Common Fixed Points for Four Self - Mappings in Symmetric Spaces

Abstract

In this paper, we obtain a unique common fixed point theorem for occasionally weakly compatible four self- mappings in symmetric spaces. Our result is a generalization and improving some of the well known comparable results are existing in this literature.

Key words: Common fixed point theorem, fixed point theorem, occasionally weakly compatible, symmetric space.

AMS subject classifications: 47H10, 54H25.

1.Introduction

In prior to 1968 Banach contraction principle was a fundamental result in fixed point theory. Later in 1968, Kannan [14] proved a fixed point result for a mapping satisfying contractive condition that no need to satisfy the continuity condition. Subsequently many authors has been extending and improving and generalizes the results in fixed point theory in many ways (see for e.g. [1-13] & [15-17]). In 1999, Hicks and Rhoades [10] obtained some commixed fixed point theorems in symmetric spaces and semi metric spaces. In 2008, Abbas and Rhoades [5] proved some common fixed point results for occasionally weakly compatible mappings satisfying a generalized contractive condition in symmetric spaces. In the present paper we obtained a common fixed point theorem for occasionally weakly compatible four self- mappings in symmetric spaces.

2. Preliminaries

The following are useful in our main results which are due to [5].

Definition 2.1.Two maps S and T are said to be weakly compatible if they commute at coincidence points.

Definition 2.2. Let X be a set , u, v self maps of X. A point x in X is called a coincidence point of u and v if and only if ux = vx. We shall call w = ux = vx a point of coincidence of u and v.

Definition 2.3. Two self- maps u, v of a set X. A point x in X are said to be occasionally weakly compatible if and only if there exists appoint x in X which is a coincidence point of u and v at which u and v are commute.

Lemma 2.4. Let X be a set, u, v are occasionally weakly compatible self maps of X. If u and v have a unique point coincidence w = ux = vx , then w is a unique common fixed point of u and v.

Note: Our results are proved in symmetric spaces, which are more general than metric spaces.

Definition 2.5. Let X be a set. A symmetric on X is a mapping $\rho : X \times X \to [0, \infty)$ such that

$\rho(x, y) = 0$ if and only if $x = y$, and $\rho(x, y) = \rho(y, x)$ for $x, y \in X$.

Let $A \in [0,\infty)$, $R_A^+ = [0, A)$. Let F: $R_A^+ \to R$ satisfy

(i) F(0) = 0 and F(t) > 0 for each $t \in (0,A)$ and
(ii) F is non decreasing on R_A^+.

Define, F{0,A)={F: $R_A^+ \to R$: F satisfies (i) - (ii)}.

Let $A \in [0,\infty)$. Let ψ: $R_A^+ \to R$ satisfies

(i) $\Psi(t) < t$ for each $t \in (0, A)$ and
(ii) Ψ is non decreasing.

Define, ψ{0,A)={ψ: $R_A^+ \to R$: F satisfies (i) - (ii) above}.

Some of the examples of mappings F: $R_A^+ \to R$: F satisfies (i) - (ii) ,we have refer to [17] .

Definition 2.6. A control function Φ is defined by $\Phi: R^+ \to R^+$ which satisfies $\Phi(t) = 0$ if and only if $t = 0$.

3. Main Results

In this section, we obtained a unique common fixed point theorem for four self-mappings for occasionally weakly compatible mappings in symmetric spaces.

Theorem 3.1. Let X be a set with symmetric ρ. Let D = Sup{ ρ(u, v) : u, v ϵ X }. Suppose that

p, q, M and N are four self- mappings of X satisfying the following conditions:

(i) $F((\rho(pu, qv))) < \psi(F(K(u, v)),$

$$\text{where, } K(u, v) = \text{Max } \{ \alpha[\rho(Mu, Nv) + \rho(Mu, pu) + \rho(Nv, qv)]\}+$$

$$\text{Max } \{\beta[\rho(Mu, qv)) + \rho(Nv, pu))/2)]\}. \quad \ldots \qquad .$$

Where $\alpha, \beta > 0$ and $\alpha + \beta < 1$.

For each u ,v ϵ X, F ϵ F([0, B) and ψ ϵ ψ [0, F((B-0)), where B = D if D = ∞ and B > D if D < ∞. And

(ii) (p, M) and (q, N) are occasionally weakly compatible.

Then p, q , M and N are having a unique common fixed point in X.

Proof. Since by (ii) (p, M) and (q, N) are each occasionally weakly compatible, then there exists two points u, v ϵ X such that pu = Mu and qv = Nv. We claim that pu = qv. For otherwise from (i) we get that

$$K(u, v) = \text{Max}\{\alpha[\rho(Mu, Nv) + \rho(Mu, pu)) + \rho(Nv, qv)]\}+ \text{Max}\{ \beta[\rho(Mu, qv))+\rho(Nv, pu)/2]\},$$

$$= \text{Max}\{\alpha[\rho(Mu, Nv) + \rho(pu, pu)) + \rho(Nv, Nv)]\}+ \text{Max}\{ \beta[\rho(Mu, Nv))+\rho(Nv, Mu)/2]\},$$

$$= \text{Max}\{\alpha[\rho(Mu, Nv) ,0]\}+ \text{Max}\{\beta[\rho(Mu, Nv))]\},$$

$$= \alpha[\rho(Mu, Nv)]+ \beta[\rho(Mu, Nv))],$$

$$= (\alpha + \beta) \rho(Mu, Nv). \quad \ldots \quad (1).$$

Then by (i) and (1) we get that

$$F((\rho(pu, qv))) < \psi(F(K(u, v))$$

$$= \psi(F(\alpha + \beta) \rho(Mu, Nv)), \text{ since } \alpha + \beta < 1.$$

$$< F(\rho(Mu, Nv)) = F(\rho(pu, qv)) ,$$

$$< F(\rho(pu, qv)),$$

which is a contradiction.

Therefore , pu = qv.

That is, pu = Mu = qv = Nv.

Moreover, if there exists another point z such that pz = Mz, then using (i) and (1) we get that pz = Mz= qv= Nv or pu = pz and w = pu = Mu is the unique point of coincidence of p and

M. By Lemma (2.4) w is the only common fixed point of p and M. By symmetry there exists a unique point $z \in X$ such that $z = qz = Mz$. Suppose $w \neq z$ by (i) and (1) we get that

$$F((\rho(w, z)) = F((\rho(pw, qz)) < \psi(F(K(w, z)))$$

$$< \psi(F \rho(w, z)),$$

$$< F(\rho(w, z))$$, which is a contradiction.

Therefore w = z and w is a common fixed point. By the previous argument it is clear that w is unique. Therefore, p, q, M and N are having a unique common fixed point in X. And this completes the proof of the theorem.

Our results are more general than the results of [5].

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