Sensitivity Analysis of an Age-Structured Model of HIV/AIDS Transmission Dynamics in Nigeria through Mathematical Modelling

Abstract

This study focuses on the estimation of parameters and sensitivity analysis of an age-structured model of HIV/AIDS transmission dynamics in Nigeria. The two age groups considered are people between 15 to 60 years old (mature susceptibles) and those less than 15 years old (immature susceptibles). Python software was used in the analysis. The variables in the model includes S_1 (mature susceptible), S_2 (immature susceptibles), E (exposed individuals), I_1 (aware infectives), I_2 (unaware infectives), T (people receiving ART treatment) and A (people in AIDS class). Sensitivity analysis reveals that the disease transmission rate (β) is the most influential parameter, positively affecting the basic reproduction number, while natural and AIDS-related death rates exert the strongest negative influences. Numerical simulations show a significant decline in susceptible populations, indicating high infection risk. The population of unaware infectives remains higher than that of aware infectives, highlighting a critical hidden reservoir for ongoing transmission. These findings underscore the importance of reducing transmission rates and increasing case detection through awareness campaigns. Hence, urgent intervention such as prevention and control measures are critically needed to mitigate the spread of the infection.

Keywords: HIV/AIDS, Age-structured model, sensitivity analysis.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

Human immunodeficiency virus (HIV) is an infection that attacks the body's immune system, specifically the white blood cells called CD4 cells. HIV destroys these CD4 cells, weakening a person's immunity against infections such as tuberculosis (TB) and cancers [1]. In (2010), Bennett et al. stated that

the Human immunodeficiency virus (HIV) is the major virus that causes acquired immune deficiency syndrome (AIDS) and weakens the immune system by destroying the T-cells until they are unable to fight off even minor illnesses [1, 2]. Scientists believe that HIV originally came from a virus peculiar to chimpanzees in West Africa during the 1930s, and originally transmitted to humans through the transfer of blood during hunting. Over the decades, the virus spread through Africa, and to other parts of the world. However, it was not until the early 1980s, when rare types of pneumonia, cancer, and other illnesses were being reported to doctors that the world became aware of HIV and AIDS [3].

Globally, HIV/AIDS was first detected in the United States of America in 1981. The World Health Organization (WHO) Trusted Source estimates there were nearly 39 million people living with HIV worldwide by the end of 2022 with the United States alone having 1.2 million people living with the virus. Many countries have been affected by the transmission of HIV/AIDS such as Nigeria (2.45 million), South Africa (9.2 million), Tanzania (2.55 million) and Mozambique (2.46 million) to mention but a few. The first two AIDS cases in Nigeria were diagnosed in 1985 and reported in 1986 in Lagos one of which was a young female sex worker aged 13 years from one of the West African countries [4].

The Nigerian Government created the National AIDS Control and Prevention Program within the Federal Ministry of Health in 1987, a Presidential AIDS Commission composed of ministers from all sectors in 1999 and the National Action Committee on AIDS based in the Office of the Presidency in 2000. The HIV/AIDS Emergency Action Plan developed in 2001, and revised in 2004, serves as the national action framework [5].

The following are the various stages of HIV/AIDS infection:

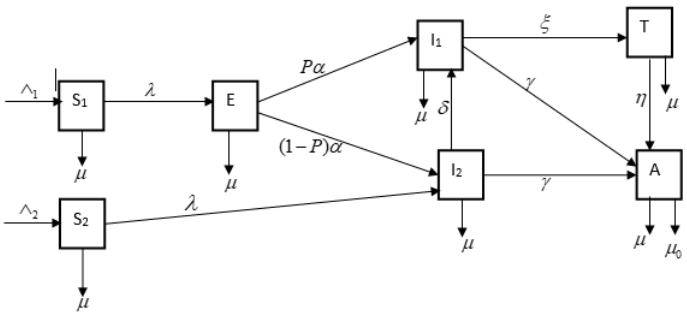
1. **Acute HIV Infection:** Acute HIV infection is the earliest stage of HIV infection, and it generally develops within 2 to 4 weeks after infection with HIV, some people in this stage have flu-like symptoms, such as fever, headache, and rash. At this stage of infection, virus multiplies rapidly and spreads throughout the body thereby attacking and destroying the body immune system [6].
2. **Chronic HIV Infection:** This is the second stage of HIV infection, during this stage, the virus continues to multiply in the body but at very low rates. People with chronic HIV infection may not have any HIV-related symptoms [6].
3. **AIDS:** AIDS is the final stage, regarded as the most severe stage of HIV infection. At this stage the virus has severely damaged the immune system, the body cannot fight off opportunistic infections [6].

