A short note on approximation properties of modified Post-Widder operators preserving exponential function

ABSTRACT

In the present article, we introduce the modified form of the Post-Widder operators preserving the test functione^{Ax}. We discuss the weighted approximation properties for the modified operators and after that, we also investigate direct quantitative estimate for these modified operators.

Keywords: Post-Widder operators, linear positive operators, weighted approximation, quantitative estimate.

1. INTRODUCTION

For the better approximation of linear positive operators, in the recent years, some convergence estimates were discussed by many researchers in [2, 6, 10, 11, 12, 13 and 14] etc.

1.1 Post-Widder operators

Post-Widder operators are defined as follows, for $n \in N, x > 0$

$$M_n(f,x) := \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1} \int_0^\infty u^n e^{-\frac{nu}{x}} f(u)du.$$

We consider $f(u) = e^{\theta u}$, $\theta \in R$, then for $f \in C(0,\infty)$, according to [7] we write

$$M_n\left(e^{\theta u},x\right) = \left(1 - \frac{x\theta}{n}\right)^{-(n+1)}. \quad \textbf{(1.1)}$$

The convergence of Post-Widder operators was investigated in [18]. The saturation and inverse results of these operators were studied by Rathore and Singh [10]. Other approximation properties of these operators were discussed by many authors in [3, 5, 8, 15 and 16].

In 1984, authors in [9] represented the function as the Post-Widder inversion operators of the generalized function.

Now considering that the operators (1.1) present the test function e^{Ax}, we have

$$\widetilde{M}_n(f,x) := \frac{1}{n!}\left(\frac{n}{a_n(x)}\right)^{n+1} \int_0^\infty u^n e^{-\frac{nu}{a_n(x)}} f(u)du. \quad \textbf{(1.2)}$$

Using (1.1), we have

$$\widetilde{M}_n(e^{Au},x) = e^{Ax} = \left(1-\frac{a_n(x)A}{n}\right)^{-(n+1)},$$

noting

$$a_n(x) = \frac{n}{A}\left(1-e^{-A\ /(n+1)}\right).$$

Hence, we write modified operators $\widetilde{M}_n$ as follows

$$\widetilde{M}_n(f,x) = \frac{1}{n!}\left[\frac{A}{\left(1-e^{-\frac{Ax}{n+1}}\right)}\right]^{(n+1)} \int_0^\infty u^n e^{-\frac{nA}{\left(1-e^{-\frac{Ax}{n+1}}\right)}} f(u)du, \quad \textbf{(1.3)}$$

where $x \in (0,\infty)$ and $\widetilde{M}_n(f,0) = f(0)$, preserving constant and the test function e^{Ax}.

2. BASIC RESULTS

Here, we obtain some lemmas essential for the proof of the main theorem.

Lemma 2.1. For $\theta > 0$, we have

$$\widetilde{M}_n(e^{\theta u},x) = \left(1-\frac{\left(1-e^{-Ax/(n+1)}\right)\theta}{A}\right)^{-(n+1)}.$$

It may be observed that $\widetilde{M}_n(e^{\theta u},x)$ can be written moment generating function and can be used in obtaining the moments of (1.3)

Let $U_k^{\widetilde{M}_n}(x) = \widetilde{M}_n(e_k,x)$, where $e_k(u) = u^k, k \in N \cup \{0\}$. The moments are given by

$$U_k^{\widetilde{M}_n}(x) = \left[\frac{\partial^k}{\partial\theta^k}\widetilde{M}_n(e^{\theta u},x)\right]_{\theta=0}$$

$$= \left[\frac{\partial^k}{\partial\theta^k}\left\{\left(1-\frac{(1-e^{-Ax/(n+1)})\theta}{A}\right)^{-(n+1)}\right\}\right]_{\theta=0}.$$

Some moments are given as

$$U_0^{\widetilde{M}_n}(x) = 1$$

$$U_1^{\widetilde{M}_n}(x) = \frac{(n+1)}{A}\left(1 - e^{-A\ /(n+1)}\right),$$

$$U_2^{\widetilde{M}_n}(x) = \frac{(n+1)(n+2)}{A^2}\left(1 - e^{-\frac{Ax}{n+1}}\right)^2.$$

