigroup $\{T(t) : t \geq 0\}$ if the condition that

$$ Ax = \lim_{h \to 0^+} \frac{1}{h} \left(T^{\beta}(h)x - x \right),\ \ \forall\, x \in X \ \text{ is satisfied.} $$

Theorem 2.16. [1] Suppose a β-semigroup $T(t)$ is a strongly continuous semigroup having infinitesimal generator $A, 0 < \beta \leq 1$. If $T(t)$ is continuously β-differentiable and $x \in D(A)$, then it follows that

$$ T^{\beta}(t)x = AT(t)x = T(t)Ax\,. $$

3. FRACTIONAL SEMIGROUP THEORY

In this section, we characterize some fractional semigroups and present some semigroup properties embedded in Riemann-Liouville fractional integral and derivative operator.

We start with the fractional translation semigroups.

3.1. Fractional translation semigroups

Fractional translation semigroups are class of translation semigroups that satisfy the fractional β-semigroup of operators presented in Definition 2.14.

Let $f : \mathbb{R} \to \mathbb{C}$ and $t \geq 0$, we call

$$(T_l(t)f)(x) \coloneqq f(x+t), \quad x \in \mathbb{R} \qquad \text{the left translation and}$$

$$(T_r(t)f)(x) \coloneqq f(x-t), \quad x \in \mathbb{R} \qquad \text{the right translation.}$$

Suppose we let $X = C_0([0,\infty))$ to be the Banach space of real-valued continuous functions on $[0,\infty)$ and for which $f(x) \to M < \infty$ as $x \to \infty$ with respect to the sup norm. Define the operator $T(t)$ on X by:

$$(T(t)f)(x) = f(x + \frac{1}{\beta}t^{\beta}).$$

Obviously, the operator $T(t)$ translates the function $f \in C_0$ to the left by t units. To show that $\{T(t)\,; t \geq 0\}$ is a fractional β-semigroup of operators, we have that

$$\left(T(t+s)^{\frac{1}{\beta}}f\right)(x) = f\left(x + \frac{1}{\beta}\left[(t+s)^{\frac{1}{\beta}}\right]^{\beta}\right)$$

$$= f\left(x + \frac{1}{\beta}t + \frac{1}{\beta}s\right)$$

$$= \left(T\left(t^{\frac{1}{\beta}}\right)T\left(s^{\frac{1}{\beta}}\right)\right)(x) \ .$$

It can be easily shown that $T(0) = I$.

Furthermore, since $\lim_{t\to 0^+} sup_{x \in \mathbb{R}} \left\| f\left(x + \frac{1}{\beta}t^{\beta}\right) - f(x)\right\| = 0$, we have that $\lim_{t\to 0^+} T(t)f = f$.

Consequently, that $\{T(t)\,; t \geq 0\}$ is a fractional translation semigroup on $\mathbb{R}$ which is continuous.

It is important to note that since $\|T(t)f\| = \|f\|$, or fractional translation semigroup forms a contraction semigroup.

We have that

$$\|T(t) - I\| \leq sup_{\|f\|\leq 1} \|T(t)f - f\|$$

$$= sup_{\|f\|\leq 1} sup_{s \in \mathbb{R}} |f(t+s) - f(s)|$$

does not converge to 0, and from Definition 2.6, the fractional translation semigroup is not uniformly continuous.

3.2. Fractional matrix semigroup

Our aim in this section is to show that matrix semigroups satisfy the properties of fractional β-semigroup of operators.

Let $X = \mathbb{C}^n$ be a finite-dmensional vector space and $M_n(\mathbb{C})$ be the space of all complex $n \times n$ matrices such that $T(.) : \mathbb{R}_+ \to M_n(\mathbb{C})$ satisfies Definition 2.4 (i) and (ii). We know from section 1 that for any $A \epsilon M_n(\mathbb{C})$ we have that

$$e^{tA} = \sum_{l=0}^{\infty} \frac{t^l}{l!} A^l, \quad t \epsilon \mathbb{R} .$$

Obviously, the series above converge and satisfies

$$\|e^{tA}\| \leq e^{t\|A\|} \quad \text{for all } t \geq 0.$$

In fact the partial sums of the series forms a Cauchy sequence.

