Original Research Article

FUZZY SOFT PRENORMAL OPERATORS

Abstract

In this paper, we introduce and examine a new class of operators called fuzzy soft prenormal operators (*FSP*). We establish several fundamental properties and characterizations of these operators and present related theorems that describe their structural behavior. Furthermore, we explore the connections between fuzzy soft prenormal operators and other known classes of operators.

Keywords: Fuzzy soft operators, Normal operators, Fuzzy soft Normal operators.

2010 Mathematics Subject Classification: Primary 47A30; Secondary 47A62, 03E72.

1 Introduction

The study of operator theory has undergone extensive generalizations, particularly within the framework of fuzzy and soft set theories, as part of an ongoing effort to model uncertainty and imprecision in mathematical structures. The fusion of fuzzy theory and soft set theory has led to the emergence of the fuzzy soft Hilbert space, which provides a rich setting for extending classical operator-theoretic notions. Within this context, many classes of operators, such as normal, hyponormal, and quasi-hyponormal operators, have been revisited and extended to their fuzzy soft analogs, thereby enriching both theoretical and applied aspects of functional analysis.

The origin of these developments can be traced back to Zadeh's pioneering concept of fuzzy sets Zadeh [1965], introduced in 1965 to model vagueness through a membership function defined in a universal domain, and to Molodtsov's theory of soft sets Molodtsov [1999], proposed in 1999 as a flexible mathematical framework to handle parameterized uncertainties. Subsequent contributions expanded this foundation through constructs such as soft normed spaces Yazar et al. [2014], soft inner product spaces Das and Samanta [2013], and soft Hilbert spaces Yazar et al. [2019]. Later, researchers merged these two paradigms to define the fuzzy soft set Maji et al. [2001], which inspired further studies on fuzzy soft points Neog et al. [2012], fuzzy soft normed spaces Beaula and Priyanga [2015], fuzzy soft inner product spaces and fuzzy soft Hilbert spaces Faried et al. [2020a]. These culminated in the definition of the fuzzy soft linear operator and the fuzzy soft self-adjoint operator Faried et al. [2020b,c], laying the groundwork for subsequent operator-theoretic explorations in the fuzzy soft setting.

Building on this foundation, Dawood and Jabur [2021] introduced the fuzzy soft normal operator, establishing essential properties and exploring its connections with related classes of operators such as self-adjoint and quasi-normal operators. This work provided a key stepping stone for the development of more complex fuzzy soft operator classes. Following this, Assi and Kabban [2024] formulated the concept of fuzzy soft quasi-normal operator $(\widetilde{T*}, \mathcal{N})$, giving several characterizations and analytical results that demonstrated its robustness within fuzzy soft Hilbert spaces. Extending this framework, Mohsen [2023] introduced the fuzzy soft $(\widetilde{k^*} - \mathcal{A})$-quasi-normal operator, presenting important theorems and relationships linking it to fuzzy soft Hermitian and fuzzy soft quasinormal operators, and providing a deeper understanding of the interplay between structure and generalization in fuzzy soft operator theory.

Further, Kadhim and Shubber [2025] investigated the fuzzy soft n-normal operator, analyzing its algebraic behavior and identifying conditions under which the addition and multiplication of such operators commute. Their results highlighted the structural flexibility of fuzzy soft n-normal operators and their potential applications in mathematical physics, medical science, and engineering. In a related advancement, Eidia and Mohsen [2022] introduced and examined the fuzzy soft $(n - \widetilde{N})$ quasi-normal operator, providing new theorems, operational conditions, and analytical properties that enrich the understanding of fuzzy soft quasi-normality. Together, these studies mark significant progress in generalizing the core concepts of normal and quasi-normal operators within the fuzzy soft environment.

