On (α, β)-*Class* (Q_n) Operators

Abstract

In this paper, we introduce a new class of operators, called the class of (α, β)-Class (Q_n) operators acting on a complex Hilbert space $\mathcal{H}$. For an integer $n \geq 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to belong to the (α, β)-Class (Q_n) if $\alpha^n T^{*n} T^n \leq (TT)^n \leq \beta^n T^n T^{*n}$, for scalars $0 \leq \alpha \leq 1 \leq \beta$. This definition extends the earlier notion of (α, β)-Class (Q) operators, which corresponds to the case $n = 2$. We investigate several fundamental properties of this generalized class through rigorous operator-theoretic analysis, employing techniques involving polar decompositions, unitary conjugations, and commutativity conditions. A key finding establishes that every (α, β)-normal operator automatically belongs to the (α, β)-Class (Q_n) for all integers $n \geq 1$, with the same coefficients α^2 and β^2 appearing uniformly across all values of n. Additional results demonstrate closure properties under scalar multiplication, unitary equivalence, and specific composition operations, while also characterizing the behavior of adjoints under polar decomposition conditions. These findings provide a comprehensive foundation for understanding the structural properties and operator-theoretic consequences that arise from this natural extension of the classical framework.

Keywords: Class (Q), Normal, (α, β)-normal, Hypernormal and (α, β)-Class (Q) operators.

Introduction

Throughout this paper, let H denote a complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators on an infinite-dimensional separable Hilbert space H. In recent years, the study of normal operators has been extensively developed and generalized by relaxing certain conditions of normality and introducing broader operator classes. For instance, the notion of (α, β)-normal operators was investigated in [5], and this was further extended to the class of p-(α, β)-normal operators in [2]. The Class (Q) of operators was introduced in [4], where several algebraic properties of this class were established. This concept was later refined to the (α, β)-Class (Q) operators in [6], where it was shown, among other things, that the class is closed under scalar multiplication. More recently, the framework has been expanded to include m-quasi-(α, β)-Class (Q) operators [1]. Motivated by these developments, in this paper we extend the concept of (α, β)-Class (Q) operators to a more general family, namely the (α, β)-Class (Q_n) operators, defined for arbitrary integers $n \geq 1$.

Definition 0.1. An operator $T \in B(H)$ is said to be:

1. Class (Q) if $T^{*2}T^2 = (T^*T)^2$.
2. (α, β)-normal if $\beta^2 T^*T \geq TT^* \geq \alpha^2 T^*T$.
3. Normal if $T^*T = TT^*$.
4. n-perinormal if $T^{*n}T^n \geq (T^*T)^n$.
5. (α, β)-Class (Q) operator if $\alpha^2 T^{*2}T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2}T^2$. If $\beta = 1$, we observe from the right inequality that this class coincides with the class of 2-perinormal operators [3].
6. (α, β)-Class (Q_n) if $\alpha^n T^{*n}T^n \leq (T^*T)^n \leq \beta^n T^{*n}T^n$, for scalars $0 \leq \alpha \leq 1 \leq \beta$. We observe that this reduces to (α, β)-Class (Q) whenever n=2 .

Main Results

Theorem 0.2. *Let $T \in B(H)$ and let $n \geq 1$ be an integer. If*

$$T \in (\alpha, \beta)\text{-Class } (Q_n), \quad i.e. \quad \alpha^n T^{*n} T^n \leq (T^*T)^n \leq \beta^n T^{*n} T^n,$$

for scalars $0 \leq \alpha \leq 1 \leq \beta$, then:

1. *$\lambda T \in (\alpha, \beta)$-Class (Q_n) for every real λ (indeed for every $\lambda \in \mathbb{C}$).*
2. *If $S \in B(H)$ is unitarily equivalent to T (so $S = UTU^*$ for some unitary U), then $S \in (\alpha, \beta)$-Class (Q_n).*