Lemma 3.2.1. Let $M_n(\mathbb{C})$ be the space of all complex $n \times n$ matrices and $A \epsilon M_n(\mathbb{C})$. Then the map $\mathbb{R}_+ \ni t \mapsto e^{tA} \epsilon M_n(\mathbb{C})$ is continuous and satisfies;

i) $e^{(t+s)^{\frac{1}{\beta}} A} = e^{(t)^{\frac{1}{\beta}} A} e^{(s)^{\frac{1}{\beta}} A}, \ t, s > 0$

ii) $e^{0A} = I$

Proof. i) We have that

$$\sum_{l=0}^{\infty} \frac{\left(\left[(t)^{\frac{1}{\beta}}\right]^{\beta}\right) A^l}{l!} \cdot \sum_{l=0}^{\infty} \frac{\left(\left[(s)^{\frac{1}{\beta}}\right]^{\beta}\right) A^l}{l!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\left(\left[(t)^{\frac{1}{\beta}}\right]^{\beta}\right)^{k-l} A^{k-l}}{(k-l)!} \cdot \frac{\left(\left[(s)^{\frac{1}{\beta}}\right]^{\beta}\right)^{l} A^{l}}{l!}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\left[(t+s)^{\frac{1}{\beta}}\right]^{\beta}\right)^{k} A^{k}}{k!} .$$

ii) The proof is a routine check.

Next is to show that $t \mapsto e^{tA}$ is continuous. So we have that

$$e^{(t+h)A} - e^{tA} = e^{tA}\left(e^{hA} - I\right), \quad \text{for all } t, h \epsilon \mathbb{R}.$$

Hence, it suffices to show that $\lim_{h \to 0} e^{hA} = I$. It follows from the fact that

$$\left\|e^{hA} - I\right\| = \left\|\sum_{l=1}^{\infty} \frac{h^l A^l}{l!}\right\| \leq \sum_{l=1}^{\infty} \frac{|h|^l \cdot \|A\|^l}{l!}$$

$$= e^{|h| \cdot \|A\|} - 1.$$

With Lemma 3.2.1 above, we say that the one parameter family $\{e^{tA};\ t \geq 0\}$ generated by the matrix $A \epsilon M_n(\mathbb{C})$ such that (i) $e^{(t+s)^{\frac{1}{\beta}}A} = e^{(t)^{\frac{1}{\beta}}A} e^{(s)^{\frac{1}{\beta}}A},\ t, s > 0$ and (ii) $e^{0A} = I$ is known as the fractional matrix semigroups.

Remark 3.2.2. We call that in [5] and [14], the following was shown

i. It is possible to find the operator $A \epsilon e^{tA}$ once the semigroup $T(t)$ is given

ii. The operator A that gives rise to respective semigroups can be determined

iii. A method of constructing the semigroup $T(t)$ if A is given.

Analogously, Lemma 3.2.1 have shown us a method of constructing the fractional matrix semigroups if A is given. For the fractional matrix semigroups, it is not an easy task to determine the operator A that give rise to respective semigroups, this is because one does not know the nature of A, i.e, the kind of operator A is (whether it is bounded or not) especially if it is unbounded. Therefore, the answer to that question is left for future work.

More so, the range of the function $t \mapsto T(t) := e^{tA} \epsilon M_n(\mathbb{C})$ is a matrix semigroup that is commutative and depends continuously on the parameter $t \epsilon \mathbb{R}_+$. Furthermore, the mapping $T(.) : \mathbb{R}_+ \to M_n(\mathbb{C})$ is a morphism, precisely a homomorphism.

3.3 Some semigroup properties of fractional operators

Here we consider the semigroup properties of Riemann-Liouville fractional integral operator and derivative operator of Definition 2.9 and Definition 2.10 respectively.

Now let $(\mathbb{C}',\ \circledast)$ be the Duhamel convolution algebra defined by

$$(h \circledast k)(t) = \int_0^t (t + 0 - s)\, h(s)\, ds, \quad \text{for all } h, k \epsilon \mathbb{C}'\ .$$

Obviously, the bilinear operation of the above algebra is a semigroup so that the Riemann-Liouville fractional integral in Definition 2.9 can now be rewritten as a convolution operator

$${}_0J_t^{\beta}\, g(t) = \frac{(t-0)^{\beta-1}}{\Gamma(\beta)} \circledast h(t)$$

where the operator ${}_0J_t^{\beta}$ is a multiplier of the Duhamel convolution algebra.