Recent works by Mohsen and Mousa [2022] and Mohsen [2025] further strengthened this theoretical landscape through the development of fuzzy soft $\mathcal{M}$-hyponormal and fuzzy soft κ quasi-hyponormal operators, each contributing crucial analytic frameworks that link classical operator theory to its fuzzy soft counterparts. Their findings underscored the unifying potential of fuzzy soft Hilbert spaces in capturing a wide range of operator behaviors influenced by fuzziness and parametric uncertainty.

Motivated by these advances, the present work introduces a new class of operators called fuzzy soft prenormal operators. This class aims to generalize the concept of normality within the fuzzy soft framework by incorporating preconditions that balance between normal and hyponormal behaviors. The fuzzy soft prenormal operator preserves fundamental spectral and analytical characteristics while offering enhanced flexibility in modeling the interplay between fuzziness, softness, and operator behavior. Consequently, this study not only extends the frontier of fuzzy soft operator theory but also establishes a foundation for further exploration of spectral properties, equivalence relations, and stability phenomena in fuzzy soft Hilbert spaces.

1.1 Basic Concepts

[Zadeh [1965]] Let $\widehat{S}$ be a fuzzy set over the universe set $\mathcal{X}$. It is characterized by a membership function

$$\mu_{\widehat{S}} : \mathcal{X} \longrightarrow \mathbb{T},$$

where $\mathbb{T} = [0, 1]$. The fuzzy set $\widehat{S}$ can be represented by an ordered pair

$$\widehat{S} = \{(x, \mu_{\widehat{S}}(x)) \mid x \in \mathcal{X},\, \mu_{\widehat{S}}(x) \in \mathbb{T}\},$$

or equivalently,

$$\widehat{S} = \left\{ \frac{\mu_{\widehat{S}}(x)}{x} : x \in \mathcal{X} \right\}.$$

Here, $\mu_{\widehat{S}}(x)$ is said to be the degree of membership of x in $\widehat{S}$. Moreover,

$$\mathfrak{T}_{\mathcal{X}} = \{\widehat{S} : \widehat{S} \text{ is a function from } \mathcal{X} \text{ into } \mathfrak{T}\}.$$

[Molodtsov [1999]] Let $\mathcal{X}$ be a universe set, and $\mathcal{Q}$ be a set of parameters, $\mathcal{P}(\mathcal{X})$ the power set of $\mathcal{X}$ and $\mathcal{S} \subseteq \mathcal{Q}$. Suppose that $\mathcal{G}$ is a mapping given by

$$\mathcal{G} : \mathcal{S} \to \mathcal{P}(\mathcal{X}),$$

where

$$\mathcal{G}_{\mathcal{S}} = \{\mathcal{G}(q) \in \mathcal{P}(\mathcal{X}) : q \in \mathcal{S}\}.$$

The pair $(\mathcal{G}, \mathcal{S})$ or $\mathcal{G}_{\mathcal{S}}$ is called a **soft set** over $\mathcal{X}$ with respect to $\mathcal{S}$.

[Maji et al. [2001]] The soft set $(\mathcal{G}, S)$ is called a **fuzzy soft set** ($\mathcal{FS}$-set) over a universe set $\mathcal{X}$ whenever $\mathcal{G}$ is a mapping

$$\mathcal{G} : S \longrightarrow \mathbb{T}^{\mathcal{X}},$$

and

$$\{\mathcal{G}(q) \in \mathbb{T}^{\mathcal{X}} : q \in S\}.$$

The family of all $\mathcal{FS}$–sets is denoted by $\mathcal{FSS}(\tilde{\mathcal{X}})$.

[Neog et al. [2012]] The $\mathcal{FS}$–set $(\mathcal{G}, S) \in \mathcal{FSS}(\tilde{\mathcal{X}})$ is called a **fuzzy soft point** over $\mathcal{X}$, denoted by $((\tilde{x}_{\mu_{\mathcal{G}(q)}}, S))$ or $\tilde{x}_{\mu_{\mathcal{G}(q)}}$, if $q \in S$ and $x \in \mathcal{X}$, where

$$\mu_{\mathcal{G}(q)}(x) = \begin{cases} \lambda, & \text{if } x = x_0 \in \mathcal{X} \text{ and } q = q_0 \in S, \\ 0, & \text{if } x \in \mathcal{X} - \{x_0\} \text{ or } q \in S - \{q_0\}, \end{cases}$$

with $\lambda \in (0, 1]$.