Mathematical modeling is a commonly used instrument to study the propagation of infectious diseases. The factor that affects the transmission of infectious diseases can be identified by mathematical models. Many mathematical models have been developed to study the transmission dynamics of HIV. Suryanto and Darti in [7] formulated an ODE model to analyze the influence of awareness, education, screening, and treatment on the spread of AIDS and the result of their work proved that education, screening, and treatment can reduce the number of infected individuals. Tripathi (2005) proposed a model for HIV/AIDS with awareness and screening as control strategies, where susceptible individuals were mainly classified by gender, and infected individuals were classified according to the level of awareness [8]. Makinde et al (2019) analyzed the effects of screening and treatment on the spread of HIV/AIDS infection in a population, and incorporated the awareness and treatment for HIV infected individuals into the model [9]. Another mathematical model of HIV/AIDS with screening and treatment as controls formulated by Nyabadza et al. in [10] showed that the number of aware infected individuals has a great impact on the spread of AIDS. Kifle and Wondinmu (2020) developed a non-linear deterministic dynamical system to investigate HIV/AIDS dynamics in Ethiopia. They incorporated age structure and considered different transmission modes [20]. Wang and Zing (2019) proposed an age –structured HIV infection model with cell -to -cell transmission. In their work, they highlighted the importance of age structure in HIV transmission dynamics [21].

Ling Xue et al, 2022 studied the transmission dynamics and optimal control strategies for HIV infection in China. They only considered the spread of HIV among people between 15 and 60 years old and fitted the number of the infected individuals to their model. They applied the annual number of new HIV cases in China from 2002 to 2019 by the Markov Chain Monte Carlo (MCMC) method to estimate the unknown parameters in their model. Also, they used the Partial Rank Correlation Coefficients (PRCC) to study the global sensitivity of the parameters of the model. The parameters they used in their model whose sensitivity they analyzed included ε (reduced infectivity of latent individuals), ρ (fraction of aware individuals), ξ (the rate at which aware infected individuals received treatment), γ (the rate at which aware and unaware infected individuals progresses to Aids), δ (the rate at which unaware individuals become aware) and β (the effective contact rate for HIV transmission). Their goal was to identify the most important parameter that affects HIV transmission. The analytical results showed the existence and stability of the disease-free equilibrium and endemic equilibrium. An optimal control model was constructed to evaluate the impacts of control measures. The simulation results showed that the optimal control strategy they proposed can eradicate AIDS by 2030. The cost-effectiveness analysis showed that the cost of the control strategy that combines screening for latent individuals and enhancing education for unaware infected individuals is the lowest [11].

2 Mathematical modelling

In this paper, we extend the model of Ling Xue et al, 2023 to an age-structured model where not only people between 15 and 60 years old (mature susceptible) are considered but also people less than 15 years old (immature susceptible) are taken into consideration. Our aim is to determine the sensitivity values of parameters of our new model.



$$\frac{dS_1}{dt} = \Lambda_1 - \mu S_1 - \lambda S_1 \tag{2.1}$$

$$\frac{dS_2}{dt} = \Lambda_2 - \mu S_2 \tag{2.2}$$

$$\frac{dE}{dt} = \lambda S_1 - (\alpha + \mu)E \tag{2.3}$$

$$\frac{dI_1}{dt} = \rho\alpha E - (\xi + \gamma + \mu)I_1 \tag{2.4}$$

$$\frac{dI_2}{dt} = (1 - \rho)\alpha E - (\delta + \gamma + \mu)I_2 \tag{2.5}$$

$$\frac{dT}{dt} = \xi I_1 - (\eta + \mu)T \tag{2.6}$$

$$\frac{dA}{dt} = \gamma(I_1 + I_2) + \eta T - (\mu + \mu_0)A \tag{2.7}$$

Let λ representing the force of infection for individuals who are infected by the HIV/AIDS be as

follows:

$$\lambda = \beta(\varepsilon E + I_2)$$

Assumptions of the Model

1. Recruitment into the S_1 class is only by immigration.
2. Recruitment into the S_2 class is only by birth.
3. All aware infectives go for treatment.
4. All susceptible recruited by birth are unaware infectives.
5. Unaware infectives are likely to infect others.
6. Immature susceptible move freely to the unaware infectives
7. The model variables and parameters are all positive.

