Quasi S-Menger Spaces and Fixed-Point Theorems via Rational Inequalities with Applications

Abstract: S-metric space is a comparatively new concept in the novel and presently there is much attention being given to the abstraction of S-metric and fixed-point theory in these spaces. The concept of S-Menger spaces was popularized in the novel as an abstraction of both S-metric space and Menger spaces. In the present paper, we define quasi-S-Menger space and prove fixed-point theorems for rational inequality in S-Menger space. Some applications are also given is support of our results. Our results extend the results of Gupta.V et.al [1] in the setting of S-Menger space.

Key Words: Quasi *S*- Menger space, S- Menger space, Rational Expression, Fixed Point.

Mathematics Subject Classification: 47H10, 54H25.

1.Introduction

The foundation for probabilistic approaches to metric spaces was laid much earlier by Karl Menger in 1942, who introduced the concept of Menger spaces by replacing the traditional distance function with a probabilistic distribution function. This idea marked a significant shift in the study of metric structures and improved the way for further generalizations in both deterministic and probabilistic settings. Fixed point theory is an essential topic in nonlinear analysis with broad applications in mathematics and enforced information. Classical contraction principles in metric spaces are often insufficient to describe concern, which led to the improvement of probabilistic metric spaces [1]. Statistical metric spaces imported by Schweizer and Sklar [4] provided the basis for probabilistic distance theory and were later continued to Menger spaces using triangular norms [5]. Sedghi et al. imported S-metric spaces, conclude metric spaces by exacting three variables [6]. By assimilation the probabilistic structure of Menger spaces with S-metric algebra, **S-Menger spaces** arise as an effective structure for fixed point study [3]. Several fixed-point results of Banach and Kannan form have been established in S-Menger spaces [3]. Rational inequalities offer a significant development of classical contraction conditions and permit the analysis of nonlinear mappings [7]. Therefore, fixed point theorems based on rational inequalities in S-Menger spaces merge and enhance results in probabilistic and S-metric settings [1,3,6].

In the present paper, we define Quasi-S-Menger space and establish a fixed-point theorem in the setting of S-Menger spaces using a rational inequality. We present integral analogue of our results.

2.Preliminaries

In this section, we have provided some preliminaries definitions and lemmas along with some examples.

Definition 2.1. [4] A map $*: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous $t-$norm if it satisfies the following conditions:

$1{:}*(\zeta, 1, 1) = *(0,0,0) = 0$

$2{:}*(\zeta, \eta, \upsilon) = *(\zeta, \upsilon, \eta) = *(\eta, \upsilon, \zeta)$

$3{:}*(\zeta_1, \eta_1, \upsilon_1) \geq *(\zeta_2, \eta_2, \upsilon_2)$,for $\zeta_1 \geq \zeta_2$, $\eta_1 \geq \eta_2$, $\upsilon_1 \geq \upsilon_2$

examples of $t-$norm are

(1): $\zeta * \eta * \upsilon = \zeta.\eta.\upsilon$ and

(2): $\zeta * \eta * \upsilon = min\{\zeta, \eta, \upsilon\}$ $(H - type)$

Definition 2.2.[2] Let X be a nonempty set. A function $S: X^3 \rightarrow [0, \infty]$ is said to be quasi-S-metric space iff for all $\zeta, \eta, \upsilon, a \in X$ the following conditions are satisfied:

(S_{q1}) $S(\zeta, \eta, \upsilon) \geq 0$;

(S_{q2}) $S(\zeta, \eta, \upsilon) = S(P\{\zeta, \eta, \upsilon\}) = 0$ iff $\zeta = \eta = \upsilon$, where P is permutation,

(S_{q3}) $S(\zeta, \eta, \upsilon) \leq [S(\zeta, \zeta, a) + S(\eta, \eta, a) + S(\upsilon, \upsilon, a)]$

The pair (X, S) is called a quasi $S-$metric space.

Example 2.3.[2] Let $X = R^+ \cup \{0\}$. Define $S: X^3 \rightarrow [0, \infty)$ by

$$S(\zeta, \eta, \upsilon) = \begin{cases} 0 & if\ \zeta = \eta = \upsilon, \\ \left|\zeta - \frac{\upsilon}{2}\right| + \left|\eta - \frac{\upsilon}{2}\right|, & \text{otherwise.} \end{cases}$$

Then (X, S) is a quasi $S-$metric space.