[Neog et al. [2012]] $\mathcal{C}(S)$ denotes the family of all $\mathcal{FS}$–complex numbers, and $\mathcal{R}(S)$ denotes the family of all $\mathcal{FS}$–real numbers.

[Beaula and Priyanga [2015]] Let $\tilde{\mathcal{X}}$ be a $\mathcal{FS}$–vector space. A mapping $\|\tilde{\cdot}\| : \tilde{\mathcal{X}} \to \mathcal{R}(S)$ is called a **fuzzy soft norm** on $\tilde{\mathcal{X}}$ (denoted $\mathcal{FSN}$) if it satisfies:

1. $\|\tilde{x}_{\mu_{\mathcal{G}(q)}}\| \geq \tilde{O}, \forall \tilde{x}_{\mu_{\mathcal{G}(q)}} \in \tilde{\mathcal{X}}$, and $\|\tilde{x}_{\mu_{\mathcal{G}(q)}}\| = \tilde{O} \iff \tilde{x}_{\mu_{\mathcal{G}(q)}} = \tilde{\theta}$.
2. $\|\tilde{r} \cdot \tilde{x}_{\mu_{\mathcal{G}(q)}}\| = |\tilde{r}| \, \|\tilde{x}_{\mu_{\mathcal{G}(q)}}\|$, $\forall \tilde{r} \in \mathcal{C}(S)$.
3. $\|\tilde{x}_{\mu_1 \mathcal{G}(q_1)} + \tilde{y}_{\mu_2 \mathcal{G}(q_2)}\| \leq \|\tilde{x}_{\mu_1 \mathcal{G}(q_1)}\| + \|\tilde{y}_{\mu_2 \mathcal{G}(q_2)}\|, \ \forall \tilde{x}_{\mu_1 \mathcal{G}(q_1)}, \tilde{y}_{\mu_2 \mathcal{G}(q_2)} \in \tilde{\mathcal{X}}$.

The $\mathcal{FS}$–vector space with fuzzy soft norm $\|\tilde{\cdot}\|$ is called a **fuzzy soft normed vector space** ($\mathcal{FSN}$–space) and is denoted by $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$.

[Faried et al. [2020a]] Let $\tilde{\mathcal{X}}$ be a $\mathcal{FSV}$–space. A mapping

$$\langle \tilde{\cdot}, \tilde{\cdot} \rangle : \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \to (\mathcal{C}(S) \text{ or } \mathcal{R}(S))$$

is called a **fuzzy soft inner product** on $\tilde{\mathcal{X}}$ (denoted $\mathcal{FST}$) if it satisfies:

1. $\langle \tilde{x}_{\mu_1 \mathcal{G}(q_1)}, \tilde{x}_{\mu_1 \mathcal{G}(q_1)} \rangle \geq \tilde{O}$, and equals $\tilde{O}$ iff $\tilde{x}_{\mu_1 \mathcal{G}(q_1)} = \tilde{\theta}$.
2. $\langle \tilde{x}_{\mu_1 \mathcal{G}(q_1)}, \tilde{y}_{\mu_2 \mathcal{G}(q_2)} \rangle = \overline{\langle \tilde{y}_{\mu_2 \mathcal{G}(q_2)}, \tilde{x}_{\mu_1 \mathcal{G}(q_1)} \rangle}$.
3. $\langle \tilde{\alpha} \tilde{x}_{\mu_1 \mathcal{G}(q_1)}, \tilde{y}_{\mu_2 \mathcal{G}(q_2)} \rangle = \tilde{\alpha} \langle \tilde{x}_{\mu_1 \mathcal{G}(q_1)}, \tilde{y}_{\mu_2 \mathcal{G}(q_2)} \rangle$, for all $\tilde{\alpha} \in \mathcal{C}(S)$.
4. $\langle \tilde{x}_{\mu_1 \mathcal{G}(q_1)} + \tilde{y}_{\mu_2 \mathcal{G}(q_2)}, \tilde{z}_{\mu_3 \mathcal{G}(q_3)} \rangle = \langle \tilde{x}_{\mu_1 \mathcal{G}(q_1)}, \tilde{z}_{\mu_3 \mathcal{G}(q_3)} \rangle + \langle \tilde{y}_{\mu_2 \mathcal{G}(q_2)}, \tilde{z}_{\mu_3 \mathcal{G}(q_3)} \rangle$.