Table 1: State variables and their descriptions

State Variables	Descriptions
S_1	Mature susceptible humans
S_2	Immature susceptible humans
E	Exposed
I_1	Aware infectives
I_2	Unaware infectives
T	Treatment
A	AIDS

3 Mathematical Analysis of derived model

3.1 Analysis of the Disease-Free Equilibrium (DFE) and basic Reproduction number R_0

At DFE, there is no disease:

$$E = I_1 = I_2 = T = A = 0.$$

Also $\lambda = 0$ since $E = I_2 = 0$. From $\frac{dS_1}{dt} = 0$ and $\lambda = 0$:

$$\Lambda_1 - \mu S_1 = 0 \Rightarrow S_1 = \frac{\Lambda_1}{\mu}$$

From $\frac{dS_2}{dt} = 0$:

$$\Lambda_2 - \mu S_2 = 0 \Rightarrow S_2 = \frac{\Lambda_2}{\mu}$$

Other equations for E, I_1, I_2, T, A are automatically satisfied with $E = I_1 = I_2 = T = A = 0$. Thus, the DFE is:

$$(S_1^0, S_2^0, E^0, I_1^0, I_2^0, T^0, A^0) = \left(\frac{\Lambda_1}{\mu}, \frac{\Lambda_2}{\mu}, 0, 0, 0, 0, 0\right)$$

The infected compartments are those that harbor the pathogen and can transmit it. From the system, these are E, I_1, and I_2. Note that T (treated) and A (AIDS) are not explicitly included in the force

Table 2: State variables and their descriptions

Model Parameter	**Meaning**	Value	Sources
Λ_1	Recruitment by birth rate	0.41999	
Λ_2	Recruitment by immigration	0.58000	
ϵ	Reduced Infectivity	0.769231	
β	Contact rate	1.7649e^{-6}	
ρ	Fraction of E that progress to I_1 Compartment	0.45584	
α	Rate at which E progress to I_1 or I_2 compartment	0.333	
ξ	Rate at which I_1 move to T Class	0.34877	
δ	Rate at which I_2 progresses to I_1 Class	0.015697	
γ	Rate at which I_1 and I_2 move to A compartment	0.056044	
η	Rate at which T move to A compartment	0.1	

of infection λ, so they are not considered part of the transmitting infected compartments for this calculation. We write the equations for the rates of change of the infected compartments, but we separate the terms that describe new infections (F) from the terms that describe transitions between compartments and removals (V) [12]. The subsystem is:

$$\frac{dE}{dt} = \lambda S_1 - [\alpha + \mu]E \tag{3.1}$$

$$\frac{dI_1}{dt} = \rho\alpha E - (\xi + \gamma + \mu)I_1 \tag{3.2}$$

$$\frac{dI_2}{dt} = (1 - \rho)\alpha E - (\delta + \gamma + \mu)I_2 \tag{3.3}$$

where $\lambda = \beta(\varepsilon E + I_2)$. In In DFE, we have $S_1^0 = \frac{\Lambda_1}{\mu}$, and $S_2^0 = \frac{\Lambda_2}{\mu}$, and all infected states are zero. The vector representing the rates of new infections into each compartment is:

$$f = \begin{pmatrix} \beta\varepsilon S_1^0 E + \beta S_1^0 I_2 \\ 0 \\ 0 \end{pmatrix}$$

The vector representing all other flows (transition and removal) is:

$$v = \begin{pmatrix} (\alpha + \mu)E \\ -\rho\alpha E + (\xi + \gamma + \mu)I_1 \\ -\rho\alpha E + (\xi + \gamma + \mu)I_1 \end{pmatrix}$$

The Jacobian matrices are powerhouse in mathematics and science. They are used to analyze systems of non linear equations and have advantages such as approximating non- linear system locally, study equilibrium points in dynamical system and compute gradients for optimization algorithms.

