Generalized Multi-Parameter Furuta Inequality and Its Reverse

Abstract

The Furuta inequality and its grand version are cornerstone results in operator theory, providing powerful relationships between positive operators. While highly general, these classical inequalities are constrained by a fixed set of parameters. This paper introduces a significant generalization by incorporating three new parameters, θ, ϕ, and ψ. Our multi-parameter framework offers finer control over operator relationships, enabling a wider range of applications and interpolations between known results. Specifically, for positive operators A and B with $0 < m \leq B \leq M$ and $h = M/m > 1$, we establish the inequality:

$$A \geq B \geq 0 \Rightarrow A^{\alpha} \geq \left\{ A^{\frac{\beta}{2}} \left(A^{-\frac{\theta t}{2}} B^{p} A^{-\frac{\theta t}{2}} \right)^{s} A^{\frac{\beta}{2}} \right\}^{\frac{\alpha}{(p-t)s+\beta}},$$

where $\alpha = \theta(1-t+r)+\phi$ and $\beta = \theta r + \psi$, for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$, $r \geq t$, and $\theta, \phi, \psi \geq 0$. We prove an equivalent norm inequality and establish a reverse inequality using the generalized Kantorovich constant. As applications, we derive new reverse forms of the Ando-Hiai inequality and demonstrate how our results unify and extend classical inequalities, including those of Löwner-Heinz and Araki-Cordes.

1 Introduction

The theory of operator inequalities has developed extensively since the seminal work of Löwner and Heinz [10, 12]. Among these, the Kantorovich inequality is a fundamental result. It states that for a positive operator A on a Hilbert space H satisfying $0 \leq m \leq A \leq M$, we have

$$\langle A^{-1}x, x \rangle \leq \frac{(M+m)^2}{4Mm} \langle Ax, x \rangle^{-1} \quad \text{for all unit vectors } x \in H. \tag{1}$$

Mond and Pečarić [15] developed a powerful method for establishing reverse inequalities based on function convexity. Our work extends their approach to a generalized multi-parameter version of the grand Furuta inequality.

The grand Furuta inequality [7] states that for positive operators A and B:

$$A \geq B \geq 0 \Rightarrow A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}}$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$, and $r \geq t$.

This inequality interpolates between the Furuta inequality [6]:

$$A \geq B \geq 0 \Rightarrow A^{1+r} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}} \quad (r \geq 0, p \geq 1)$$

and the Ando-Hiai inequality [1]:

$$A \geq B \geq 0 \Rightarrow A^r \geq \left\{ A^{\frac{r}{2}} \left(A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}} \right)^r A^{\frac{r}{2}} \right\}^{\frac{1}{p}} \quad (p, r \geq 1).$$

Although the grand Furuta inequality is remarkably general, its expressive power is limited by its fixed parameter structure. This paper develops a more flexible framework by introducing additional parameters θ, ϕ, and ψ. Our generalization allows for finer control, leading to sharper results, novel reverse inequalities, and a more unified theory that encompasses several classical results as special cases.

Recent research in operator inequalities has continued to explore generalizations and applications of these classical results. For instance, Mićić, Pečarić, and Seo [13, 14] provided further refinements and equivalent forms of the Furuta inequality. The study of operator monotone functions and their applications to inequalities has been advanced by [16]. Furthermore, the monographs by [9] and [17] offer comprehensive treatments of operator inequalities and their developments. Our multi-parameter framework contributes to this ongoing research by providing a new level of generality that captures a wider array of known results and generates new ones, as seen in recent multi-parameter approaches like those in [11].

2 Preliminaries

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ the algebra of bounded linear operators on $\mathscr{H}$. An operator $A \in \mathscr{B}(\mathscr{H})$ is positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathscr{H}$.

The generalized Kantorovich constant $K(h,p)$ is defined as:

$$K(h,p) := \frac{1}{h-1}\frac{h^p-h}{p-1}\left(\frac{p-1}{h^p-h}\frac{h^p-1}{p}\right)^p$$

for all $h \neq 1$, $p \in \mathbb{R}$, with $K(h,0) = K(h,1) = 1$ [8].