The $\mathcal{FS}$–vector space $\tilde{\mathcal{X}}$ with $\mathcal{FST}$ is called a **fuzzy soft inner product space** ($\mathcal{FST}$–space), denoted by $(\tilde{\mathcal{X}}, \langle \tilde{\cdot}, \tilde{\cdot} \rangle)$.

[Khameneh et al. [2013]] A sequence of $\mathcal{FS}$–vectors $\{\tilde{x}_{\mu_n \mathcal{G}(q_n)}\}_n$ in the $\mathcal{FSN}$–space $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$ is called **fuzzy soft convergent** and converges to $\tilde{x}_{\mu_0 \mathcal{G}(q_0)}$ if

$$\lim_{n \to \infty} \|\tilde{x}_{\mu_n \mathcal{G}(q_n)} - \tilde{x}_{\mu_0 \mathcal{G}(q_0)}\| = \tilde{O},$$

i.e., $\forall \tilde{\varepsilon} > \tilde{O}, \exists n_0 \in \mathbb{N}$ such that

$$\|\tilde{x}_{\mu_n \mathcal{G}(q_n)} - \tilde{x}_{\mu_0 \mathcal{G}(q_0)}\| < \tilde{\varepsilon}, \ \forall n \geq n_0.$$

It is denoted by

$$\lim_{n\to\infty} \tilde{x}_{\mu_n\mathcal{G}(q_n)} = \tilde{x}_{\mu_0\mathcal{G}(q_0)} \quad \text{or} \quad \tilde{x}_{\mu_n\mathcal{G}(q_n)} \to \tilde{x}_{\mu_0\mathcal{G}(q_0)} \text{ as } n \to \infty.$$

[Khameneh et al. [2013]] A sequence of $\mathcal{FS}$–vectors $\{\tilde{x}_{\mu_n\mathcal{G}(q_n)}\}_n$ in the $\mathcal{FSN}$–space $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$ is called a **fuzzy soft Cauchy sequence** if for every $\tilde{\varepsilon} > \tilde{O}$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\tilde{x}_{\mu_n\mathcal{G}(q_n)} - \tilde{x}_{\mu_m\mathcal{G}(q_m)}\| < \tilde{\varepsilon}, \quad \forall n, m \geq n_0,\ n > m.$$

That is,

$$\|\tilde{x}_{\mu_n\mathcal{G}(q_n)} - \tilde{x}_{\mu_m\mathcal{G}(q_m)}\| \to \tilde{O} \quad \text{as } n, m \to \infty.$$

[Khameneh et al. [2013]] The $\mathcal{FSN}$–space $(\tilde{\mathcal{X}}, \|\tilde{\cdot}\|)$ is called **fuzzy soft complete** (or **$\mathcal{FS}$–complete**) if every $\mathcal{FS}$–Cauchy sequence is a $\mathcal{FS}$–convergent sequence in it.

