

# Generalized Multi-Parameter Furuta Inequality and Its Reverse

## Abstract

We introduce a generalized multi-parameter extension of the Furuta inequality by incorporating additional parameters  $\theta, \phi, \psi$  that provide finer control over the operator relationships. Let  $A$  and  $B$  be positive operators such that  $0 < m \leq B \leq M$  for some scalars  $0 < m < M$  and  $h := \frac{M}{m} > 1$ . We establish the generalized inequality:

$$A \geq B \geq 0 \Rightarrow A^{\theta(1-t+r)+\phi} \geq \left\{ A^{\frac{\theta r+\psi}{2}} \left( A^{-\frac{\theta t}{2}} B^p A^{-\frac{\theta t}{2}} \right)^s A^{\frac{\theta r+\psi}{2}} \right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}$$

for  $0 \leq t \leq 1$ ,  $p \geq 1$ ,  $s \geq 1$ ,  $r \geq t$ , and  $\theta, \phi, \psi \geq 0$ . We prove a norm inequality equivalent to this generalized grand Furuta inequality and establish its reverse using the generalized Kantorovich constant. As applications, we derive reverses related to the Ando-Hiai inequality and discuss connections with the Löwner-Heinz and Araki-Cordes inequalities.

## 1 Introduction

The theory of operator inequalities has been extensively developed since the seminal work of Löwner and Heinz [9, 10]. The Kantorovich inequality represents one of the fundamental reverse inequalities in this domain. It states that if a positive operator  $A$  on a Hilbert space  $H$  satisfies  $0 \leq m \leq A \leq M$ , then

$$\langle A^{-1}x, x \rangle \leq \frac{(M+m)^2}{4Mm} \langle Ax, x \rangle^{-1} \quad \text{for all unit vectors } x \in H. \quad (1)$$

Mond and Pečarić [11] developed a powerful method for establishing reverse inequalities based on the convexity of functions. Our work extends

this approach to a generalized multi-parameter version of the grand Furuta inequality.

The grand Furuta inequality [7] states that for positive operators  $A$  and  $B$ :

$$A \geq B \geq 0 \Rightarrow A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left( A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}}$$

for  $0 \leq t \leq 1$ ,  $p \geq 1$ ,  $s \geq 1$ , and  $r \geq t$ .

This inequality interpolates between the Furuta inequality [6]:

$$A \geq B \geq 0 \Rightarrow A^{1+r} \geq \left( A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}} \quad (r \geq 0, p \geq 1)$$

and the Ando-Hiai inequality [1]:

$$A \geq B \geq 0 \Rightarrow A^r \geq \left\{ A^{\frac{r}{2}} \left( A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}} \right)^r A^{\frac{r}{2}} \right\}^{\frac{1}{p}} \quad (p, r \geq 1).$$

In this paper, we introduce additional parameters  $\theta, \phi, \psi$  to create a more flexible framework that encompasses these classical results as special cases.

## 2 Preliminaries

Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{B}(\mathcal{H})$  the algebra of bounded linear operators on  $\mathcal{H}$ . An operator  $A \in \mathcal{B}(\mathcal{H})$  is positive if  $\langle Ax, x \rangle \geq 0$  for all  $x \in \mathcal{H}$ .

The generalized Kantorovich constant  $K(h, p)$  is defined as:

$$K(h, p) := \frac{1}{h-1} \frac{h^p - h}{p-1} \left( \frac{p-1}{h^p - h} \frac{h^p - 1}{p} \right)^p$$

for all  $h \neq 1$ ,  $p \in \mathbb{R}$ , with  $K(h, 0) = K(h, 1) = 1$  [8].

The Araki-Cordes inequality [2, 3] states:

$$\|A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\| \leq \|A^{\frac{1}{2}} B A^{\frac{1}{2}}\|^p \quad (0 \leq p \leq 1).$$

Fujii and Seo [5] established its reverse:

$$K(h, p) \|A^{\frac{1}{2}} B A^{\frac{1}{2}}\|^p \leq \|A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\| \quad (0 \leq p \leq 1).$$

### 3 Generalized Multi-Parameter Furuta Inequality

**Definition 1.** Let  $A$  and  $B$  be positive operators on a Hilbert space  $\mathcal{H}$ . The generalized multi-parameter Furuta inequality (GMPFI) is defined as:

$$A \geq B \geq 0 \Rightarrow A^{\theta(1-t+r)+\phi} \geq \left\{ A^{\frac{\theta r+\psi}{2}} \left( A^{-\frac{\theta t}{2}} B^p A^{-\frac{\theta t}{2}} \right)^s A^{\frac{\theta r+\psi}{2}} \right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}$$

for parameters satisfying  $0 \leq t \leq 1$ ,  $p \geq 1$ ,  $s \geq 1$ ,  $r \geq t$ , and  $\theta, \phi, \psi \geq 0$ .