Lemma 2.2. Here we find the moments of arbitrary order, satisfying the following

$$U_r^{\widetilde{M}_n}(x) = \frac{(n+1)_r}{A^r}\left(1 - e^{-Ax/(n+1)}\right)^r, r = 0,1, \ldots$$

Where the Pochhammer symbol is given by

$$(d)_0 = 1, (d)_r = d(d+1)\cdots(d+r-1).$$

We have the following lemma by using linearity property and Lemma 2.2,

Lemma 2.3. The central moments $\mu_k^{\widetilde{M}_n}(x) = \widetilde{M}_n((u-x)^k, x)$ are given below:

$$\mu_r^{\widetilde{M}_n}(x) = \sum_{v=0}^{r} (-1)^{r-v}\binom{r}{v} x^{r-v}\left(1 - e^{-Ax/(n+1)}\right)^v \frac{(n+1)_v}{A^v}, r = 0,1, \ldots.$$

For $n \in N$, we have

$$\mu_1^{\widetilde{M}_n}(x) = \frac{(n+1)}{A}\left(1 - e^{-A\ /(n+1)} - 1\right) - x$$

$$\mu_2^{\widetilde{M}_n}(x) = \frac{(n+1)(n+2)}{A^2}\left(1 - e^{-A\ /(n+1)}\right)^2 + x^2 - 2x\frac{(n+1)}{A}\left(1 - e^{-A\ /(n+1)}\right).$$

Lemma 2.4. The central moments$\mu_{2r}^{\widetilde{M}_n}(x) = \widetilde{M}_n((u-x)^{2r}, x)$, we have

$$\mu_{2r}^{\widetilde{M}_n}(x) = O(n^{-r}), n \to \infty, r = 1,2,3, \cdots$$

Proof. We analyze

$$\widetilde{M}_n(f, x) = M_n(f, \alpha_n(x)),$$

where

$$a_n(x) = \frac{n}{A}\left(1 - e^{-Ax/(n+1)}\right).$$

It can be seen that $j > 1 - e^{-j} > j - \frac{j^2}{2}$ for $j \in [0, \infty)$. We set $j = Ax/(n+1)$ and get

$$x\left(\frac{n}{n+1}\right) > \alpha_n(x) > x\left(\frac{n}{n+1}\right) - \left(\frac{Ax}{n+1}\right)^2 \cdot \frac{n}{2A}.$$

Therefore

$\frac{x}{n+1} < x - \alpha_n(x) < \frac{x}{n+1} + \frac{Ax^2 n}{2(n+1)^2} = O(n^{-1})$, for $x \in [0, \infty)$.

Hence

$$\widetilde{M}_n((u-x)^{2r}, x) = M_n((u-x)^{2r}, \alpha_n(x))$$

$$= M_n((u - \alpha_n(x) + \alpha_n(x) - x)^{2r}, \alpha_n(x))$$

$$\leq K(r) M_n((u - \alpha_n(x))^{2r}, \alpha_n(x)) + M_n\big((x - \alpha_n(x)^{2r}, \alpha_n(x)\big)$$

$$\leq K(\mathrm{r}) \cdot \frac{1}{n^r} + \big(x - \alpha_n(x)\big)^{2r} = O(n^{-r}).$$

Hence the proof of Lemma 2.4.

3. MAIN RESULTS

Definition 3.1 (WEIGHTED APPROXIMATION)

Here we observe the behavior of the operators on some weighted spaces.