The Matrices F and V for f and v were computed respectively, where new infections matrix F is the Jacobian of f with respect to the infected states (E, I_1, I_2) and evaluated at the DFE while the transition matrix V is the Jacobian of v with respect to the infected states (E, I_1, I_2), evaluated at the DFE. First, we find the inverse of V. Since V is a lower triangular matrix, its inverse is also lower triangular.

$$V^{-1} = \begin{pmatrix} 1/(\alpha+\mu) & 0 & 0 \\ \rho\alpha/[(\alpha+\mu)(\xi+\gamma+\mu)] & 1/(\xi+\gamma+\mu) & 0 \\ (1-\rho)\alpha/[(\alpha+\mu)(\delta+\gamma+\mu)] & 0 & 1/(\delta+\gamma+\mu) \end{pmatrix}$$

Now, compute the next-generation matrix $K = FV^{-1}$ [12], hence the basic reproduction number R_0 is the dominant eigenvalue (spectral radius) of the next-generation matrix $K = (FV^{-1})$. Since K is a rank-one matrix with only one non-zero row, its only non-zero eigenvalue is the entry in the first row, first column. Therefore,

$$R_0 = \frac{\beta\varepsilon S_1^0}{\alpha+\mu} + \frac{\beta S_1^0(1-\rho)\alpha}{(\alpha+\mu)(\delta+\gamma+\mu)}$$

Substituting $S_1^0 = \Lambda_1/\mu$, we get the final expression:

$$R_0 = \frac{\beta\Lambda_1}{\mu(\alpha+\mu)} \times \frac{\varepsilon+\alpha(1-\rho)}{(\delta+\gamma+\mu)}$$

3.2 Sensitivity Analysis of the HIV/AIDS Model

The significance and importance of mathematical modelling cannot be over emphasized. It helps to understand, predict and optimize complex systems. The types of mathematical models we have include deterministic model, stochastic model, discrete model and continuous model. The model in this study is deterministic model. Deterministic models predict outcomes with certainty.The importance of deterministic models are as follows:

1. Provide clear, unambiguous predictions based on given inputs
2. Help understand underlying mechanisms and relationships
3. Are often simpler to analyze and interpret.

The algorithms of deterministic models involves (1) solving equations such as ordinary differential equations and partial differential equations (2) optimization techniques such as gradient descent and (3) simulations. The method of non-linear least squares which depends on nonlinearity of the residual employed as adopted by [17]. The non-linear problems are generally used in the iterative method of refinement in which the reported data were fitted to the model using the curve fitting model in python programming language. The HIV/AIDS infection data report was collected from the Ebonyi State Ministry of Health, Abakaliki. The estimated parameter values are tabulated in table (2).

The sensitivity analysis of the basic reproduction number R_0 is performed to identify the most influential parameters in the transmission dynamics of HIV/AIDS. The normalized forward sensitivity index of R_0 with respect to a parameter p as applied in [13] is defined as:

$$\gamma_p^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0}$$

The basic reproduction number is given by:

$$R_0 = \frac{\beta\Lambda_1}{\mu(\alpha+\mu)} \times \frac{\varepsilon+\alpha(1-\rho)}{(\delta+\gamma+\mu)}$$

for transmission rate β

$$\frac{\partial R_0}{\partial \beta} = \frac{\Lambda_1}{\mu(\alpha+\mu)} \times \frac{\varepsilon+\alpha(1-\rho)}{(\delta+\gamma+\mu)}$$

$$\gamma_{\beta}^{R_0} = \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} = 1$$

Similarly, other values were obtained and presented in the table below:

Table 3: Sensitivity Result.