Proof. (1) Let $\lambda \in \mathbb{R}$ (the same argument works for $\lambda \in \mathbb{C}$ using $|\lambda|$). Compute

$$(\lambda T)^{*n} (\lambda T)^n = \overline{\lambda}^n \lambda^n \, T^{*n} T^n = |\lambda|^{2n} T^{*n} T^n,$$

and

$$\big((\lambda T)^* (\lambda T)\big)^n = \big(|\lambda|^2 T^* T\big)^n = |\lambda|^{2n} (T^*T)^n.$$

Multiplying the assumed inequality

$$\alpha^n T^{*n} T^n \leq (T^*T)^n \leq \beta^n T^{*n} T^n$$

by the positive scalar $|\lambda|^{2n}$ yields

$$\alpha^n |\lambda|^{2n} T^{*n} T^n \leq |\lambda|^{2n} (T^*T)^n \leq \beta^n |\lambda|^{2n} T^{*n} T^n,$$

which is precisely

$$\alpha^n (\lambda T)^{*n} (\lambda T)^n \leq \big((\lambda T)^* (\lambda T)\big)^n \leq \beta^n (\lambda T)^{*n} (\lambda T)^n.$$

Hence $\lambda T \in (\alpha, \beta)$-Class (Q_n).

(2) Let $S = UTU^*$ with U unitary. Note first that

$$S^* = UT^*U^*, \qquad S^{*n} = UT^{*n}U^*, \qquad S^n = UT^nU^*,$$

so

$$S^{*n} S^n = UT^{*n} T^n U^*, \qquad (S^*S)^n = U(T^*T)^n U^*.$$

Conjugating the assumed inequality by U (i.e. applying $X \mapsto UXU^*$) gives

$$\alpha^n UT^{*n} T^n U^* \leq U(T^*T)^n U^* \leq \beta^n UT^{*n} T^n U^*,$$

which is exactly

$$\alpha^n S^{*n} S^n \leq (S^*S)^n \leq \beta^n S^{*n} S^n.$$

Thus $S \in (\alpha, \beta)$-Class (Q_n). □

Theorem 0.3. *Let $T \in B(H)$ and let $n \geq 1$ be an integer. If T is (α, β)–normal, i.e.*

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$$

for scalars $0 \leq \alpha \leq 1 \leq \beta$, then for every integer $n \geq 1$ we have

$$\alpha^2 T^{*n} T^n \leq (T^*T)^n \leq \beta^2 T^{*n} T^n,$$

that is, $T \in (\alpha, \beta)$-Class (Q_n) (with the same coefficients α^2, β^2 appearing for all n).

Proof. Assume

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T.$$

Fix an integer $n \geq 1$. We shall conjugate the inequality by suitable powers of T and T^*.

First we prove the identity

$$T^{*(k-1)}(TT^*)T^{k-1} = (T^*T)^k \qquad \text{for every } k \geq 1. \tag{1}$$

The identity holds for $k = 1$ since both sides equal TT^*. Assume it holds for some $k \geq 1$. Left multiplying by T^* and right multiplying by T we obtain

$$T^{*k}(TT^*)T^k = T^*\big(T^{*(k-1)}(TT^*)T^{k-1}\big)T = T^*(T^*T)^k T = (T^*T)^{k+1},$$

so the identity holds for $k+1$. Hence (1) holds for all $k \geq 1$.

Conjugating the assumed inequality by $T^{*(n-1)}$ on the left and by T^{n-1} on the right. Because the map $X \mapsto T^{*(n-1)}XT^{n-1}$ preserves the order for positive operators, we get

$$\alpha^2\, T^{*(n-1)}(T^*T)T^{n-1} \;\leq\; T^{*(n-1)}(TT^*)T^{n-1} \;\leq\; \beta^2\, T^{*(n-1)}(T^*T)T^{n-1}.$$

But

$$T^{*(n-1)}(T^*T)T^{n-1} = T^{*n}T^n,$$

and by (1)

$$T^{*(n-1)}(TT^*)T^{n-1} = (T^*T)^n.$$

Substituting these equalities into the conjugated inequality yields

$$\alpha^2 T^{*n}T^n \leq (T^*T)^n \leq \beta^2 T^{*n}T^n,$$

. Thus $T \in (\alpha, \beta)$-Class (Q_n) for every $n \geq 1$. □

Theorem 0.4. *Let $T \in B(H)$ and let $n \geq 1$ be an integer. Assume T belongs to the (α, β)–Class (Q). Suppose further that T admits a polar decomposition $T = U|T|$ with U unitary. Then T^* belongs to the (α, β)–Class (Q_n) for every integer $n \geq 1$,*