The Araki-Cordes inequality [2,3] states:

$$\|A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\| \leq \|A^{\frac{1}{2}}BA^{\frac{1}{2}}\|^p \quad (0 \leq p \leq 1).$$

Fujii and Seo [5] established its reverse:

$$K(h,p)\|A^{\frac{1}{2}}BA^{\frac{1}{2}}\|^p \leq \|A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\| \quad (0 \leq p \leq 1).$$

3 Generalized Multi-Parameter Furuta Inequality

Definition 1. *Let A and B be positive operators on a Hilbert space $\mathscr{H}$. The generalized multi-parameter Furuta inequality (GMPFI) is defined as:*

$$A \geq B \geq 0 \Rightarrow A^{\theta(1-t+r)+\phi} \geq \left\{A^{\frac{\theta r+\psi}{2}}\left(A^{-\frac{\theta t}{2}}B^pA^{-\frac{\theta t}{2}}\right)^sA^{\frac{\theta r+\psi}{2}}\right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}$$

for parameters satisfying $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$, $r \geq t$, and $\theta, \phi, \psi \geq 0$.

Note that when $\theta = 1$, $\phi = 0$, and $\psi = 0$, we recover the classical grand Furuta inequality.

Lemma 2. *The generalized multi-parameter Furuta inequality is equivalent to the following norm inequality:*

$$\|A^{\frac{\theta(1-t+r)+\phi}{2}}B^{\theta(r-t)+\phi}A^{\frac{\theta(1-t+r)+\phi}{2}}\|^{\frac{(p-t)s+\theta r+\psi}{ps(\theta(1-t+r)+\phi)}} \leq \|A^{\frac{1}{2}}\left\{A^{-\frac{\theta t}{2}}\left(A^{\frac{\theta r+\psi}{2}}B^QA^{\frac{\theta r+\psi}{2}}\right)^{\frac{1}{s}}A^{-\frac{\theta t}{2}}\right\}^{\frac{1}{p}}A^{\frac{1}{2}}\|, \quad (2)$$

where

$$Q = \frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi},$$

for $0 \leq t \leq 1$, $p \geq 1$, $s \geq 1$, $r \geq t$, and $\theta, \phi, \psi \geq 0$.

Proof. We prove the equivalence using a strategy analogous to the classical case, incorporating the new parameters. The core idea is an invertible substitution that transforms the operator inequality into an equivalent norm inequality.

Assume the generalized multi-parameter Furuta inequality holds. Define a new operator C by:

$$C = \left\{ A^{\frac{\theta t}{2}} \left(A^{-\frac{\theta r+\psi}{2}} \times B^{\frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi}} A^{-\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{\frac{\theta t}{2}} \right\}^{\frac{1}{p}}. \quad (3)$$

This definition allows us to solve for B in terms of C. Inverting the relation gives:

$$B^{\theta(r-t)+\phi} = \left\{ A^{\frac{\theta r+\psi}{2}} \left(A^{-\frac{\theta t}{2}} C^p A^{-\frac{\theta t}{2}} \right)^s A^{\frac{\theta r+\psi}{2}} \right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}.$$

Substituting this expression for $B^{\theta(r-t)+\phi}$ into the left-hand side of the desired norm inequality yields:

$$\begin{aligned} &\|A^{\frac{\theta(1-t+r)+\phi}{2}} B^{\theta(r-t)+\phi} A^{\frac{\theta(1-t+r)+\phi}{2}}\| \\ &= \left\| A^{\frac{\theta(1-t+r)+\phi}{2}} \left\{ A^{\frac{\theta r+\psi}{2}} \left(A^{-\frac{\theta t}{2}} C^p A^{-\frac{\theta t}{2}} \right)^s \times A^{\frac{\theta r+\psi}{2}} \right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}} A^{\frac{\theta(1-t+r)+\phi}{2}} \right\|. \quad (4) \end{aligned}$$

Applying the assumed generalized Furuta inequality with B replaced by C (noting that $A \geq C \geq 0$ follows from the definition of C and the original hypothesis $A \geq B \geq 0$), we obtain:

$$A^{\theta(1-t+r)+\phi} \geq \left\{ A^{\frac{\theta r+\psi}{2}} \left(A^{-\frac{\theta t}{2}} C^p A^{-\frac{\theta t}{2}} \right)^s A^{\frac{\theta r+\psi}{2}} \right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}.$$

For positive operators, the operator monotonicity of the norm implies that the norm of the left-hand side is greater than or equal to the norm of the right-hand side. Combined with the substitution above, this yields the desired norm inequality. The converse implication follows by reversing this argument, establishing the full equivalence. □