Let $(\mathbb{C}',\ \circledast)$ be the Duhamel convolution algebra, a mapping $\sigma : \mathbb{C}' \to \mathbb{C}'$ is a multiplier of $(\mathbb{C}',\ \circledast)$ if $(\sigma h) \circledast k = h \circledast \sigma k$, for all $h, k \epsilon \mathbb{C}'$.

It can be easily shown that if γ denote the set of all multipliers of $(\mathbb{C}',\ \circledast)$, then γ form a sub algebra which is commutative of the algebra of all continuous linear operators $\rho : \mathbb{C}' \to \mathbb{C}'$.

More so, it is important to note that it contains the identity operator which is not necessarily a convolution operator.

Furthermore, for $\sigma : \mathbb{C}' \to \mathbb{C}'$, we have that

$$\sigma(h \circledast k) = (\sigma h) \circledast k, \quad \text{for all } h, k \in \mathbb{C}'.$$

Suppose $h \circledast$ represent the multiplier operator $(h \circledast)k = h \circledast k$, then we have that $\sigma(h \circledast) = (h \circledast)\sigma$.

Consequently, we have that

$$[\sigma(h \circledast)]k = [(h \circledast)\sigma]k$$
$$\Longrightarrow \sigma(h \circledast k) = h \circledast \sigma k, \text{ for } k \in \mathbb{C}'.$$

Lemma 3.3.1. Suppose $\sigma : \mathbb{C}' \to \mathbb{C}'$ is a continuous linear operation such that $\sigma\, {}_0J_t^{\beta} = {}_0J_t^{\beta}\, \sigma$. Then the convolution algebra $(\mathbb{C}', \circledast)$ has a multiplier σ.

Proof. Let I be the identity operator. Then we have that

$$\sigma I \circledast I = I \circledast \sigma I.$$

For any arbitrary non-negative integers u and v, we have that

$$\left({}_0J_t^{\beta}\right)^{u+v} = {}_0J_t^{u\beta}\ {}_0J_t^{v\beta} .$$

Hence, $({}_0J_t^{u\beta}\, \sigma I) \circledast {}_0J_t^{v\beta} I = {}_0J_t^{u\beta} I \circledast ({}_0J_t^{v\beta}\, \sigma I)$

$$\Longrightarrow \sigma({}_0J_t^{u\beta}\, I) \circledast {}_0J_t^{v\beta}\, I = {}_0J_t^{u\beta}\, I \circledast \sigma({}_0J_t^{v\beta}\, \sigma I).$$

Since ${}_0J_t^{\beta}\, I = \frac{(t-0)^{\beta}}{\Gamma(\beta+1)}$, for $\beta > 0$.

Consequently, we have that

$$\sigma\left[(t-0)^{u\beta}\right] \circledast \left[(t-0)^{v\beta}\right]$$
$$= \left[(t-0)^{u\beta}\right] \circledast \sigma\left[(t-0)^{v\beta}\right],$$

for $u, v \in \mathbb{N}$. Thus σ is a multiplier of $(\mathbb{C}', \circledast)$.

Remark 3.3.2. Unlike the fractional integral operator, Riemann-Liouville fractional derivatives of Definition 2.10 do not necessarily satisfy the semigroup property or the commutative law.

For example, let $h(t) = t^{\frac{1}{2}}$, $\beta = \frac{1}{2}$, $\gamma = \frac{3}{2}$. Then we have that

$${}_0^{Rl}D_t^{\beta}\, h = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} , \quad {}_0^{Rl}D_t^{\gamma}\, h = 0 .$$

Now, ${}_0^{Rl}D_t^{\beta}\left({}_0^{Rl}D_t^{\gamma}\, h\right) = 0$ and ${}_0^{Rl}D_t^{\gamma}\left({}_0^{Rl}D_t^{\beta}\, h\right) = -\frac{1}{4}t^{-\frac{3}{2}}$.

Also, we have that ${}_0^{Rl}D_t^{\beta+\gamma}\, h = -\frac{1}{4}t^{-\frac{3}{2}}$.

Consequently, we have that

$${}_0^{Rl}D_t^{\beta}\left({}_0^{Rl}D_t^{\gamma}\, h\right) \neq {}_0^{Rl}D_t^{\gamma}\left({}_0^{Rl}D_t^{\beta}\, h\right) \qquad \text{(commutativity fails).}$$
$${}_0^{Rl}D_t^{\beta}\left({}_0^{Rl}D_t^{\gamma}\, h\right) \neq {}_0^{Rl}D_t^{\beta+\gamma}\, h \qquad \text{(semigroup property fails)}$$

It is worthy to note Cong in [11] established partial semigroup property of Riemann-Liouville and Caputo fractional differential operators.