Definition 2.4.[3] The 3-tuple $((X, S, *))$ is said to be S-Menger space if X is a non-empty set, S is a function defined on X^3 to the set of distribution function and * is a continuous third order t-norm such that the following conditions are satisfied:

(i) $S_{(\zeta,\eta,\upsilon)}(0) = 0$ for all $\zeta, \eta, \upsilon \in X$

(ii) $S_{(\zeta,\zeta,\eta)}(t) < 1$ for $t > 0$ with $\zeta \neq \eta$,

(iii) $S_{(\zeta,\eta,\upsilon)}(t) = 1$ for all $t > 0$, if and only if $\zeta = \eta = \upsilon$,

(iv) $S_{(\zeta,\eta,\upsilon)}(t) \geq T(S_{(\zeta,\zeta,a)}(t_1), S_{(\eta,\eta,a)}(t_2), S_{(\upsilon,\upsilon,a)}(t_3))$, Where $t = t_1 + t_2 + t_3$ and $t, t_1, t_2, t_3 > 0$ for all $\zeta, \eta, \upsilon, a \in X$

Example 2.5.[3] Let $X = R$, $S(\zeta, \eta, \upsilon)$ be defined as

$$S(\zeta, \eta, \upsilon) = \begin{cases} \frac{t}{t + |\eta - \upsilon| + |\upsilon - \zeta|}, & t > 0 \\ 0 & t = 0 \end{cases},$$

For all $\zeta, \eta, \upsilon \in X$, $t > 0$ and $*$ be a 3rd order minimum t-norm. Then $(X, S, *)$ is an S −Menger space.

Definition 2.6.[3] Let $(X, S, *)$ be an S-Menger space then a sequence $\{\eta_n\} \in X$ is said to be convergent to a point $\eta \in X$ if $\lim_{n\to\infty} S_{(\eta_n, \eta_n, \eta)}(t) \geq 1 - \lambda$.

Definition 2.7.[3] Let $(X, S, *)$ be an S-Menger space then a sequence $\{\eta_n\} \in X$ is called a Cauchy sequence if $\eta \in X$ if $\lim_{n\to\infty} S_{(\eta_n, \eta_n, \eta_{n+p})}(t) = 1$ for $p = 1,2,3..$ for each $t > 0$.

Lemma 2.8. [3] Let $(X, S, *)$ be an S-Menger space with continuous third order t −norm $*$. Then $S_{(\zeta,\zeta,\eta)}(t)$ is non-decreasing with respect to t,for all $\zeta, \eta \in X$.

Lemma 2.9.[3] In every S-Menger space $(X, S, *)$,where $*$ is a continuous third order t −norm we have $S_{(\zeta,\zeta,\eta)}(t) = S_{(\eta,\eta,\zeta)}(t)$ for all $\zeta, \eta \in X$ and $t > 0$.

3. Main Results

In this section, we will define quasi S −Menger space with example and prove lemma and fixed-point theorems for rational inequality.

Quasi S −Menger space

Definition 3.1 A quasi −S −Menger space is a triplet $(X, S_q, *)$ where X is a nonempty set, * is continuous t −norm and S_q is a distribution function on X^3 satisfying the following conditions:

(S_{q1}) $S_{q(\zeta,\eta,\upsilon)}(t) = 0$, for all $\zeta, \eta, \upsilon \in X$

(S_{q2}) $S_{(\zeta,\zeta,\eta)}(t) < 1$ for $t > 0$ with $\zeta \neq \eta$,

(S_{q3}) $S_{q(\zeta,\eta,\upsilon)}(t) = S_{q(P(\zeta,\eta,\upsilon)}(t) = 1$ iff $\zeta = \eta = \upsilon$ where P is permutation,

(S_{q4}) $S_{q(\zeta,\eta,\upsilon)}(r + s + t) \geq S_{q(\zeta,\zeta,a)}(t) * S_{q(\eta,\eta,a)}(t) * S_{q(\upsilon,\upsilon,a)}(t)$,

Example 3.2. Let (X, S) be a quasi-S-metric as defined in example 2.3. we define a distribution function S_q on X^3 such that

$$S_{q(\zeta,\eta,\upsilon)}(t) = \begin{cases} \frac{t}{t + S(\zeta, \eta, \upsilon)}, & t > 0 \\ 0 & t = 0. \end{cases}$$

For all $\zeta, \eta, \upsilon \in X$, $t > 0$ and $*$ be a 3rd order minimum t-norm. Then $(X, S_q, *)$ is a quasi-S −Menger space.