Proof. Let $T = U|T|$ with U unitary and $|T| = (T^*T)^{1/2}$. Since U is unitary we have the conjugation identities

$$TT^* = U|T|^2U^*, \qquad (TT^*)^n = U|T|^{2n}U^*,$$

and, using $T^{*n}T^n = |T|^{2n}$,

$$T^nT^{*n} = U|T|^{2n}U^*.$$

by assumption and the positivity of operator T^*T, we obtain for every integer $n \geq 1$

$$\alpha^n T^{*n}T^n \leq (T^*T)^n \leq \beta^n T^{*n}T^n,$$

Conjugating this inequality by the unitary U yields

$$\alpha^n UT^{*n}T^nU^* \leq U(T^*T)^nU^* \leq \beta^n UT^{*n}T^nU^*.$$

Using $UT^{*n}T^nU^* = T^nT^{*n}$ and $U(T^*T)^nU^* = (TT^*)^n$ we get

$$\alpha^n T^nT^{*n} \leq (TT^*)^n \leq \beta^n T^nT^{*n},$$

implying that $T^* \in (\alpha, \beta)$-Class (Q_n). □

Theorem 0.5. *Let $T \in B(H)$ and let $n \geq 1$ be an integer. If T belongs to the (α, β)-Class (Q_n) and P is a unitary operator on H such that $TP = PT$, then the operator $K := TP$ also belongs to the (α, β)-Class (Q_n).*

Proof. Since P is unitary and commutes with T, it also commutes with T^* and with all powers of T and T^*. In particular, for every integer $k \geq 1$,

$$PT^k = T^kP, \qquad PT^{*k} = T^{*k}P, \qquad P^*T^k = T^kP^*, \qquad P^*T^{*k} = T^{*k}P^*.$$

computing the powers of K and K^*:

$$K^n = (TP)^n = T^nP^n, \qquad K^{*n} = (TP)^{*n} = (P^*T^*)^n = P^{*n}T^{*n},$$

hence ;

$$K^{*n}K^n = P^{*n}T^{*n}T^nP^n = P^{*n}\big(T^{*n}T^n\big)P^n,$$

and

$$(K^*K)^n = \big((TP)^*(TP)\big)^n = \big(P^*T^*TP\big)^n = P^{*n}(T^*T)^nP^n,$$

conjugating the assumed inequality

$$\alpha^nT^{*n}T^n \leq (T^*T)^n \leq \beta^nT^{*n}T^n$$

by $P^{*n}P^n$ we obtain

$$\alpha^nP^{*n}T^{*n}T^nP^n \leq P^{*n}(T^*T)^nP^n \leq \beta^nP^{*n}T^{*n}T^nP^n.$$

Using the identities above for $K^{*n}K^n$ and $(K^*K)^n$ this becomes

$$\alpha^nK^{*n}K^n \leq (K^*K)^n \leq \beta^nK^{*n}K^n,$$

implying that $K \in (\alpha, \beta)$-Class (Q_n). □

Theorem 0.6. *Let $S, T \in B(H)$ and let $n \geq 1$ be an integer. Assume that S and T* * commute, *i.e.*

$$ST = TS, \quad S^*T = TS^*, \quad ST^* = T^*S,$$

. If both S and T belong to the (α, β)–Class (Q_n), then the product ST also belongs to (α, β)–Class (Q_n)

Proof. Because of the *-commutation hypothesis, every pair drawn from $\{S, S^*, T, T^*\}$ commutes. In particular the positive operators S^*S and T^*T commute, hence so do their powers; therefore

$$\big((ST)^*(ST)\big)^n = \big(T^*S^*ST\big)^n = (T^*)^n(S^*S)^nT^n = (T^*T)^n(S^*S)^n,$$

