

# On $(\alpha, \beta)$ -Class $(Q_n)$ Operators

## Abstract

In this paper, we introduce a new class of operators, called the class of  $(\alpha, \beta)$ -Class  $(Q_n)$  operators acting on a complex Hilbert space  $H$ . For an integer  $n \geq 1$ , an operator  $T \in B(H)$  is said to belong to the  $(\alpha, \beta)$ -Class  $(Q_n)$  if

$$\alpha^n T^{*n} T^n \leq (T^* T)^n \leq \beta^n T^{*n} T^n,$$

for scalars  $0 \leq \alpha \leq 1 \leq \beta$ . This definition extends the earlier notion of  $(\alpha, \beta)$ -Class  $(Q)$  operators, which corresponds to the case  $n = 2$ . We investigate several fundamental properties of this generalized class and explore operator-theoretic consequences that arise from this extension.

**Keywords:** Class  $(Q)$ , Normal,  $(\alpha, \beta)$ -normal, Hyponormal and  $(\alpha, \beta)$ -Class  $(Q)$  operators.

## Introduction

Throughout this paper, let  $H$  denote a complex Hilbert space and  $B(H)$  the Banach algebra of all bounded linear operators on an infinite-dimensional separable Hilbert space  $H$ . In recent years, the study of normal operators has been extensively developed and generalized by relaxing certain conditions of normality and introducing broader operator classes. For instance, the notion of  $(\alpha, \beta)$ -normal operators was investigated in [5], and this was further extended to the class of  $p$ - $(\alpha, \beta)$ -normal operators in [2]. The Class  $(Q)$  of operators was introduced in [4], where several algebraic properties of this class were established. This concept was later refined to the  $(\alpha, \beta)$ -Class  $(Q)$  operators in [6], where it was shown, among other things, that the class is closed under scalar multiplication. More recently, the framework has been expanded to include  $m$ -quasi- $(\alpha, \beta)$ -Class  $(Q)$  operators [1]. Motivated by these developments, in this paper we extend the concept of  $(\alpha, \beta)$ -Class  $(Q)$  operators to a more general family, namely the  $(\alpha, \beta)$ -Class  $(Q_n)$  operators, defined for arbitrary integers  $n \geq 1$ .

**Definition 0.1.** An operator  $T \in B(H)$  is said to be:

1. Class  $(Q)$  if  $T^{*2} T^2 = (T^* T)^2$ .
2.  $(\alpha, \beta)$ -normal if  $\beta^2 T^* T \geq T T^* \geq \alpha^2 T^* T$ .
3. Normal if  $T^* T = T T^*$ .
4.  $n$ -perinormal if  $T^{*n} T^n \geq (T^* T)^n$ .
5.  $(\alpha, \beta)$ -Class  $(Q)$  operator if  $\alpha^2 T^{*2} T^2 \leq (T^* T)^2 \leq \beta^2 T^{*2} T^2$ . If  $\beta = 1$ , we observe from the right inequality that this class coincides with the class of 2-perinormal operators [3].
6.  $(\alpha, \beta)$ -Class  $(Q_n)$  if  $\alpha^n T^{*n} T^n \leq (T^* T)^n \leq \beta^n T^{*n} T^n$ , for scalars  $0 \leq \alpha \leq 1 \leq \beta$ . We observe that this reduces to  $(\alpha, \beta)$ -Class  $(Q)$  whenever  $n=2$ .

## Main Results

**Theorem 0.2.** *Let  $T \in B(H)$  and let  $n \geq 1$  be an integer. If*

$$T \in (\alpha, \beta)\text{-Class } (Q_n), \quad \text{i.e.} \quad \alpha^n T^{*n} T^n \leq (T^* T)^n \leq \beta^n T^{*n} T^n,$$

for scalars  $0 \leq \alpha \leq 1 \leq \beta$ , then:

1.  $\lambda T \in (\alpha, \beta)\text{-Class } (Q_n)$  for every real  $\lambda$  (indeed for every  $\lambda \in \mathbb{C}$ ).
2. If  $S \in B(H)$  is unitarily equivalent to  $T$  (so  $S = UTU^*$  for some unitary  $U$ ), then  $S \in (\alpha, \beta)\text{-Class } (Q_n)$ .

*Proof.* (1) Let  $\lambda \in \mathbb{R}$  (the same argument works for  $\lambda \in \mathbb{C}$  using  $|\lambda|$ ). Compute

$$(\lambda T)^{*n} (\lambda T)^n = \bar{\lambda}^n \lambda^n T^{*n} T^n = |\lambda|^{2n} T^{*n} T^n,$$

and

$$((\lambda T)^*(\lambda T))^n = (|\lambda|^2 T^* T)^n = |\lambda|^{2n} (T^* T)^n.$$

Multiplying the assumed inequality

$$\alpha^n T^{*n} T^n \leq (T^* T)^n \leq \beta^n T^{*n} T^n$$

by the positive scalar  $|\lambda|^{2n}$  yields

$$\alpha^n |\lambda|^{2n} T^{*n} T^n \leq |\lambda|^{2n} (T^* T)^n \leq \beta^n |\lambda|^{2n} T^{*n} T^n,$$

which is precisely

$$\alpha^n (\lambda T)^{*n} (\lambda T)^n \leq ((\lambda T)^*(\lambda T))^n \leq \beta^n (\lambda T)^{*n} (\lambda T)^n.$$

Hence  $\lambda T \in (\alpha, \beta)\text{-Class } (Q_n)$ .