Lemma 3.3. Let $(X,S,*)$ be an S-Menger space. *If* there exists $k \in (0,1)$ such that $S_{(\zeta,\zeta,\eta)}(kt) \geq S_{(\zeta,\zeta,\eta)}(t)$ for all $\zeta,\eta \in X, t > 0$ and

$$\lim_{t\to\infty} S_{(\zeta,\eta,\upsilon\ )}(t) = 1.$$

then $\zeta = \eta$.

Proof. Suppose that there exist $k \in (0,1)$ such that $S_{(\zeta,\zeta,\eta)}(kt) \geq S_{(\zeta,\zeta,\eta)}(t)$ for all $\zeta,\eta \in X, t > 0$.

Then

$$S_{(\zeta,\zeta,\eta)}(t) \geq S_{(\zeta,\zeta,\eta)}(\frac{t}{k})$$

and also

$$S_{(\zeta,\zeta,\eta)}(t) \geq S_{(\zeta,\zeta,\eta)}(\frac{t}{k^n})$$

For positive integer n. Taking limit as $n \to \infty$, $S_{(\zeta,\zeta,\eta)}(t) \geq 1$ and hence $\zeta = \eta$.

Theorem 3.4. Let $(X,S,*)$ be a complete S-Menger space and $f: X \to X$ be a mapping satisfying

$$\lim_{n\to\infty} S_{(\zeta,\eta,\upsilon)}(t) = 1 \tag{3.4.1}$$

and

$$S_{(f(\zeta),f(\zeta),f(\ \eta)}(\ kt) \geq \lambda_{(\ \zeta,\zeta,\eta)}(\ t) \tag{3.4.2}$$

Where

$$\lambda_{(\zeta,\zeta,\eta)}(\ t) = min\left\{\frac{S_{(\eta,\eta,f(\eta)}(t)[1+S_{(\zeta,\zeta,f(\zeta)}(t)]}{[1+S_{(\zeta,\zeta,\eta)}(t)]}, S_{(\zeta,\zeta,\eta)}(t)\right\} \tag{3.4.3}$$

For all $\zeta,\eta \in X$ and $k \in \left(0,\frac{1}{3}\right)$. then f has a fixed point.

Proof: Let us consider $\zeta \in X$ be any arbitrary point in X. Now construct a sequence $\{\zeta_n\} \in X$ such that

$f(\zeta_n) = \zeta_{n+1}$ for all $n \in N$.

Claim. $\{\zeta_n\}$ is a Cauchy sequence.

Let us take $\zeta = \zeta_{n-1}$ and $\eta = \zeta_n$ in (3.4.2), we get

$$S_{(\zeta_n,\zeta_n,\zeta_{n-1})}(kt) = S_{(f(\zeta_{n-1}),f(\zeta_{n-1}),f(\zeta_n)}(kt) \geq \lambda_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t) \tag{3.4.4}$$

Now

$$\lambda_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t) = min\left\{\frac{S_{(\zeta_n,\zeta_n,f(\zeta_n)}(t)[1+S_{(\zeta_{n-1},\zeta_{n-1},f(\zeta_{n-1})]}(t)]}{[1+S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)]}, S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)\right\}$$

$$\lambda_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t) = min\left\{\frac{S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(t)[1+S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)]}{[1+S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)]}, S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)\right\}$$

$\lambda_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t) = min\{S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(t), S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)\}$

Now if $S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(t) \leq S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)$ then form equation (3.4.4)

$$S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(kt) \geq S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(t)$$

Hence form lemma (3.3), our claim follows immediately. Now suppose

$$S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(t) \geq S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)$$

Then again from equation (3.4.4)

$$S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(kt) \geq S_{(\zeta_{n-1},\zeta_{n-1},\zeta_n)}(t)$$

Now by simple induction, for all n and $t > 0$, we get

$$S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(kt) \geq S_{(\zeta_0,\zeta_0,\zeta_1)}(\frac{t}{k^{n-1}}) \tag{3.4.5}$$