In comparison, it was proved in Theorem 3 and Theorem 4 of [11] that there is a partial semigroup property of Riemann-Liouville and Caputo Fractional differential operators.

Generally, our results have shown that the semigroup property or commutative law do not hold for Riemann-Liouville fractional derivatives.

More so, it is essential to understand that the semigroup property enables the reduction of high-order ordinary differential equations. This approach benefits from a well-established theory already developed for such systems.

To conclude this section, we note that for the Riemann-Liouville fractional integral operator, the following properties are valid;

i) Suppose $\beta = n \in \mathbb{N}$, then we have that

$$ {}_0J_t^{\beta}\, h(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} h(s)\, ds $$
$$ = \int_0^{t_n}\int_0^{t_{n-1}} \dots \int_0^{t_1} h(s) ds_1 \dots ds_n, $$

this means that ${}_0J_t^n\, h$ is an n-fold integral of h.

ii) the semigroup property and commutativity hold i.e.

$$ {}_0J_t^{\beta}\ {}_0J_t^{\beta}\, g = {}_0J_t^{\beta+\gamma}\, h \text{ and } {}_0J_t^{\gamma}\ {}_0J_t^{\beta} = {}_0J_t^{\beta}\ {}_0J_t^{\gamma}\ . $$

4. FRACTIONAL EVOLUTION EQUATIONS

In this section, we consider the fractional evolution equation, precisely, fractional abstract Cauchy problem. In particular, we will employ our knowledge of fractional β-semigroups to solve the fractional Cauchy problem.

Suppose X be a Banach space and let $\mathcal{L}(X)$ the set of all bounded (continuous) operators $A : D(A) = X \to X$ with $x \in X$. The abstract Cauchy problem

$$ \begin{cases} u^{\beta}(t) = Au(t), & t \geq o \\ u(0) = x \end{cases} $$

is said to have a solution $u : [0,\infty) \to X$ whenever the following conditions are satisfied;

i) u is continuous in $[0,\infty)$

ii) if u is continuously β-differentiable in the interval $(0,\infty)$

iii) $u(t) \in D(A)$ for $t > 0$.

With the above, we now state the following theorem;

Theorem 4.1. Suppose X is a Banach space and A is the infinitesimal generator as in Definition 2.17 of a C_0 - semigroup $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$. If $x \in D(A)$, then $u(t) = T(t)x$ is the unique solution of the fractional abstract Cauchy problem.

Proof. Obviously, $u(t) = T(t)x$ is the solution of the abstract Cauchy problem. Next is to show that the solution is unique. Let u be the solution of the abstract Cauchy problem, then we have that

$$
\begin{aligned}
[T(t-s)\,u(s)]^{\beta} &= T(t-s)u^{\beta}(s) - AT(t-s)u(s) \\
&= T(t-s)u^{\beta}(s) - T(t-s)A\,u(s) \\
&= T(t-s)\left[u^{\beta}(s) - Au(s)\right] \\
&= 0\,.
\end{aligned}
$$

Consequently, we have that

$$
\begin{aligned}
&T(t-t)u(t) - T(t)x = 0 \\
&= T(0)u(t) - T(t)x = 0 \quad (T(0) = I \text{ from Definition 2.4 ii)}) \\
&= u(t) - T(t)x = 0
\end{aligned}
$$

which implies that $u(t) = T(t)x$ and the proof is completed.

5. CONCLUSION

The primary objective of this paper was to provide a foundational understanding of fractional semigroup theory and fractional evolution equations. The objective was accomplished by establishing various fractional semigroups and exploring associated properties of fractional operators.

Our focus was directed towards strongly continuous semigroups of bounded linear operators, leading to the development of techniques for deriving the fractional β-semigroup within an appropriate Banach space framework.

We investigated the behavior of Riemann-Liouville fractional integral and derivative operators in relation to the semigroup property, contrasting our findings with previous studies utilizing Caputo fractional differential operators (see [11]).

In conclusion, our work demonstrates the utility of the fractional β-semigroup as an effective tool for solving fractional evolution equations.