[Faried et al. [2020a]] The $\mathcal{FS}$–space $(\tilde{\mathcal{X}}, \langle\tilde{\cdot}, \tilde{\cdot}\rangle)$ is called a **fuzzy soft Hilbert space** ($\mathcal{FSH}$–space) if it is $\mathcal{FS}$–complete in the induced $\mathcal{FSN}$ given by

$$\|\tilde{x}_{\mu_{\mathcal{G}(q)}}\| = \sqrt{\langle \tilde{x}_{\mu_{\mathcal{G}(q)}}, \tilde{x}_{\mu_{\mathcal{G}(q)}} \rangle}.$$

It is denoted by $(\tilde{\mathcal{H}}, \langle\tilde{\cdot}, \tilde{\cdot}\rangle)$.

[Faried et al. [2020b]] Let $\tilde{\mathcal{H}}$ be a $\mathcal{FSH}$–space and $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be a fuzzy soft operator. Then $\tilde{\mathcal{T}}$ is called a **fuzzy soft linear operator** ($\mathcal{FSL}$–operator) if:

1. $\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)} + \tilde{y}_{\mu_2\mathcal{G}(q_2)}) = \tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) + \tilde{\mathcal{T}}(\tilde{y}_{\mu_2\mathcal{G}(q_2)})$, for all $\tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \in \tilde{\mathcal{H}}$.
2. $\tilde{\mathcal{T}}(\tilde{\beta}\tilde{x}_{\mu_1\mathcal{G}(q_1)}) = \tilde{\beta}\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)})$, for all $\tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}}$ and $\tilde{\beta} \in \mathcal{C}(S)$.

That is,

$$\tilde{\mathcal{T}}(\tilde{\alpha}\tilde{x}_{\mu_1\mathcal{G}(q_1)} + \tilde{\beta}\tilde{y}_{\mu_2\mathcal{G}(q_2)}) = \tilde{\alpha}\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) + \tilde{\beta}\tilde{\mathcal{T}}(\tilde{y}_{\mu_2\mathcal{G}(q_2)}),$$

for all $\tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \in \tilde{\mathcal{H}}$ and fuzzy soft scalars $\tilde{\alpha}, \tilde{\beta}$.

[Faried et al. [2020b]] Let $\tilde{\mathcal{H}}$ be a $\mathcal{FSH}$–space and $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be a $\mathcal{FS}$–operator. Then $\tilde{\mathcal{T}}$ is called a **fuzzy soft bounded operator** ($\mathcal{FSB}$–operator) if there exists $\tilde{m} \in \mathcal{R}(S)$ such that

$$\|\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)})\| \leq \tilde{m}\|\tilde{x}_{\mu_1\mathcal{G}(q_1)}\|, \quad \forall \tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}}.$$

The collection of all $\mathcal{FS}$–linear and bounded operators is denoted by $\tilde{\mathcal{B}}(\tilde{\mathcal{H}})$.

[Faried et al. [2020b]] Let $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}$ be two $\mathcal{FSH}$–spaces. Then:

1. The **range** of the $\mathcal{FS}$–operator $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{K}}$ is denoted by $\mathrm{Ran}(\tilde{\mathcal{T}})$ and defined as

$$\mathrm{Ran}(\tilde{\mathcal{T}}) = \{\tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) \in \tilde{\mathcal{K}} : \tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}}\}.$$

2. The **null space** (kernel) of $\tilde{\mathcal{T}}$ is denoted by $\mathrm{Ker}(\tilde{\mathcal{T}})$ and defined as

$$\mathrm{Ker}(\tilde{\mathcal{T}}) = \{\tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}} \ : \ \tilde{\mathcal{T}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) = \tilde{0}\}.$$

Example 1.1 (Faried et al. [2020b]). *The $\mathcal{FS}$–operator $\tilde{\mathcal{I}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ defined by*

$$\tilde{\mathcal{I}}(\tilde{x}_{\mu_1\mathcal{G}(q_1)}) = \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \quad \forall \tilde{x}_{\mu_1\mathcal{G}(q_1)} \in \tilde{\mathcal{H}},$$