By using equation (3.4.5), for any positive integer ‘p’ we get

$$S_{(\zeta_n,\zeta_n,\zeta_{n+p})}(t) \geq S_{(\zeta_n,\zeta_n,\zeta_{n+1})}\left(\frac{t}{3}\right) * S_{(\zeta_n,\zeta_n,\zeta_{n+1})}\left(\frac{t}{3}\right) * S_{(\zeta_{n+p},\zeta_{n+p},\zeta_{n+1})}\left(\frac{t}{3}\right) \text{ [by lemma.3.3]}$$

$$S_{(\zeta_n,\zeta_n,\zeta_{n+p})}(t) = S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(\frac{t}{3}) * S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(\frac{t}{3}) * S_{(\zeta_{n+1},\zeta_{n+1},\zeta_{n+p})}\left(\frac{t}{3}\right)$$

$$S_{(\zeta_n,\zeta_n,\zeta_{n+p})}(t) \geq S_{(\zeta_n,\zeta_n,\zeta_{n+1})}\left(\frac{t}{3}\right) * S_{(\zeta_n,\zeta_n,\zeta_{n+1})}\left(\frac{t}{3}\right) * S_{(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2})}(\frac{t}{(3)^2}) * S_{(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2})}(\frac{t}{(3)^2}) * S_{(\zeta_{n+p},\zeta_{n+p},\zeta_{n+2})}(\frac{t}{(3)^2})$$

$$S_{(\zeta_n,\zeta_n,\zeta_{n+p})}(t) = S_{(\zeta_n,\zeta_n,\zeta_{n+1})}\left(\frac{t}{3}\right) * S_{(\zeta_n,\zeta_n,\zeta_{n+1})}\left(\frac{t}{3}\right) * S_{(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2})}(\frac{t}{(3)^2}) * S_{(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2})}(\frac{t}{(3)^2}) * S_{(\zeta_{n+2},\zeta_{n+2},\zeta_{n+p})}(\frac{t}{(3)^2}) \text{ [by lemma. 3.3]}$$

$$S_{(\zeta_n,\zeta_n,\zeta_{n+p})}(t) = S_{(\zeta_0,\zeta_0,\zeta_1)}(\frac{t}{k^n 3}) * S_{(\zeta_0,\zeta_0,\zeta_1)}(\frac{t}{k^n 3}) * S_{(\zeta_0,\zeta_0,\zeta_1)}(\frac{t}{k^{n+1}(3)^2}) * S_{(\zeta_0,\zeta_0,\zeta_1)}(\frac{t}{k^{n+1}(3)^2}) * \ldots.$$

By the definition of S-Menger k-contraction (i.e., k<1) together with condition (3.4.1) and letting $n\to\infty$, we get

$$\lim_{n\to\infty} S_{(\zeta_n,\zeta_n,\zeta_{n+p})}(t) = 1 * 1 * 1 * 1 \ldots. * 1 * 1 = 1$$

Hence, $\{\zeta_n\}$ is Cauchy sequence. Since $(X, S, *)$ is a complete S−MS, there exists $\mu \in X$ such that

$$\lim_{n\to\infty} \zeta_n = \mu \tag{3.4.6}$$

Claim. μ is a fixed point of f.

Consider

$$S_{(\mu,\mu,f(\mu))}(t) \geq S_{(\mu,\mu,f(\zeta_n))}\left(\frac{t}{3}\right) * S_{(\mu,\mu,f(\zeta_n))}\left(\frac{t}{3}\right) * S_{(f(\mu),f(\mu),f(\zeta_n))}(\frac{t}{3}) \tag{3.4.7}$$

$$= S_{(\mu,\mu,\zeta_{n+1})}(\tfrac{t}{3}) * S_{(\mu,\mu,\zeta_{n+1})}(\tfrac{t}{3}) * S_{(f(\zeta_n),f(\zeta_n),f(\mu))}(\tfrac{t}{3})$$

$$= S_{(\mu,\mu,\zeta_{n+1})}(\tfrac{t}{3}) * S_{(\mu,\mu,\zeta_{n+1})}(\tfrac{t}{3}) * \lambda_{(\zeta_n,\zeta_n,\mu)}(\tfrac{t}{3k})$$