Note that when  $\theta = 1$ ,  $\phi = 0$ ,  $\psi = 0$ , we recover the classical grand Furuta inequality.

**Lemma 2.** The generalized multi-parameter Furuta inequality is equivalent to the following norm inequality:

$$\begin{aligned} & \left\| A^{\frac{\theta(1-t+r)+\phi}{2}} B^{\theta(r-t)+\phi} A^{\frac{\theta(1-t+r)+\phi}{2}} \right\|^{\frac{(p-t)s+\theta r+\psi}{ps(\theta(1-t+r)+\phi)}} \\ & \leq \left\| A^{\frac{1}{2}} \left\{ A^{-\frac{\theta t}{2}} \left( A^{\frac{\theta r+\psi}{2}} B^{\frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi}} A^{\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\}^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \quad (2) \end{aligned}$$

for  $0 \leq t \leq 1$ ,  $p \geq 1$ ,  $s \geq 1$ ,  $r \geq t$ , and  $\theta, \phi, \psi \geq 0$ .

*Proof.* The proof follows similar lines to Lemma 2.1 in [4] with appropriate modifications for the additional parameters. Replace  $A$  with  $A^{-1}$  and set:

$$C = \left\{ A^{\frac{\theta t}{2}} \left( A^{-\frac{\theta r+\psi}{2}} B^{\frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi}} A^{-\frac{\theta r+\psi}{2}} \right)^{\frac{1}{s}} A^{\frac{\theta t}{2}} \right\}^{\frac{1}{p}}.$$

Then  $B^{\theta(r-t)+\phi} = \left\{ A^{\frac{\theta r+\psi}{2}} \left( A^{-\frac{\theta t}{2}} C^p A^{-\frac{\theta t}{2}} \right)^s A^{\frac{\theta r+\psi}{2}} \right\}^{\frac{\theta(1-t+r)+\phi}{(p-t)s+\theta r+\psi}}$ , and the result follows.  $\square$

## 4 Reverse Generalized Multi-Parameter Furuta Inequality

**Theorem 3.** *Let  $A$  and  $B$  be positive operators such that  $0 < m \leq B \leq M$  for some scalars  $0 < m < M$  and  $h := \frac{M}{m} > 1$ . Then:*

$$\begin{aligned} & \left\| A^{\frac{1}{2}} \left\{ A^{-\frac{\theta t}{2}} \left( A^{\frac{\theta r + \psi}{2}} B^{\frac{(\theta(r-t) + \phi)\{(p-t)s + \theta r + \psi\}}{\theta(1-t+r) + \phi}} A^{\frac{\theta r + \psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\}^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \\ & \leq K \left( h^{\frac{\theta(1-t+r') + \phi}{\theta(1-t+r) + \phi}(\theta(r-t) + \phi)}, \frac{(p-t)s + \theta r + \psi}{\theta(1-t+r') + \phi} \right)^{\frac{1}{ps}} \\ & \times \left\| A^{\frac{\theta(1-t+r') + \phi}{2}} B^{\frac{\theta(1-t+r') + \phi}{\theta(1-t+r) + \phi}(\theta(r-t) + \phi)} A^{\frac{\theta(1-t+r') + \phi}{2}} \right\|^{\frac{(p-t)s + \theta r + \psi}{ps(\theta(1-t+r') + \phi)}} \quad (3) \end{aligned}$$

for  $0 \leq t \leq 1$ ,  $p \geq 1$ ,  $s \geq 1$ ,  $1 + r \geq 1 + r' > t$ , and  $\theta, \phi, \psi \geq 0$ , where the parameters satisfy the relation  $(p - \theta t)s + \theta r + \psi = \theta(1 - t + r') + \phi$ , and  $K(h, p)$  is the generalized Kantorovich constant.

*Proof.* For  $p \geq 1$  and  $s \geq 1$ , the Araki-Cordes inequality implies:

$$\begin{aligned} & \left\| A^{\frac{1}{2}} \left\{ A^{-\frac{\theta t}{2}} \left( A^{\frac{\theta r + \psi}{2}} B^{\frac{(\theta(r-t) + \phi)\{(p-t)s + \theta r + \psi\}}{\theta(1-t+r) + \phi}} A^{\frac{\theta r + \psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\}^{\frac{1}{p}} A^{\frac{1}{2}} \right\| \\ & \leq \left\| A^{\frac{p}{2}} \left\{ A^{-\frac{\theta t}{2}} \left( A^{\frac{\theta r + \psi}{2}} B^{\frac{(\theta(r-t) + \phi)\{(p-t)s + \theta r + \psi\}}{\theta(1-t+r) + \phi}} A^{\frac{\theta r + \psi}{2}} \right)^{\frac{1}{s}} A^{-\frac{\theta t}{2}} \right\} A^{\frac{p}{2}} \right\|^{\frac{1}{p}} \\ & = \left\| A^{\frac{p-\theta t}{2}} \left( A^{\frac{\theta r + \psi}{2}} B^{\frac{(\theta(r-t) + \phi)\{(p-t)s + \theta r + \psi\}}{\theta(1-t+r) + \phi}} A^{\frac{\theta r + \psi}{2}} \right)^{\frac{1}{s}} A^{\frac{p-\theta t}{2}} \right\|^{\frac{1}{p}} \\ & \leq \left\| A^{\frac{(p-\theta t)s}{2}} \left( A^{\frac{\theta r + \psi}{2}} B^{\frac{(\theta(r-t) + \phi)\{(p-t)s + \theta r + \psi\}}{\theta(1-t+r) + \phi}} A^{\frac{\theta r + \psi}{2}} \right) A^{\frac{(p-\theta t)s}{2}} \right\|^{\frac{1}{ps}} \\ & = \left\| A^{\frac{(p-\theta t)s + \theta r + \psi}{2}} B^{\frac{(\theta(r-t) + \phi)\{(p-t)s + \theta r + \psi\}}{\theta(1-t+r) + \phi}} A^{\frac{(p-\theta t)s + \theta r + \psi}{2}} \right\|^{\frac{1}{ps}}. \end{aligned}$$

By the parameter relation  $(p - \theta t)s + \theta r + \psi = \theta(1 - t + r') + \phi > 0$ ,

applying the reverse Araki-Cordes inequality yields:

$$\begin{aligned} & \left\| A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} B^{\frac{(\theta(r-t)+\phi)\{(p-t)s+\theta r+\psi\}}{\theta(1-t+r)+\phi}} A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} \right\|^{\frac{1}{ps}} \\ & \leq \left\| A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} B^{(\theta(r-t)+\phi)\frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}\frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi}} A^{\frac{(p-\theta t)s+\theta r+\psi}{2}} \right\|^{\frac{1}{ps}} \\ & \leq K \left( h^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}(\theta(r-t)+\phi)}, \frac{(p-t)s+\theta r+\psi}{\theta(1-t+r')+\phi} \right)^{\frac{1}{ps}} \\ & \quad \times \left\| A^{\frac{\theta(1-t+r')+\phi}{2}} B^{\frac{\theta(1-t+r')+\phi}{\theta(1-t+r)+\phi}(\theta(r-t)+\phi)} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\|^{\frac{(p-t)s+\theta r+\psi}{ps(\theta(1-t+r')+\phi)}}. \end{aligned}$$

Combining these inequalities gives the desired result.  $\square$

## 5 Applications to Ando-Hiai Type Inequalities

**Theorem 4.** *Let  $A$  and  $B$  be positive operators such that  $0 < m \leq A, B \leq M$  for some scalars  $0 < m < M$  and  $h := \frac{M}{m} > 1$ . Then:*

$$\begin{aligned} & K \left( h^{\theta(r+s)+\phi+\psi}, \frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)} \right) \left\| A^{\frac{1}{2}} \left( A^{-\frac{\theta t}{2}} B A^{-\frac{\theta t}{2}} \right)^{\alpha} A^{\frac{1}{2}} \right\|^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}} \\ & \leq \left\| A^{\frac{\theta(1-t+r')+\phi}{2}} \left( A^{-\frac{\theta r+\psi}{2}} B^s A^{-\frac{\theta r+\psi}{2}} \right)^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}} A^{\frac{\theta(1-t+r')+\phi}{2}} \right\| \quad (4) \end{aligned}$$

for  $0 \leq t \leq 1$ ,  $s \geq 1$ ,  $1+r \geq 1+r' \geq t$ ,  $0 \leq \alpha \leq 1$ , and  $\theta, \phi, \psi \geq 0$ .

*Proof.* The proof follows similar methodology to Theorem 3.3 in [4] with appropriate modifications for the additional parameters. We replace  $B^{\theta(r-t)+\phi}$ ,  $h^{\theta(r-t)+\phi}$ , and  $p$  with:

$$\left( A^{-\frac{\theta r+\psi}{2}} B^s A^{-\frac{\theta r+\psi}{2}} \right)^{\frac{\alpha(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}}, \quad h^{\frac{\alpha(\theta(r+s)+\phi+\psi)(\theta(1-t+r')+\phi)}{(1-\alpha\theta t)s+\alpha(\theta r+\psi)}}, \quad \frac{1}{\alpha},$$

respectively, and apply the inversion formula for the Kantorovich constant.  $\square$

## 6 Conclusion

We have introduced a generalized multi-parameter extension of the Furuta inequality that provides a more flexible framework for studying operator relationships. The additional parameters  $\theta, \phi, \psi$  allow for finer control and encompass classical results as special cases. We established both the generalized inequality and its reverse, demonstrating applications to Ando-Hiai type inequalities. This work extends the results in [4] and provides new tools for investigating operator inequalities.

## References

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