REFERENCES

[1] Atimad, H, Said, M and Lalla, S.C (2020). Fuzzy conformable fractional semigroups of operators. *International Journal of differential Equations*, Article ID 8: 36011, 1 – 6.

[2] Bochner, S (1949). Diffusion equations and stochastic processes, *Proc. Nat. Acad, Sciences*, U.S.A, 35, Pp. 368 – 370.

[3] Balakrishman, A. V (1960). Fractional powers of closed operators and the semigroups generated by them. *Pacific Journal of Mathematics*, 10, Pp 419 – 439.

[4] Capinteri, A and Mainardi, F (1977). Fractals and fractional calculus in continuum mechanisms. Springer-*Verlag Wien GmbH*, No. 378.

[5] Engel, K and Nagel, R (1995). One parameter semigroups for linear evolution equations, Graduate Tests in Mathematics, Springer, New York.

[6] Fernandez, A, Ozarslan, M. A and Baleanu, D (2019). On fractional calculus with general analytic kernels. *Applied Mathematics and Computation*, 354, 248 – 265.

[7] Horani, Al . M (2010) Projection method for solving degenerate first order identification problem. *J. Math. And App*. Vol. 364, Pp 204 -208.

[8] Ndubuisi, R. U, Abubakar, R. B, Udoaka, O. G and Ugbene, I. J (2021). Characterization of a heart-oriented paraletrix. *Journal of Mathematical and computation*, 354, 248 – 265.

[9] Nwajeri, U. K, Omame, A and Onyenegecha, C. P (2021). Analysis of a fractional order model for HPV and CT Co-infection. *Results in Physics*, 28, 104643.

[10] Nwajeri, U. K, Asamoah, K. K. J, Ndubuisi, R. U Omame, A and Jin, Z (2023). A mathematical model of corruption dynamics endowed with fractal-fractional derivative. *Results in Physics*, 52, 106894.

[11] N. D. Cong (2023). Semigroup property of fractional differential operators and its applications. *Discrete and continuous dynamical systems* - B, 28(1), 1-19.

[12] Omame, A, Okonghae, D, Nwajeri, U. K and Onyenegecha, C.P (2021). A fractional-order multi-vaccination model for Covid-19 with non-singular kernel. *Alexandria Engineering Journal*, 61(8), 6089 – 6104.

[13] Omame, A, Nwajeri, U. K, Abbas, M and Onyenegecha, C. P (2022). A fractional order control model for diabetes and Covid – 19 Co-dynamics with Mittag-Leffler. *Alexandria Engineering Journal*, 61(10), 7619 – 7635.

[14] Pazy, A (1983). Semigroups of linear operators and applications to partial differential equations. *Applied Mathematical Sciences*, Springer-Verlag, New York.

[15] Prilepko, I, Orlorsky, G and Vasin, A (2022). Methods for solving inverse problems in Mathematical Physics, *Marcel Delkar Inc*. New York.

[16] Popescu, E (2010). On the fractional Cauchy problem associated with a feller semigroup, *Romanian Academy mathematical reports*, Vol. 12 (62), No. 2, Pp 181- 188.

[17] R. U. Ndubuisi, U. K. Nwajeri, C. P. Onyenegecha, K. M. Patil, O. G. Udoaka and W. I. Osuji (2022). Linear mappings in paraletrix spaces and their application to fractional calculus. *Notes on Number Theory and Discrete Mathematics*, 28(4), 698 – 709.

[18] Noureddine, B, Abderrahmane, B, Abdelkader, B, Svetlin, G.G, Zayd, H and Khaled, Z (2023). Global existence and energy decay for a transmission problem under a boundary fractional derivative type (J). *AIMS Mathematics*, 8(11), 27605 - 27625.

[19] Naimi, A, Brahim, T and Khaled, Z (2023). Existence and stability results of the solution for non-linear fractional differential problem. *DOI: https://doi.org/10.5269/bspm.52043*.

[20] Svetlin, G.G, Khaled, Z, Wien, A.S, Amal, S.M.Y, Aymen, E, Sulima, A.M.Z and Najla, E.B.O (2022). Classical solutions for a BVP for a class impulsive fractional partial differential equation. *https://doi.org/10.1142/s0218348X22402642*.

[21] Serrai,H, Tellab, B and Zennir, Kh (2023). On two-order fractional boundary value problem with generalized Riemann-Liouvile derivative. *Ufa Mathematical Journal*, Vol. 15 (2), 135 – 156.