. Similarly, using commutativity of the adjoint-powers,

$$(ST)^{*n}(ST)^n = (T^{*n}S^{*n})(S^nT^n) = T^{*n}(S^{*n}S^n)T^n = (T^{*n}T^n)(S^{*n}S^n).$$

Hence ;

$$\alpha^nS^{*n}S^n \leq (S^*S)^n \leq \beta^nS^{*n}S^n$$

and

$$\alpha^nT^{*n}T^n \leq (T^*T)^n \leq \beta^nT^{*n}T^n,$$

and using the fact that $(S^{*n}S^n)$ commutes with $(T^{*n}T^n)$ and likewise for $(S^*S)^n$ and $(T^*T)^n$, we may multiply the left–hand inequalities and the right–hand inequalities to obtain

$$\alpha^n(T^{*n}T^n)(S^{*n}S^n) \;\leq\; (T^*T)^n(S^*S)^n \;\leq\; \beta^n(T^{*n}T^n)(S^{*n}S^n).$$

Using the identities noted above this is exactly

$$\alpha^n(ST)^{*n}(ST)^n \leq \big((ST)^*(ST)\big)^n \leq \beta^n(ST)^{*n}(ST)^n,$$

which proves that $ST \in (\alpha, \beta)$-Class (Q_n). □

Conclusion

In this paper, we have successfully extended the theory of (α, β)-Class (Q) operators to a more general framework by introducing the class of (α, β)-Class (Q_n) operators for arbitrary positive integers $n \geq 1$. The main theoretical contributions include establishing fundamental properties that parallel and generalize known results for the $n = 2$ case.

The key findings demonstrate that the (α, β)-Class (Q_n) operators exhibit robust structural properties. Specifically, we have shown that this class is closed under scalar multiplication and unitary equivalence, providing essential algebraic stability. The relationship between (α, β)-normal operators and (α, β)-Class (Q_n) operators has been clarified, with every (α, β)-normal operator belonging to (α, β)-Class (Q_n) for all $n \geq 1$ using the same coefficients α^2 and β^2.

Furthermore, we established conditions under which the adjoint of an operator in (α, β)-Class (Q) belongs to (α, β)-Class (Q_n) when the operator admits a unitary polar decomposition. The behaviour under composition with commuting unitary operators and the multiplicative properties for $*$-commuting operators extend the utility of this operator class in applications involving operator products and transformations.

These results provide a solid foundation for future investigations into the spectral properties, invariant subspaces, and approximation theory aspects of (α, β)-Class (Q_n) operators. The generalization from $n = 2$ to arbitrary n opens new avenues for studying higher-order relationships between T^nT^n and $(TT)^n$, which may yield insights into the fine structure of non-normal operators and their generalizations.

The systematic approach developed here also suggests that further extensions to related operator classes, such as m-quasi-(α, β)-Class (Q_n) operators, may be fruitful areas for continued research in operator theory.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] AL-Ahmadi, A. Mohammed Ayed, N. Maqbul Saqer Alruwaili, and S.Ahmed Ould Ahmed Magmoud , Class of Operators Related to (α, β)-Class (Q) Operators." Bulletin of Mathematical Analysis and Applications 15.4 (2023): 12-20.

[2] D. Senthilkumar, On p-(α, β)-Normal Operators, *Applied Mathematical Sciences*, **8**(2014), 2041-2052.

[3] Hongliang Zuo and Fei Zuo, A note on n-perinormal operators, *Acta Mathematica Scientia*, **34B**(2014), 194-198.

[4] A. A. Jibril, On n-power normal operators, *The Arabian Journal for Science and Engineering*, **33**(2008), 247-251.

[5] E. Rasoul, M. Farzollah and Ali Morassaei, More on (α, β)-Normal operators in Hilbert spaces, *Abstract and Applied Analysis*, **2012**(2012).

[6] W.Victor, and A. M. Nyongesa, On (α, β)-Class (Q) Operators, Int. J. Math. And Appl.,9(2)(2021), 111–113.