Now

$$\lambda_{(\zeta_n,\zeta_n,\mu)}(\tfrac{t}{3k}) = min\left\{\frac{S_{(\mu,\mu,f(\mu))}(\frac{t}{3k})\left[1+S_{(\zeta_n,\zeta_n,f(\zeta_n))}(\frac{t}{3k})\right]}{\left[1+S_{(\zeta_n,\zeta_n,\mu)}(\frac{t}{3k})\right]}, S_{(\zeta_n,\zeta_n,\mu)}(\tfrac{t}{3k})\right\}$$

$$\lambda_{(\zeta_n,\zeta_n,\mu)}(\tfrac{t}{3k}) = min\left\{\frac{S_{(\mu,\mu,f(\mu))}(\frac{t}{3k})[1+S_{(\zeta_n,\zeta_n,\zeta_{n+1})}(\frac{t}{3k})]}{[1+S_{(\zeta_n,\zeta_n,\mu)}(\frac{t}{3k})]}, S_{(\zeta_n,\zeta_n,\mu)}(\tfrac{t}{3k})\right\}$$

Taking $\lim_{n\to\infty}$. In above inequality and using (3.4.1), we get

$$\lambda_{(\zeta_n,\zeta_n,\mu)}(\tfrac{t}{3k}) = min\left\{S_{(\mu,\mu,f(\mu))}(\tfrac{t}{3k}), 1\right\}$$

Now if $S_{(\mu,\mu,f(\mu))}\frac{t}{3k}) \geq 1$ then $\lambda_{(\mu,\mu,\mu)}(\frac{t}{3k}) = 1$.

Therefore from (3.4.7) and using definition we get μ is a fixed point of f.

Now if $S_{(\mu,\mu,f(\mu))}(\frac{t}{3k}) \leq 1$ then $\lambda_{(\mu,\mu,\mu)}(\frac{t}{3k}) = S_{(\mu,\mu,f(\mu))}(\frac{t}{3k})$.

Hence from equation (3.4.7), we get

$$S_{(\mu,\mu,f(\mu))}(t) \geq S_{(\mu,\mu,f(\mu))}(\tfrac{t}{3k}) \tag{3.4.8}$$

Since $k \in \left(0,\frac{1}{3}\right)$ therefor by lemma (3.3), we get $f(\mu) = \mu$.

This completes the proof of theorem 3.4.

Let us define $\theta = \{\phi: [0,1] \to [0,1]\}$ is a continuous function such that $\phi(1) = 1, \phi(0) = 0, \phi(a) > a$ for each $0 < a < 1$.

Theorem 3.5. Let $(X, S,*)$ is a *S*-Menger space with

$$\lim_{n\to\infty} S_{(\zeta,\eta,\upsilon)}(t) = 1 \tag{3.5.1}$$

and $f: X \to X$ be a mapping satisfying

$$S_{(f(\zeta),f(\zeta),f(\eta))}(kt) \geq \phi\{\lambda_{(\zeta,\zeta,\eta)}(t)\} \tag{3.5.2}$$

Where

$$\lambda_{(\zeta,\zeta,\eta)}(t) = min\left\{\frac{S_{(\eta,\eta,f(\eta))}(t)\left[1+S_{(\zeta,\zeta,f(\zeta))}(t)\right]}{\left[1+S_{(\zeta,\zeta,\eta)}(t)\right]}, S_{(\zeta,\zeta,\eta)}(t)\right\}$$

For all $\zeta, \eta \in X$ and $k \in (0,1/3), \phi \in \theta$.

Then f has a fixed point.

Proof: Since $\phi \in \theta$. This implies that $\phi(a) > a$ for each $0 < a < 1$. Thus from (3.5.2)

$S_{(f(\zeta),f(\zeta),f(\eta))}(kt) \geq \phi\{\lambda_{(\zeta,\zeta,\eta)}(t)\} \geq \lambda_{(\zeta,\zeta,\eta)}(t)$

Now, applying theorem 3.5, we obtain the desired result.

4. Applications

In this section, we give some applications related to our results. Let us define $\psi: [0,\infty] \rightarrow [0,\infty), as\ \ \psi(t) = \int_0^t \varphi(t)dt \quad t > 0$, be a non- decreasing and continuous function. Moreover, for each $\epsilon > 0$, $\varphi(\epsilon) > 0$. Also implies that $\varphi(t) = 0$ iff $t = 0$. In the following, we prove integral analogue of theorem 3.4 in S-Menger Space.