Take $\psi(x) = 1 + e^{Ax}, x$ is real and positive. Considering the following weighted spaces

$$G_\psi(R^+) = \{f: R^+ \to R: | \, f(x) \leq K_1(1 + e^{Ax})\},$$

$$K_\psi(R^+) = G_\psi(R^+) \cap K(R^+)$$

$K_\psi^r(R^+) = \left\{ f \in K_\psi(R^+): \lim_{x\to\infty} \frac{f(x)}{1+e^{Ax}} = K_2 < \infty \right\}$,

where K_1, K_2 are constants which depend on f. The norm is defined as

$$\| f \|_\psi = \sup_{x \in R^+} \frac{|f(x)|}{1 + e^{Ax}}.$$

Theorem 3.2. For each $f \in K_\psi^r(R^+)$, we have

$$\lim_{n\to\infty} \|\widetilde{M}_n f - f\|_\psi = 0.$$

Proof [1]**:** In order to get the proof, we have to prove

$$\lim_{n\to\infty} \|\widetilde{M}_n\big(e^{iAu/2}\big) - e^{iAx/2}\|_\psi = 0, i = 0,1,2$$

The result is true for $i = 0$ and $i = 2$. To verify it for $i = 1$, by Lemma 2.1 we have

$\| \widetilde{M}_n\left(e^{Au/2}\right) - e^{Ax/2}\|_{\psi}$=

$$\sup_{x\in R^+} \frac{\left|\left(1-\frac{\left(1-e^{-Ax/(n+1)}\right)}{2}\right)^{-(n+1)} - e^{Ax/2}\right|}{1+e^{Ax}}$$

$$= \sup_{x\in R^+} \frac{\left|\left(1+e^{-Ax/(n+1)}\right)^{-(n+1)}2^{n+1} - e^{Ax/2}\right|}{1+e^{Ax}}$$

$$= \sup_{x\in R^+} \frac{\left|e^{Ax}\left(1+e^{Ax/(n+1)}\right)^{-(n+1)}2^{n+1} - e^{Ax/2}\right|}{1+e^{Ax}}$$

$$= \sup_{x\in R^+} \left[\frac{e^{Ax}}{1+e^{Ax}}\right] \left|\left(\frac{2}{1+e^{Ax/(n+1)}}\right)^{n+1} - e^{Ax/2}\right|. \qquad \textbf{(3.1)}$$

Clearly $\frac{e^{Ax}}{1+e^{Ax}} \in \left[\frac{1}{2}, 1\right), A > 0, x > 0.$

We write $u = e^{Ax/2}, u \in [1,\infty)$ for $x \in (0,\infty)$. Then from (3.1)

$$\left|\left(\frac{2}{1+u^{2/(n+1)}}\right)^{n+1} - u^{-1}\right| = u^{-1}\left|\left(\frac{2u^{1/(n+1)}}{1+u^{2/(n+1)}}\right)^{n+1} - 1\right| = g(u). \qquad \textbf{(3.2)}$$

In equation (3.2), we set $u^{1/(n+1)} = j \in [1,\infty)$. Hence

$$g(u) = h(j) = j^{-(n+1)}\left|\left(\frac{2j}{1+j^2}\right)^{n+1} - 1\right|$$

$$= \left|\left(\frac{2}{1+j^2}\right)^{n+1} - j^{-(n+1)}\right|$$

$$= j^{-(n+1)} - \left(\frac{2}{1+j^2}\right)^{n+1}. \qquad \textbf{(3.3)}$$

We set $h(1) = 0, h(+\infty) = \lim_{j\to\infty} h(j) = 0$. To obtain the global maxima of $h(j)$ we solve the equation $h'(j) = 0$. Simple computations give that $h'(j_0) = 0$ for j_0 satisfying the equation

$$\frac{2}{1+j_0^2} = j_0^{-(n+3)/(n+2)}, j_0 \in (1,\infty). \qquad \textbf{(3.4)}$$

From equations (3.3) and (3.4)

$$h(j) \le h(j_0) = j_0^{-(n+1)} - j_0^{-\frac{(n+3)(n+1)}{n+2}}. \qquad \textbf{(3.5)}$$

If we show the following, the proof will be completed

$$h(j_0) < \frac{1}{2(n+3)}, n \to \infty \ . \quad \textbf{(3.6)}$$

If we set in (3.5)$j_0^{n+1} = y_0 \in (1, +\infty)$. Then $h(y_0) = y_0^{-1} - y_0 < max l(y)$ with $l(y) = y^{-1} - y^{-(n+3)/(n+2)}$. We find that $l^{'}(y_1)$ for $y_1 = \left(\frac{n+3}{n+2}\right)^{n+2}$.

