ON (n,m)-SQUARE METRICALLY EQUIVALENT OPERATORS.

Original Research Article

Abstract

We introduce and study a new equivalence relation among bounded linear operators, termed (n,m)-Square Metrically Equivalent Operators. Given positive integers n and m, two bounded linear operators $\mathcal{A}$ and $\mathcal{B}$ are said to be (n,m)-square metrically equivalent if they satisfy the relation $\mathcal{A}^{*n}\mathcal{A}^{m} = \mathcal{B}^{*n}\mathcal{B}^{m}$. This definition generalizes the classical notions of metric and square-metric equivalence, extending them to a broader framework that captures deeper algebraic and spectral similarities between operators. We show that this relation forms an equivalence class and investigate its algebraic, spectral, and structural properties. Furthermore, we explore how (n,m)-square metric equivalence interacts with well-known operator classes such as (n,m)-normal and quasi-similar operators, and we establish conditions under which important properties including Bishop's property, isoloid, and polaroid behaviors are preserved.

Keywords: n-square normal operators; metrically equivalent operators; square metrically equivalent operators;square normal operators; unitary operators.

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1 Introduction

Equivalence relations among bounded linear operators on Hilbert spaces provide a fundamental framework in functional analysis and operator theory. These relations allow researchers to group operators into classes that preserve essential structural and spectral properties. Classical equivalence notions such as unitary equivalence, similarity, and metric equivalence have been widely used to analyze and classify operators Furuta (2001); Kreyszig (1991). For example, unitary equivalence preserves both the spectrum and inner product structure, while similarity preserves algebraic behavior but not necessarily the norm. Metric equivalence, in contrast, emphasizes norm preservation, distinguishing itself from the former by its more geometric nature.

The concept of metric equivalence has been particularly influential. Nzimbi et al. (2013) established that two operators $\mathcal{A}, \mathcal{B} \in B(H)$ are metrically equivalent if and only if $\|\mathcal{A}x\| = \|\mathcal{B}x\|$ for all $x \in H$. This relation provided a clear and practical criterion for comparing operators through their action on vectors. However, as operator theory expanded into new applications such as quantum mechanics and large-scale data analysis, it became evident that more flexible notions of equivalence were needed. This motivated several generalizations of metric equivalence.

One such extension was the introduction of n-metric equivalence by . Victor et al. (2020), where equivalence is determined by higher powers of operators: $\|\mathcal{A}^n x\| = \|\mathcal{B}^n x\|$ for some fixed $n \in \mathbb{N}$. This refinement allowed the study of deeper operator behavior across iterations. Subsequently,Victor and Nyongesa (2021) proposed the (n, m)-metric equivalence, which captures an even broader class of operator relations. In this case, two operators $\mathcal{A}, \mathcal{B} \in B(H)$ are said to be (n, m)-metrically equivalent if they satisfy

$$\mathcal{A}^{*n}\mathcal{A}^m = \mathcal{B}^{*n}\mathcal{B}^m,$$

for positive integers n and m. This definition generalizes both metric equivalence and n-metric equivalence, providing a flexible framework for comparing operator pairs through algebraic and spectral relationships.

In parallel, researchers have investigated related classes such as square metrically equivalent operators, recently introduced by Wanjala and Nyongesa (2024). This class was shown to preserve important operator properties under unitary equivalence and revealed deeper structural similarities between operator pairs. However, the literature still lacks a systematic generalization of this concept to the (n, m)-framework. The extension to (n, m)-square metrically equivalent operators provides such a generalization and offers new tools for exploring equivalence among broader families of operators.

The significance of developing the (n, m)-square metric equivalence lies in its potential to unify and extend several strands of operator theory. First, it provides a natural bridge between classical metric equivalence, n-metric equivalence, and square-metric equivalence. Second, it allows us to investigate how key spectral properties behave under this broader relation. For instance, one may ask whether (n, m)-square equivalence preserves Bishop's property, isoloid, or polaroid properties, which are central in spectral theory. Third, it offers a framework for studying the stability of equivalence under common operator constructions such as direct sums and tensor products, which are essential in multi-operator systems.