Parameters	Values	Sensitivity index
β	0.000002	1.00000
Λ_1	0.419990	1.00000
α	0.333000	0.03996
ε	0.001906733	0.00090
μ	0.014286	-1.20702
ρ	0.455840	-0.83703
γ	0.056044	-0.65092
δ	0.015697	-0.18233

3.3 Numerical Simulation

Numerical simulation is the use of computer programs to provide solutions for complex mathematical models that represent real- world physical system, thereby giving scientists and engineers opportunity to forcast behavior, test design and comprehend phenomena which are too hard or expensive for experiments by rounding off solutions to equations. The graphical simulation performed with Python programming language is significant as it allows the modelling of complex systems safely and efficiently and predicts results based on data and algorithms. In addition, it models real-world system without risk. The application of python program was utilized in the numerical simulation of this study. The following graphs were obtained as:

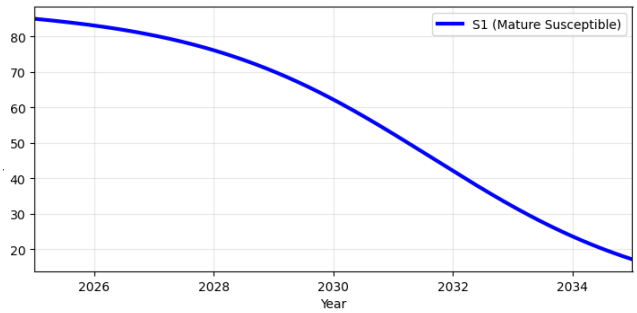


Figure 1: Mature Susceptible (S_1)

4 Discussion of Results

The sensitivity indices values in table (3) have both negative and positive signs. Based on the sensitivity indices provided, it follows that the most sensitive parameters for the HIV/AIDS model are: β with positive sign is the most influential parameter which implies that a 1% increase in β

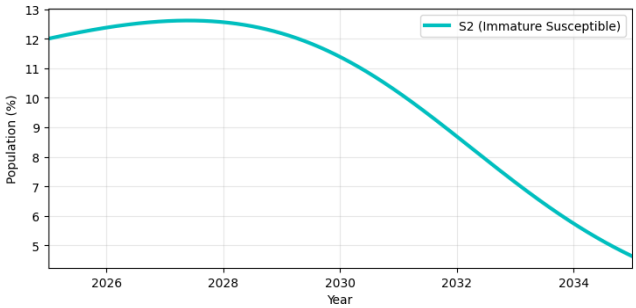


Figure 2: Immature Susceptible (S_2)

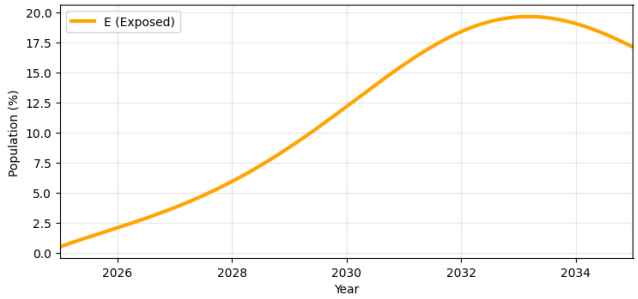


Figure 3: Exposed

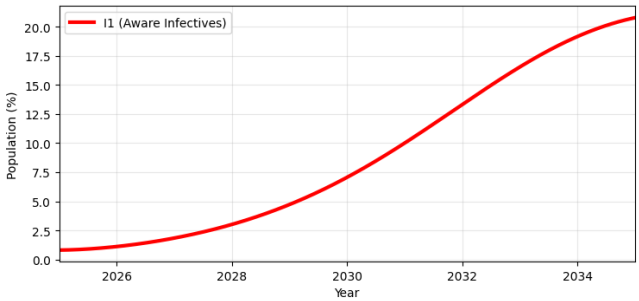


Figure 4: HIV/AIDS Dynamics for Aware Infectives

increases the reproduction number by 3.4% while the Natural death rate and AIDS-related death rate are the most significant negative influences which implies that an increase in either death rate reduces the reproduction number. The parameters ξ and ε, have a moderately strong positive effect on the reproduction number. Other parameters $(\rho, \alpha, \delta, \gamma, \eta)$ have relatively low sensitivity indices (magnitude < 0.4) and thus a weaker influence.