Therefore

$$l(y_1) = \left(\frac{n+3}{n+2}\right)^{-(n+2)} - \left(\frac{n+3}{n+2}\right)^{-(n+3)}$$

$$= \left(\frac{n+3}{n+2}\right)^{-(n+2)} \left[1 - \left(\frac{n+3}{n+2}\right)^{-1}\right]$$

$$= \left(1 + \frac{1}{n+2}\right)^{-(n+2)} \frac{1}{n+3} < \frac{1}{2(n+3)},$$

due to $\lim_{n\to\infty} \left(1 + \frac{1}{n+2}\right)^{-(n+2)} = e^{-1} < 1/2$.

Definition 3.3. (A DIRECT QUANTITATIVE ESTIMATE)

In this section, our aim is to obtain a quantitative form of the statement in Theorem 3.2. Simply we modify the weight function and we consider $\psi(x) = e^{Ax}, x \in R^+$, instead of $\psi(x) = 1 + e^{Ax}, x \in R^+$. For continuous functions on $[0, \infty)$ with exponential growth i.e.

$$\| f \|_A := \sup_{x\in[0,\infty)} |f(x) \cdot e^{-Ax}| < \infty, A > 0. \quad \textbf{(3.7)}$$

We observe that

$$\|\widetilde{M}_n f\|_A \leq \| f \|_A. \quad \textbf{(3.8)}$$

If the following function series is uniformly convergent on $[0, \infty)$

$$Q(x) = \sum_{r=0}^{\infty} t_r(x), x \in [0, \infty),$$

Then

$$\widetilde{M}_n(Q)(u), x) = \sum_{r=0}^{\infty} \widetilde{M}_n(t_r(u), x), x \in [0, \infty). \quad \textbf{(3.9)}$$

The last series is also uniformly convergent. To achieve our goal, in this section we need the first order exponential modulus of continuity, given by Ditzian in [4] and defined as

$$\omega_1(f, \delta, A) := \sup_{h\leq\delta, 0\leq x<\infty} |f(x) - f(x+h)|e^{-Ax}.$$

Let the sequence of operators$\widetilde{M}_n: E \to C[0,\infty)$, where the domain of the operator $\widetilde{M}_n$ contains the space of functions f with exponential growth, i.e. $\| f \|_A < \infty$.

Statement of our main results follows as:

Theorem 3.4. Let $\widetilde{M}_n: E \to C[0,\infty)$ be the sequence of linear positive operators of Post-Widder type defined in (1.3). Then

$$\left|\widetilde{M}_n(f,x) - f(x)\right| \le e^{Ax}[3 + K(n,x)]\omega_1\left(f, \sqrt{\mu_2^{\widetilde{M}_n}(x)}, A\right),$$

where

$$K(n,x) = 2\sum_{r=1}^{\infty} \frac{A^r}{r!}\sqrt{\mu_{2r}^{\widetilde{M}_n}(x)}, n \to \infty \text{ for fixed } x \in [0,\infty).$$

Proof. We notify that

$$|f(t) - f(x)| \le \begin{cases} e^{Ax}\omega_1(f,\delta,A), |u-x| \le \delta \\ e^{Ax}\omega_1(f,k\delta,A), \delta \le |u-x| \le r\delta \end{cases} \quad \textbf{(3.10)}$$

where r is the smallest natural number in the above upper bound. Following [17], we show

$$\omega_1(f,r\delta,A) \le re^{A(r-1)\delta}\omega_1(f,\delta,A)$$
$$\le \omega_1(f,\delta,A)\left[\frac{|u-x|}{\delta} + 1\right]e^{A|u-x|}. \quad \textbf{(3.11)}$$