The aim of this paper is therefore threefold:

1. To formally introduce and define the class of (n, m)-square metrically equivalent operators.
2. To examine its algebraic, structural, and spectral properties, establishing its role as an equivalence relation.
3. To explore its interaction with well-known operator classes such as normal, quasi-similar, isoloid, and polaroid operators, and to identify conditions under which Bishop's property is preserved.

By extending the square-metric equivalence to the general (n, m)-case, this work provides a new perspective on operator equivalence. The results not only enrich the theoretical foundations of functional analysis but also open new avenues for applications in spectral theory, quantum mechanics, and operator classification. For literature related to class (Q), we refer the reader to Jibril (2018); Manikandan and Veluchamy (2018); Panayappan and Sivamani (2012); Rasimi and Gjoka (2013); Obiero and Victor (2021); Victor and Nyongesa (2021).

2 Key Definitions

Definition 2.1. Ould (2014) An operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ has *Bishop's property* if, for every sequence of analytic functions $f_p : U \to H$, where $U \subset \mathbb{C}$ is open, $(\lambda - A)f_p(\lambda) \to 0$ uniformly on compact subsets of U and $f_p(\lambda) \to 0$ locally uniformly in U as $p \to \infty$.

Definition 2.2. Muneo and Biljana (2018) An operator $\mathcal{A}$ is *isoloid* if each isolated point of $\sigma(A)$ belongs to the point spectrum $\sigma_p(A)$.

Definition 2.3. Muneo and Biljana (2018) An operator $\mathcal{A}$ is *polaroid* if every isolated point of $\sigma(A)$ is a pole of the resolvent of $\mathcal{A}$.

Definition 2.4. Furuta (2001) An operator $\mathcal{A}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$ is referred to as: *normal* if $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$.

Definition 2.5. Mahmoud (2016) A is *square-normal* if $\mathcal{A}^2(\mathcal{A}^*)^2 = (\mathcal{A}^*)^2\mathcal{A}^2$.

Definition 2.6. Nzimbi et al. (2013) $\mathcal{A}$ and $\mathcal{B}$ are metrically equivalent if $\mathcal{A}^*\mathcal{A} = \mathcal{B}^*\mathcal{B}$.

Definition 2.7. Eiman and Mustafa (2016) An operator is said to be n-Normal Operato if$\mathcal{A}^*\mathcal{A}^n$ = $\mathcal{A}^n\mathcal{A}^*$ for $n \in \mathbb{Z}^+$.

Lemma 2.1. *Vijayalakshmia and Maryb (2016) Let $T \in \mathcal{B}(\mathcal{H})$. Then T is n-power normal if and only if T^n is normal.*

Proposition 2.1. *Mahmoud (2016) If $\mathcal{A}$ is a normal operator, then $\mathcal{A}$ is a square-normal operator.*

Theorem 2.2. *Muneo and Biljana (2018) Suppose $\mathcal{Q} \in \mathcal{B}(\mathcal{H})$ is an n-normal operator. Then $\mathcal{Q}$ is both isoloid and polaroid.*

These definitions underpin our analysis of the structural and spectral characteristics of n-square metrically equivalent operators.

3 Main Results

Definition 3.1. Two operators $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ are said to be (n, m)-Square Metrically Equivalent Operators, denoted by $\mathcal{A} \sim_{m^{2(n,m)}} \mathcal{B}$, provided:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{B}^{*2m}\mathcal{B}^{2n}.$$

$$\forall n, m \in \mathbb{R}^+$$

Theorem 3.1. *Let $\mathcal{A}$ be (n, m)-square-normal operator and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ is unitarily equivalent to $\mathcal{A}$, then $\mathcal{B}$ is (n, m)-square-normal.*

Proof. Since $\mathcal{B} = \mathcal{U}^*\mathcal{A}\mathcal{U}$, with $\mathcal{U}$ being unitary and $\mathcal{A}$ (n, m)-square-normal, we have:

$$\begin{aligned}
\mathcal{B}^{*2m}\mathcal{B}^{2n} &= (\mathcal{U}^*\mathcal{A}^{*m}\mathcal{U})^2(\mathcal{U}^*\mathcal{A}^n\mathcal{U})^2 \\
&= (\mathcal{U}^*\mathcal{A}^{*2m}\mathcal{A}^{2n}\mathcal{U}) \\
&= (\mathcal{U}^*\mathcal{A}^{2n}\mathcal{A}^{*2m}\mathcal{U}) \\
&= (\mathcal{B}^{2n}\mathcal{U}^*\mathcal{A}^{*2m}\mathcal{U}) \\
&= \mathcal{B}^{2n}\mathcal{U}^*\mathcal{U}\mathcal{B}^{*2m} \\
&= \mathcal{B}^{2n}\mathcal{B}^{*2m}. \\
&\forall n, m \in \mathbb{R}^+
\end{aligned}$$