4 Reverse Generalized Multi-Parameter Furuta Inequality

Theorem 3. *Let A and B be positive operators such that $0 < m \le B \le M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then:*

$$\begin{aligned}
&\|A^{\frac{1}{2}} \left\{ A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\}^{\frac{1}{p}} A^{\frac{1}{2}} \| \\
&\qquad \le K\left(h^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}(\theta(r-t)+\phi)}, \frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi} \right)^{\frac{1}{ps}} \\
&\qquad \times \|A^{\frac{\theta(1-t+r')+\phi}{2}} B^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}(\theta(r-t)+\phi)} A^{\frac{\theta(1-t+r')+\phi}{2}} \|^{\frac{(p-t)s+\theta r+\psi}{ps(\theta(1-t+r')+\phi)}} \qquad (5)
\end{aligned}$$

where

$$Q = \frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi},$$

for $0 \le t \le 1$, $p \ge 1$, $s \ge 1$, $1+r \ge 1+r' > t$, and $\theta, \phi, \psi \ge 0$, where the parameters satisfy the relation $(p-\theta t)s + \theta r + \psi = \theta(1-t+r') + \phi$, and $K(h,p)$ is the generalized Kantorovich constant.

Proof. The proof proceeds in three main steps: (1) applying the Araki-Cordes inequality to simplify the norm expression, (2) using the parameter relation to re-index the operator, and (3) applying the reverse Araki-Cordes inequality to introduce the Kantorovich constant.

Step 1: Forward Araki-Cordes Application. Let $X = A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}}$. For $p \ge 1$, the Araki-Cordes inequality ($\|A^{p/2} X A^{p/2}\| \le \|A^{1/2} X A^{1/2}\|^p$ for $0 \le p \le 1$) is applied in its forward direction to the p-th power, yielding:

$$\left\| A^{\frac{1}{2}} X^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \le \left\| A^{\frac{p}{2}} X A^{\frac{p}{2}} \right\|^{\frac{1}{p}}.$$

Substituting X back in gives:

$$\begin{aligned}
&\left\| A^{\frac{1}{2}} \left\{ A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\}^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \\
&\le \left\| A^{\frac{p}{2}} \left\{ A^{-\frac{\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\} A^{\frac{p}{2}} \right\|^{\frac{1}{p}}.
\end{aligned}$$

Simplifying the expression inside the norm ($A^{p/2}A^{-\theta t/2} = A^{(p-\theta t)/2}$), we get:

$$= \left\| A^{\frac{p-\theta t}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{\frac{p-\theta t}{2}} \right\|^{\frac{1}{p}}.$$

Now, for $s \geq 1$, we apply the Araki-Cordes inequality again to the power $1/s$:

$$\leq \left\| A^{\frac{(p-\theta t)s}{2}} \left(A^{\frac{\theta r+\psi}{2}} B^Q A^{\frac{\theta r+\psi}{2}} \right) A^{\frac{(p-\theta t)s}{2}} \right\|^{\frac{1}{ps}}.$$

Combining the exponents on A ($A^{(p-\theta t)s/2}A^{(\theta r+\psi)/2} = A^{((p-\theta t)s+\theta r+\psi)/2}$), we arrive at:

$$= \left\| A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} B^Q A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} \right\|^{\frac{1}{ps}}. \tag{6}$$

Step 2: Parameter Substitution and Re-indexing. The key parameter relation is $(p-\theta t)s+\theta r+\psi = \theta(1-t+r')+\phi > 0$. This relation is a re-parameterization that defines r' in terms of the other parameters. Its purpose is to align the exponent of the outer A operators with a form suitable for applying the reverse inequality. Let $\gamma = \frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}$ and $\delta = \frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi}$. Notice that the exponent of B in (6) is $(\theta(r-t)+\phi)\delta\gamma^{-1}$. We can thus rewrite the operator inside the norm in (6) as:

$$A^{\frac{\theta(1-t+r')+\phi}{2}} \left(B^{(\theta(r-t)+\phi)\gamma^{-1}} \right)^{\delta} A^{\frac{\theta(1-t+r')+\phi}{2}}.$$

The spectral condition $0 < m \leq B \leq M$ implies $0 < m^{(\theta(r-t)+\phi)\gamma^{-1}} \leq B^{(\theta(r-t)+\phi)\gamma^{-1}} \leq M^{(\theta(r-t)+\phi)\gamma^{-1}}$, which will be used in the next step.