(2) Let  $S = UTU^*$  with  $U$  unitary. Note first that

$$S^* = UT^*U^*, \quad S^{*n} = UT^{*n}U^*, \quad S^n = UT^nU^*,$$

so

$$S^{*n} S^n = UT^{*n} T^n U^*, \quad (S^* S)^n = U(T^* T)^n U^*.$$

Conjugating the assumed inequality by  $U$  (i.e. applying  $X \mapsto UXU^*$ ) gives

$$\alpha^n UT^{*n} T^n U^* \leq U(T^* T)^n U^* \leq \beta^n UT^{*n} T^n U^*,$$

which is exactly

$$\alpha^n S^{*n} S^n \leq (S^* S)^n \leq \beta^n S^{*n} S^n.$$

Thus  $S \in (\alpha, \beta)\text{-Class } (Q_n)$ . □

**Theorem 0.3.** *Let  $T \in B(H)$  and let  $n \geq 1$  be an integer. If  $T$  is  $(\alpha, \beta)$ -normal, i.e.*

$$\alpha^2 T^* T \leq T T^* \leq \beta^2 T^* T$$

for scalars  $0 \leq \alpha \leq 1 \leq \beta$ , then for every integer  $n \geq 1$  we have

$$\alpha^2 T^{*n} T^n \leq (T^* T)^n \leq \beta^2 T^{*n} T^n,$$

that is,  $T \in (\alpha, \beta)\text{-Class } (Q_n)$  (with the same coefficients  $\alpha^2, \beta^2$  appearing for all  $n$ ).

*Proof.* Assume

$$\alpha^2 T^* T \leq T T^* \leq \beta^2 T^* T.$$

Fix an integer  $n \geq 1$ . We shall conjugate the inequality by suitable powers of  $T$  and  $T^*$ .

First we prove the identity

$$T^{*(k-1)}(T T^*)T^{k-1} = (T^* T)^k \quad \text{for every } k \geq 1. \quad (1)$$

The identity holds for  $k = 1$  since both sides equal  $T T^*$ . Assume it holds for some  $k \geq 1$ . Left multiplying by  $T^*$  and right multiplying by  $T$  we obtain

$$T^{*k}(T T^*)T^k = T^*(T^{*(k-1)}(T T^*)T^{k-1})T = T^*(T^* T)^k T = (T^* T)^{k+1},$$

so the identity holds for  $k + 1$ . Hence (1) holds for all  $k \geq 1$ .

Conjugating the assumed inequality by  $T^{*(n-1)}$  on the left and by  $T^{n-1}$  on the right. Because the map  $X \mapsto T^{*(n-1)} X T^{n-1}$  preserves the order for positive operators, we get

$$\alpha^2 T^{*(n-1)}(T^* T)T^{n-1} \leq T^{*(n-1)}(T T^*)T^{n-1} \leq \beta^2 T^{*(n-1)}(T^* T)T^{n-1}.$$

But

$$T^{*(n-1)}(T^* T)T^{n-1} = T^{*n} T^n,$$

and by (1)

$$T^{*(n-1)}(T T^*)T^{n-1} = (T^* T)^n.$$

Substituting these equalities into the conjugated inequality yields

$$\alpha^2 T^{*n} T^n \leq (T^* T)^n \leq \beta^2 T^{*n} T^n,$$

. Thus  $T \in (\alpha, \beta)$ -Class  $(Q_n)$  for every  $n \geq 1$ . □

**Theorem 0.4.** *Let  $T \in B(H)$  and let  $n \geq 1$  be an integer. Assume  $T$  belongs to the  $(\alpha, \beta)$ -Class  $(Q)$ . Suppose further that  $T$  admits a polar decomposition  $T = U|T|$  with  $U$  unitary. Then  $T^*$  belongs to the  $(\alpha, \beta)$ -Class  $(Q_n)$  for every integer  $n \geq 1$ ,*

*Proof.* Let  $T = U|T|$  with  $U$  unitary and  $|T| = (T^* T)^{1/2}$ . Since  $U$  is unitary we have the conjugation identities

$$T T^* = U|T|^2 U^*, \quad (T T^*)^n = U|T|^{2n} U^*,$$

and, using  $T^{*n} T^n = |T|^{2n}$ ,

$$T^n T^{*n} = U|T|^{2n} U^*.$$

by assumption and the positivity of operator  $T^* T$ , we obtain for every integer  $n \geq 1$

$$\alpha^n T^{*n} T^n \leq (T^* T)^n \leq \beta^n T^{*n} T^n,$$

Conjugating this inequality by the unitary  $U$  yields

$$\alpha^n U T^{*n} T^n U^* \leq U (T^* T)^n U^* \leq \beta^n U T^{*n} T^n U^*.$$