Theorem 4.1. Let $(X, S, *)$ be a complete S-Menger space with

$\lim_{n\to\infty} S_{(\zeta,\eta,\upsilon)}(t) = 1$ and $f: X \rightarrow X$ be a mapping satisfying

$$\int_0^{S_{(f(\zeta),f(\zeta),f(\eta))}(kt)} \varphi(t)dt \geq \int_0^{\lambda_{(\zeta,\zeta,\eta)}(t)} \varphi(t)dt$$

Where

$$\lambda_{(\zeta,\zeta,\eta)}(t) = min\left\{\frac{S_{(\eta,\eta,f(\eta))}(t)[1+S_{(\zeta,\zeta,f(\zeta))}(t)]}{[1+S_{(\zeta,\zeta,\eta)}(t)]}, S_{(\zeta,\zeta,\eta)}(t)\right\}$$

For all $\zeta, \eta \in X$, $\varphi \in \psi$ and $k \in \left(0, \frac{1}{3}\right)$.

This f has a fixed point.

Proof: By taking $\varphi(t) = 1$ and applying theorem 3.4, we obtain the result.

Theorem 4.2. Let $(X, S, *)$ be a complete S -Menger space with

$\lim_{n\to\infty} S_{(\zeta,\eta,\upsilon\ )}(t) = 1$ and $f: X \rightarrow X$ be a mapping satisfying

$$\int_0^{S_{(f(\zeta),f(\zeta),f(\beta))}(kt)} \varphi(t)dt \geq \phi\{\int_0^{\lambda_{(\zeta,\zeta,\beta)}(t)} \varphi(t)dt\}$$

Where

$$\lambda_{(\zeta,\zeta,\eta)}(t) = min\left\{\frac{S_{(\eta,\eta,f(\eta))}(t)\left[1+S_{(\zeta,\zeta,f(\eta))}(t)\right]}{\left[1+S_{(\zeta,\zeta,\eta)}(t)\right]}, S_{(\zeta,\zeta,\eta)}(t)\right\}$$

For all $\zeta, \eta \in X$, $\varphi \in \psi$ and $k \in \left(0, \frac{1}{3}\right)$.

This f has a fixed point.

Proof: Since $\phi(a) > a$ for each $0 < a < 1$, therefore result follows immediately from theorem 4.1.

5. Conclusion

In this paper, we introduced Quasi S- Menger space and proved fixed point theorems using rational expression. In the last section we have also given integral analogue of fixed-point theorems. Our results extend and generalize the results of Gupta V et al [1] in the structure of S-Menger space

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

References

[1] Gupta V., Fixed Point Theorems and its Applications in Fuzzy Metric Spaces. https://www.researchgate.net/publication/256496646.

[2] Fernandez, J., Işık, H., Malviya, N., & Jarad, F. (2023). Nb-fuzzy metric spaces with topological properties and applications. AIMS Mathematics, 8, 5879-5898. https://doi.org/10.3934/math.2023296.

[3] Sarkar, K. K., Das, K., & Pramanik, A. (2023). Generalized fixed point result of Banach and Kannan type in S-Menger spaces. South East Asian Journal of Mathematics & Mathematical Sciences, 19(1), 121-134. https://doi.org/10.56827/SEAJMMS.2023.1901.11.

[4] Schweizer, B., & Sklar, A. (1960). Statistical metric spaces. Pacific Journal of Mathematics, 10(1), 313-334. https://doi.org/10.2140/pjm.1960.10.313.

[5] Hadžić, Olga, and Endre Pap. "Fixed point theory in probabilistic metric spaces." *(No Title)* (2001). https://doi.org/10.1007/978-94-017-1560-7_2.

[6] Sedghi, Shaban, et al. "Some properties of S-metric spaces and fixed-point results." *Kyungpook Math. J* 54.1 (2014): 113-122. https://doi.org/10.5666/kmj.2014.54.1.113.

[7] Saluja, G. S., et al. "Common Fixed-Point Theorems on S-Metric Spaces for Integral Type Contractions Involving Rational Terms and Application to Fractional Integral Equation." *Journal of Function Spaces* 2024.1 (2024): 5108481. https://onlinelibrary.wiley.com/doi/abs/10.1155/2024/5108481.