Now (3.10) and (3.11) imply

$$|f(u) - f(x)| \le \left[1 + \left(\frac{|u-x|}{\delta} + 1\right)e^{A.|u-x|}\right]e^{Ax}\omega_1(f,\delta,A) \quad \textbf{(3.12)}$$

For fixed $x \in [0,\infty)$, we observe that the following series is uniformly convergent for $u \in [0,\infty)$

$$Q_1(u,x) = e^{A\cdot|u-x|} = \sum_{r=0}^{\infty} \frac{(A|u-x|)^r}{r!}$$

$$\frac{|u-x|}{\delta}Q_1(t,x) = \frac{|u-x|}{\delta} + \frac{1}{\delta}\sum_{r=1}^{\infty} \frac{A^r|u-x|^{r+1}}{r!}. \quad \textbf{(3.13)}$$

Clearly for our operators $\widetilde{M}_n$ using (3.10), (3.12) and (3.13), we obtain

$$|\ \widetilde{M}_n(f(u) - f(x)\ | \le \ \widetilde{M}_n(|f(u) - f(x)|, x)$$

$$\le e^{Ax}\left\{1 + \widetilde{M}_n(Q_1(u,x),x) + \frac{1}{\delta}\widetilde{M}_n(|u-x|,x)\right.$$

$$\left. + \frac{1}{\delta}\sum_{r=1}^{\infty} \frac{A^r\widetilde{M}_n(|u-x|^{r+1},x)}{r!}\right\}\omega_1(f,\delta,A). \quad \textbf{(3.14)}$$

Using Cauchy Schwarz inequality, we have

$$\widetilde{M}_n(|u-x|^{r+1},x) \leq \sqrt{\widetilde{M}_n((u-x)^2,x)}\sqrt{\widetilde{M}_n((u-x)^{2r},x)}$$

$$= \sqrt{\mu_2^{\widetilde{M}_n}(x)}\sqrt{\mu_{2r}^{\widetilde{M}_n}(x)}. \qquad \textbf{(3.15)}$$

Further

$$Q_1(u,x) = 1 + A|u-x| + \sum_{r=2}^{\infty} \frac{(A|u-x|)^r}{r!}.$$

Hence

$$\widetilde{M}_n(Q_1(u,x),x) \leq 1 + A\sqrt{\mu_2^{\widetilde{M}_n}(x)} + \sum_{r=2}^{\infty} \frac{A^r\sqrt{\mu_{2r}^{\widetilde{M}_n}(x)}}{r!} \qquad \textbf{(3.16)}$$

From Lemma 2.4, for fixed $x \in [0,\infty)$, we have

$$\mu_{2r}^{\widetilde{M}_n}(x) = O(n^{-r}), n \to \infty \qquad \textbf{(3.17)}$$

We set in (3.14) that

$$\delta = \sqrt{\mu_2^{\widetilde{M}_n}(x)} = O\left(n^{-1/2}\right), n \to \infty \qquad \textbf{(3.18)}$$

Therefore estimates (3.14) -(3.18) imply

$$\left|\widetilde{M}_n(f,x) - f(x)\right| \leq e^{Ax}[3 + K(n,x)]\omega_1\left(f, \sqrt{\mu_2^{\widetilde{M}_n}(x)}), A\right),$$

where

$$K(n,x) = A\sqrt{\mu_2^{\widetilde{M}_n}(x)} + \sum_{r=2}^{\infty} \frac{A^r\sqrt{\mu_2^{\widetilde{M}_n}(x)}}{r!} + \sum_{r=1}^{\infty} \frac{A^r\sqrt{\mu_{2r}^{\widetilde{M}_n}(x)}}{r!} = O\left(n^{-1/2}\right), n \to \infty$$

by fixed $x \in [0,\infty)$. This completes the proof of theorem.

CONCLUSION: Author has discussed the weighted approximation properties for the modified Post-Widder operators. We also investigated a direct quantitative estimate for above discussed operators. Some authors have shown the approximation graphically for linear positive operators, similarly one can find the approximation results graphically for these operators for different test functions.

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