Using  $U T^{*n} T^n U^* = T^n T^{*n}$  and  $U (T^* T)^n U^* = (T T^*)^n$  we get

$$\alpha^n T^n T^{*n} \leq (T T^*)^n \leq \beta^n T^n T^{*n},$$

implying that  $T^* \in (\alpha, \beta)$ -Class  $(Q_n)$ . □

**Theorem 0.5.** *Let  $T \in B(H)$  and let  $n \geq 1$  be an integer. If  $T$  belongs to the  $(\alpha, \beta)$ -Class  $(Q_n)$  and  $P$  is a unitary operator on  $H$  such that  $TP = PT$ , then the operator  $K := TP$  also belongs to the  $(\alpha, \beta)$ -Class  $(Q_n)$ .*

*Proof.* Since  $P$  is unitary and commutes with  $T$ , it also commutes with  $T^*$  and with all powers of  $T$  and  $T^*$ . In particular, for every integer  $k \geq 1$ ,

$$PT^k = T^k P, \quad PT^{*k} = T^{*k} P, \quad P^* T^k = T^k P^*, \quad P^* T^{*k} = T^{*k} P^*.$$

computing the powers of  $K$  and  $K^*$ :

$$K^n = (TP)^n = T^n P^n, \quad K^{*n} = (TP)^{*n} = (P^* T^*)^n = P^{*n} T^{*n},$$

hence ;

$$K^{*n} K^n = P^{*n} T^{*n} T^n P^n = P^{*n} (T^{*n} T^n) P^n,$$

and

$$(K^* K)^n = ((TP)^*(TP))^n = (P^* T^* TP)^n = P^{*n} (T^* T)^n P^n,$$

conjugating the assumed inequality

$$\alpha^n T^{*n} T^n \leq (T^* T)^n \leq \beta^n T^{*n} T^n$$

by  $P^{*n} P^n$  we obtain

$$\alpha^n P^{*n} T^{*n} T^n P^n \leq P^{*n} (T^* T)^n P^n \leq \beta^n P^{*n} T^{*n} T^n P^n.$$

Using the identities above for  $K^{*n} K^n$  and  $(K^* K)^n$  this becomes

$$\alpha^n K^{*n} K^n \leq (K^* K)^n \leq \beta^n K^{*n} K^n,$$

implying that  $K \in (\alpha, \beta)$ -Class  $(Q_n)$ . □

**Theorem 0.6.** *Let  $S, T \in B(H)$  and let  $n \geq 1$  be an integer. Assume that  $S$  and  $T^*$  commute, i.e.*

$$ST = TS, \quad S^* T = T S^*, \quad ST^* = T^* S,$$

*. If both  $S$  and  $T$  belong to the  $(\alpha, \beta)$ -Class  $(Q_n)$ , then the product  $ST$  also belongs to  $(\alpha, \beta)$ -Class  $(Q_n)$*

*Proof.* Because of the  $*$ -commutation hypothesis, every pair drawn from  $\{S, S^*, T, T^*\}$  commutes. In particular the positive operators  $S^* S$  and  $T^* T$  commute, hence so do their powers; therefore

$$((ST)^*(ST))^n = (T^* S^* ST)^n = (T^*)^n (S^* S)^n T^n = (T^* T)^n (S^* S)^n,$$

*. Similarly, using commutativity of the adjoint-powers,*

$$(ST)^{*n} (ST)^n = (T^{*n} S^{*n})(S^n T^n) = T^{*n} (S^{*n} S^n) T^n = (T^{*n} T^n)(S^{*n} S^n).$$

Hence ;

$$\alpha^n S^{*n} S^n \leq (S^* S)^n \leq \beta^n S^{*n} S^n$$

and

$$\alpha^n T^{*n} T^n \leq (T^* T)^n \leq \beta^n T^{*n} T^n,$$

and using the fact that  $(S^{*n} S^n)$  commutes with  $(T^{*n} T^n)$  and likewise for  $(S^* S)^n$  and  $(T^* T)^n$ , we may multiply the left-hand inequalities and the right-hand inequalities to obtain

$$\alpha^n (T^{*n} T^n)(S^{*n} S^n) \leq (T^* T)^n (S^* S)^n \leq \beta^n (T^{*n} T^n)(S^{*n} S^n).$$

Using the identities noted above this is exactly

$$\alpha^n (ST)^{*n} (ST)^n \leq ((ST)^*(ST))^n \leq \beta^n (ST)^{*n} (ST)^n,$$

which proves that  $ST \in (\alpha, \beta)$ -Class  $(Q_n)$ . □

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