*is called the **fuzzy soft identity operator**.*

[Faried et al. [2020b]] Let $\tilde{\mathcal{H}}$ be a $\mathcal{FSH}$–space and $\tilde{\mathcal{T}} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be a $\mathcal{FSB}$–operator. Then the **fuzzy soft adjoint operator** $\tilde{\mathcal{T}}^*$ is defined by

$$\langle \tilde{\mathcal{T}}\tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \rangle = \langle \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{\mathcal{T}}^*\tilde{y}_{\mu_2\mathcal{G}(q_2)} \rangle, \quad \forall \tilde{x}_{\mu_1\mathcal{G}(q_1)}, \tilde{y}_{\mu_2\mathcal{G}(q_2)} \in \tilde{\mathcal{H}}.$$

[Faried et al. [2020c]] The $\mathcal{FS}$–operator $\widetilde{\mathcal{T}}$ of the $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ is called a **fuzzy soft self-adjoint operator** ($\mathcal{FS}$–self-adjoint operator) if
$$\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}^*.$$

[Dawood and Jabur [2021]] Let $\widetilde{\mathcal{T}}$ be an $\mathcal{FS}$–operator on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$. Then $\widetilde{\mathcal{T}}$ is called a **fuzzy soft normal operator** ($\mathcal{FSN}$–operator) if
$$\widetilde{\mathcal{T}}\,\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\,\widetilde{\mathcal{T}}.$$

[Dawood and Jabur [2021]] A $\mathcal{FS}$–operator $\widetilde{\mathcal{U}}$ on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ is called a **fuzzy soft unitary operator** ($\mathcal{FSU}$–operator) if
$$\widetilde{\mathcal{U}}\widetilde{\mathcal{U}}^* = \widetilde{\mathcal{U}}^*\widetilde{\mathcal{U}} = \widetilde{\mathcal{I}}.$$

2 Main results

Let $\widetilde{\mathcal{T}}$ be an $\mathcal{FS}$–operator on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$. Then $\widetilde{\mathcal{T}}$ is called a **fuzzy soft prenormal operator** ($\mathcal{FSP}$–operator) if
$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2.$$

Let $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{S}}$ be $\mathcal{FSP}$–operators on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ satisfying
$$\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}^* = \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}} \quad \text{and} \quad \widetilde{\mathcal{S}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}}.$$
Then both $\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}$ are $\mathcal{FSP}$–operators.

Proof. Since $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{S}}$ are $\mathcal{FSP}$–operators, we have
$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2, \qquad (\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)^2 = (\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}})^2. \tag{1}$$
Also, by assumption,
$$\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}^* = \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}}, \qquad \widetilde{\mathcal{S}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}}. \tag{2}$$
(i) To show that $\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}$ is an $\mathcal{FSP}$–operator.

We first compute
$$(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* + \widetilde{\mathcal{T}}\widetilde{\mathcal{S}}^* + \widetilde{\mathcal{S}}\widetilde{\mathcal{T}}^* + \widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*.$$
Using (2), this becomes
$$(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}} + \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}} + \widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*. \tag{3}$$
Similarly,
$$(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}) = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}} + \widetilde{\mathcal{T}}^*\widetilde{\mathcal{S}} + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}}. \tag{4}$$
Now, squaring both sides of (3) and (4), and applying (1) together with the commutation relations in (2), we obtain
$$\left[(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^*\right]^2 = \left[(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})\right]^2.$$
Hence, $\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}}$ is an $\mathcal{FSP}$–operator.

(ii) To show that $\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}$ is an $\mathcal{FSP}$–operator. We have
$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}(\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)\widetilde{\mathcal{T}}^*.$$
By (1),
$$(\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)^2 = (\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}})^2,$$
and so
$$\left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*\right]^2 = \widetilde{\mathcal{T}}(\widetilde{\mathcal{S}}\widetilde{\mathcal{S}}^*)^2\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}(\widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}})^2\widetilde{\mathcal{T}}^*.$$

Moreover,

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}) = \widetilde{\mathcal{S}}^*\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}^*(\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})\widetilde{\mathcal{S}},$$

and hence

$$\left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})\right]^2 = \widetilde{\mathcal{S}}^*(\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2\widetilde{\mathcal{S}}.$$

Since $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2$, we finally get

$$\left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*\right]^2 = \left[(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})^*(\widetilde{\mathcal{T}}\widetilde{\mathcal{S}})\right]^2.$$

Therefore, $\widetilde{\mathcal{T}}\widetilde{\mathcal{S}}$ is also an $\mathcal{FSP}$–operator.