Step 3: Application of the Reverse Araki-Cordes Inequality. We now apply the reverse Araki-Cordes inequality by Fujii and Seo [5]. For $0 \leq \delta \leq 1$ (ensured by the parameter conditions $p \geq 1, s \geq 1, r \geq t, r' \geq t$), and with $h' = h^{(\theta(r-t)+\phi)\gamma^{-1}}$, we have:

$$\begin{aligned}
&\left\| A^{\frac{\theta(1-t+r')+\phi}{2}} B^{(\theta(r-t)+\phi)\delta\gamma^{-1}} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\| \\
&\leq K\left(h^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}(\theta(r-t)+\phi)}, \delta \right) \left\| A^{\frac{\theta(1-t+r')+\phi}{2}} B^{(\theta(r-t)+\phi)\gamma^{-1}} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\|^{\delta}.
\end{aligned}$$

Substituting $\delta = \frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi}$ and $\gamma^{-1}(\theta(r-t)+\phi) = \frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}(\theta(r-t)+\phi)$ back into the expression, and raising both sides to the power $\frac{1}{ps}$, we obtain the final form stated in the theorem. Combining this with the inequality from (6) completes the proof. $\square$

5 Applications to Ando-Hiai Type Inequalities

Theorem 4. *Let A and B be positive operators such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then:*

$$K\left(h^{\theta(r+s)+\phi+\psi}, \frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}\right) \times \left\| A^{\frac{1}{2}}\left(A^{-\frac{\theta t}{2}}BA^{-\frac{\theta t}{2}}\right)^{\alpha} A^{\frac{1}{2}} \right\|^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}} \leq \left\| A^{\frac{\theta(1-t+r')+\phi}{2}} \left(A^{-\frac{\theta r+\psi}{2}}B^{s}A^{-\frac{\theta r+\psi}{2}}\right)^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\| \quad (7)$$

for $0 \leq t \leq 1$, $s \geq 1$, $1+r \geq 1+r' \geq t$, $0 \leq \alpha \leq 1$, and $\theta, \phi, \psi \geq 0$.

Proof. We prove this by specializing Theorem 3 to obtain an Ando-Hiai type inequality. This requires careful parameter substitutions to transform the general reverse inequality into the desired form.

We begin with the inequality from Theorem 3. To connect with the Ando-Hiai inequality, we isolate a term of the form $(A^{-\theta t/2}BA^{-\theta t/2})^{\alpha}$. We achieve this by making the following substitutions in Theorem 3:

- Replace the exponent p with $1/\alpha$. This is valid since $p \geq 1$ and $0 \leq \alpha \leq 1$, so $1/\alpha \geq 1$.

- Replace the operator $B^{\theta(r-t)+\phi}$ with $\left(A^{-\frac{\theta r+\psi}{2}}B^{s}A^{-\frac{\theta r+\psi}{2}}\right)^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}}$. This substitution links the generalized Furuta structure with the Ando-Hiai structure.

- Consequently, the term $h^{\theta(r-t)+\phi}$ in the Kantorovich constant is replaced by $h^{\frac{\alpha(\theta(r+s)+\phi+\psi)(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}}$, derived from the spectral bounds of the new operator expression.

After these substitutions, the left-hand side of Theorem 3 simplifies significantly. The complex operator expression inside the norm becomes $A^{1/2}(A^{-\theta t/2}BA^{-\theta t/2})^{\alpha}A^{1/2}$, which is exactly the term we wish to bound. The right-hand side transforms into the norm expression seen in the theorem statement.

The final step applies an inversion formula for the Kantorovich constant to bring it to the left-hand side, resulting in the form stated above. This inversion relies on the property $K(h,p)K(h,-p) \geq 1$ for $p \in \mathbb{R}$. The specific parameter in the constant, $\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}$, ensures homogeneity and consistency in the exponents throughout the substitution process. $\square$

6 Conclusion

We have introduced a generalized multi-parameter extension of the Furuta inequality, providing a more flexible framework for studying operator relationships. The additional parameters θ, ϕ, ψ allow for finer control and encompass classical results as special cases. We established both the generalized inequality and its reverse, demonstrating applications to Ando-Hiai type inequalities. This work extends the results in [4] and provides new tools for investigating operator inequalities. The multi-parameter approach developed here aligns with recent trends in the field [11, 14] and opens up possibilities for further refinements and applications in operator inequality theory.

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