The graphs in figures 2 and 3 Showed a steady decrease over time in the population of mature and immature susceptible humans respectively. This significant decline, suggested that these groups face higher infection risk due to less protection. The graph of Exposed (E) individuals relatively stable but persistent, indicating continuous new infections as evident in figure 4. For the Aware Infectives (I_1), there exist rapid increase in the population due to lack of treatment as shown in figure 5 while the

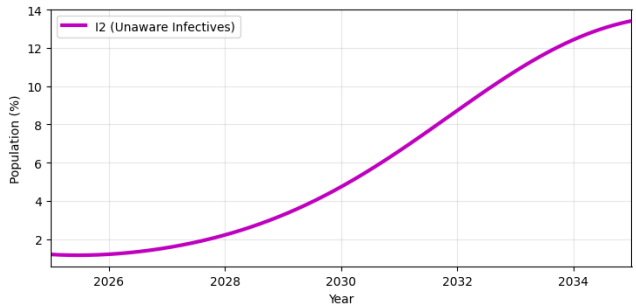


Figure 5: HIV/AIDS Dynamics for Unaware Infectives

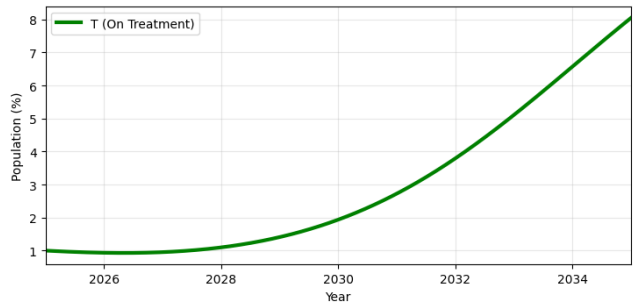


Figure 6: HIV/AIDS Dynamics for Treated Individuals

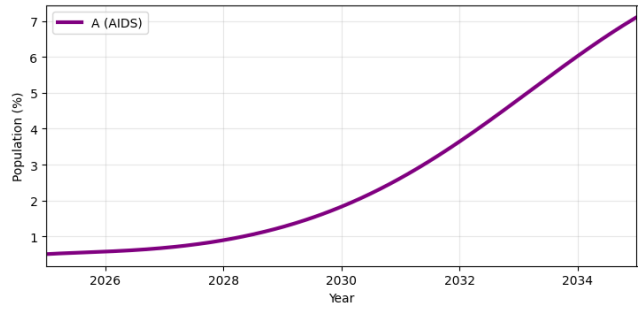


Figure 7: HIV/AIDS Dynamics for AIDS

population of Unaware Infectives (I_2), tends higher than (I_1), representing the "hidden" transmission reservoir in the society.

5 Conclusion

The model projects an uncontrolled HIV epidemic with explosive growth over 2025-2035. Key findings indicate; epidemic expansion showing that all infected compartments increases rapidly and AIDS cases increase 85.0%, indicating severe health system burden. Python programming software was

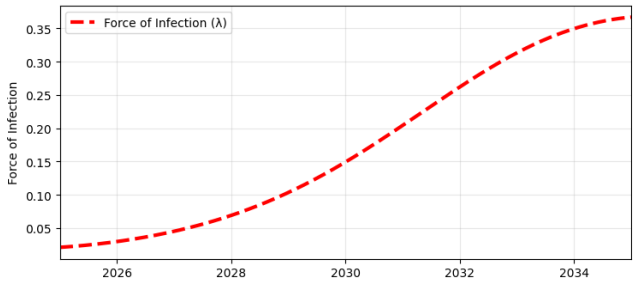


Figure 8: HIV/AIDS Dynamics for the force of infection

used to analyze its sensitivity. The sensitivity analysis identified the key parameters driving HIV/AIDS dynamics, with the transmission rate (β) being the most critical. The numerical simulations corroborate this, showing a high risk to susceptible populations and a persistent, large reservoir of unaware infectives that sustains the epidemic. Therefore, effective control strategies must prioritize reducing transmission through preventative measures and shrinking this hidden reservoir by enhancing awareness and screening programs to identify and treat infected individuals.

Hence, urgent intervention such as prevention and control measures are critically needed to mitigate the spread of the infection. The result of this study will be of benefit to the Nigerian government, health experts and policy makers. However, the result of this study is limited as it may not be generalizable to other populations or settings due to differences in demographics and healthcare systems. Moreover, the result may be highly dependent on estimate on key parameters such as transmission rate which may be uncertain or context –specific.

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