□

An $\mathcal{FS}$–operator $\widetilde{\mathcal{T}}$ on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ is an $\mathcal{FSN}$–operator if and only if

$$\|\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \|\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\| \quad \text{for every } \widetilde{x}_{\mu_1\mathcal{G}(e_1)} \in \widetilde{\mathcal{H}}.$$

Proof. (⇒) Suppose that $\widetilde{\mathcal{T}}$ is an $\mathcal{FSN}$–operator, that is, $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$. Then, for any $\widetilde{x}_{\mu_1\mathcal{G}(e_1)} \in \widetilde{\mathcal{H}}$, we have

$$\|\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\|^2 = \langle \widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle = \langle \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle.$$

Since $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$, it follows that

$$\langle \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle = \langle \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}, \widetilde{x}_{\mu_1\mathcal{G}(e_1)}\rangle = \|\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\|^2.$$

Hence,

$$\|\widetilde{\mathcal{T}}\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\| = \|\widetilde{\mathcal{T}}^*\widetilde{x}_{\mu_1\mathcal{G}(e_1)}\|.$$

(⇐) Conversely, suppose that

$$\|\widetilde{\mathcal{T}}\widetilde{x}\| = \|\widetilde{\mathcal{T}}^*\widetilde{x}\| \quad \text{for every } \widetilde{x} \in \widetilde{\mathcal{H}}.$$

Then

$$\langle (\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})\widetilde{x}, \widetilde{x}\rangle = 0, \quad \forall\, \widetilde{x} \in \widetilde{\mathcal{H}}.$$

Since $\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* - \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}}$ is self–adjoint, it must be the zero operator. Therefore,

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}},$$

and thus $\widetilde{\mathcal{T}}$ is an $\mathcal{FSN}$–operator. □

Let $\widetilde{\mathcal{T}}$ be an $\mathcal{FSP}$–operator on an $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ and let $\widetilde{\alpha} \in \mathcal{C}(A)$. Then the operator $\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I}$ is also an $\mathcal{FSP}$–operator.

Proof. Since $\widetilde{\mathcal{T}}$ is an $\mathcal{FSP}$–operator, we have

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2.$$

Consider $\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I}$:

$$\left((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*\right)^2 = \left((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})(\widetilde{\mathcal{T}}^* - \overline{\widetilde{\alpha}}\widetilde{I})\right)^2.$$

Expanding, we get

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* - \widetilde{\alpha}\widetilde{\mathcal{T}}^* - \overline{\widetilde{\alpha}}\widetilde{\mathcal{T}} + |\widetilde{\alpha}|^2\widetilde{I})^2.$$

Similarly,

$$\left((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})\right)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{\mathcal{T}}^* - \overline{\widetilde{\alpha}}\widetilde{\mathcal{T}} + |\widetilde{\alpha}|^2\widetilde{I})^2.$$

Since $(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2$ and scalar multiples commute with operators, it follows that

$$\left((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*\right)^2 = \left((\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})^*(\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I})\right)^2.$$

Hence, $\widetilde{\mathcal{T}} - \widetilde{\alpha}\widetilde{I}$ is also an $\mathcal{FSP}$–operator. □

The class of $\mathcal{FSP}$–operators is closed in the strong operator topology.

Proof. Let $\{\widetilde{\mathcal{T}}_n\}$ be a sequence of $\mathcal{FSP}$–operators on $\mathcal{FSH}$–space $\widetilde{\mathcal{H}}$ that converges strongly to $\widetilde{\mathcal{T}} \in \widetilde{\mathcal{B}}(\widetilde{\mathcal{H}})$. That is,

$$\widetilde{\mathcal{T}}_n\widetilde{x} - \widetilde{\mathcal{T}}\widetilde{x} \to 0 \quad \text{as } n \to \infty \text{ for each } \widetilde{x} \in \widetilde{\mathcal{H}}.$$

Since each $\widetilde{\mathcal{T}}_n$ is $\mathcal{FSP}$, we have

$$(\widetilde{\mathcal{T}}_n\widetilde{\mathcal{T}}_n^*)^2 = (\widetilde{\mathcal{T}}_n^*\widetilde{\mathcal{T}}_n)^2.$$

By the continuity of operator multiplication in the strong operator topology,

$$(\widetilde{\mathcal{T}}_n\widetilde{\mathcal{T}}_n^*)^2 \to (\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 \quad \text{and} \quad (\widetilde{\mathcal{T}}_n^*\widetilde{\mathcal{T}}_n)^2 \to (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2 \quad \text{strongly as } n \to \infty.$$

Therefore, taking the limit as $n \to \infty$ in the equality

$$(\widetilde{\mathcal{T}}_n\widetilde{\mathcal{T}}_n^*)^2 = (\widetilde{\mathcal{T}}_n^*\widetilde{\mathcal{T}}_n)^2$$

gives

$$(\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*)^2 = (\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})^2.$$

Hence, $\widetilde{\mathcal{T}}$ is an $\mathcal{FSP}$–operator. This proves that the class of $\mathcal{FSP}$–operators is closed in the strong operator topology. □

If $\widetilde{\mathcal{T}} \in (\mathcal{FSP})$ is a fuzzy soft isometry, then $\widetilde{\mathcal{T}}$ is a fuzzy soft unitary operator.

Proof. Since $\widetilde{\mathcal{T}} \in (\mathcal{FSP})$, by definition of a fuzzy soft prenormal operator, there exists a fuzzy soft unitary operator $\widetilde{\mathcal{U}}$ on the fuzzy soft Hilbert space $\widetilde{\mathcal{H}}$ such that

$$(\widetilde{\mathcal{T}}^{*2})(\widetilde{\mathcal{T}}^2) = \widetilde{\mathcal{U}}(\widetilde{\mathcal{T}}^{*2}\widetilde{\mathcal{T}}^2)\widetilde{\mathcal{U}}^* \quad \text{and} \quad (\widetilde{\mathcal{T}}^2)(\widetilde{\mathcal{T}}^{*2}) = \widetilde{\mathcal{U}}(\widetilde{\mathcal{T}}^2\widetilde{\mathcal{T}}^{*2})\widetilde{\mathcal{U}}^*.$$

Since $\widetilde{\mathcal{T}}$ is a fuzzy soft isometry, we have

$$\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} = \widetilde{\mathcal{I}},$$

where $\widetilde{\mathcal{I}}$ denotes the fuzzy soft identity operator.

Then,

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}(\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}})\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{I}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*.$$

By the $\mathcal{FSP}$ property, it follows that

$$\widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{I}}.$$

Hence,

$$\widetilde{\mathcal{T}}^*\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^* = \widetilde{\mathcal{I}}.$$

Therefore, $\widetilde{\mathcal{T}}$ is a fuzzy soft unitary operator.

□

3 CONCLUSIONS

From the foregoing results, it is evident that the class of fuzzy soft prenormal operators $(\mathcal{FSP})$ forms a stable extension of the fuzzy soft normal operators. In particular, every fuzzy soft isometry that satisfies the $\mathcal{FSP}$ condition is necessarily a fuzzy soft unitary operator. This establishes that the $\mathcal{FSP}$ framework preserves the essential spectral and structural properties associated with normality in the fuzzy soft Hilbert space setting.

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