This proves the claim. □

Corollary 3.2. *An operator $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ is (n, m)-square-normal if and only if $\mathcal{B}$ and $\mathcal{B}^*$ are (n, m)-square metrically equivalent.*

Proof. The proof follows from Theorem 3.1. □

Theorem 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be (n,m)-Square Metrically Equivalent then $\mathcal{A}^{*2m}\mathcal{A}^{2n}$ and $\mathcal{B}^{*2m}\mathcal{B}^{2n}$ are unitarily equivalent.*

Proof. The proof is simple and follows from

$$\begin{aligned}\mathcal{A}^{*2m}\mathcal{A}^{2n} &= \mathcal{U}\mathcal{B}^{*2m}\mathcal{B}^{2n}\mathcal{U}^* \\ &= \mathcal{B}^{*2m}\mathcal{U}\mathcal{U}^*\mathcal{B}^{2n} \\ &= \mathcal{B}^{*2m}\mathcal{U}^*\mathcal{U}\mathcal{B}^{2n} \\ &= \mathcal{B}^{*2m}\mathcal{B}^{2n}\end{aligned}$$

for all positive integers n, m and unitary operator $\mathcal{U}$. □

Theorem 3.4. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$ be (n,m)-Square Metrically Equivalent Operators. If $\mathcal{A}^n$ and $\mathcal{B}^m$ share a common isolated eigenvalue λ with finite multiplicity, then both $\mathcal{A}$ and $\mathcal{B}$ are polaroid at λ.*

Proof. By assumption and equivalence, λ is isolated in both $\sigma(\mathcal{A}^n)$ and $\sigma(\mathcal{B}^m)$ and an eigenvalue. The finite multiplicity ensures the resolvent has a pole of finite order, which implies both operators are polaroid at λ. □

Theorem 3.5. *(n,m)-Square metric equivalence is an equivalence relation.*

Proof. (i) $\mathcal{A} \sim_{m^{2(n,m)}} \mathcal{A}$ since:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{U}\mathcal{A}^{*2m}\mathcal{A}^{2n}\mathcal{U}^*.$$

(ii) If $\mathcal{A} \sim_{m^{2(n,m)}} \mathcal{B}$, it means:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{U}\mathcal{B}^{*2m}\mathcal{B}^{2n}\mathcal{U}^* \tag{3.1}$$

Pre-multiplying 3.1 on both sides by $\mathcal{U}^*$ and post-multiplying the same by $\mathcal{U}$, we get:

$$\mathcal{U}^*\mathcal{A}^{*2m}\mathcal{A}^{2n}\mathcal{U} = \mathcal{U}^*\mathcal{U}\mathcal{B}^{*2m}\mathcal{B}^{2n}\mathcal{U}^*\mathcal{U} = \mathcal{U}^*\mathcal{A}^{*2m}\mathcal{A}^{2n}\mathcal{U} = \mathcal{B}^{*2m}\mathcal{B}^{2n} \tag{3.2}$$

Hence, $\mathcal{B} \sim_{m^{2(n,m)}} \mathcal{A}$.

(iii) We need to illustrate that: if $\mathcal{A} \sim_{m^{2(n,m)}} \mathcal{Q}$ and $\mathcal{Q} \sim_{m^{2(n,m)}} \mathcal{B}$, it follows that $\mathcal{A} \sim_{m^{2(n,m)}} \mathcal{B}$. That is:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{U}\mathcal{Q}^{*2m}\mathcal{Q}^{2n}\mathcal{U}^* \quad \text{and} \quad \mathcal{Q}^{*2m}\mathcal{Q}^{2n} = \mathcal{G}\mathcal{B}^{*2m}\mathcal{B}^{2n}\mathcal{G}^*,$$

where $\mathcal{U}$ and $\mathcal{G}$ are unitary operators. Then:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{U}\mathcal{Q}^{*2m}\mathcal{Q}^{2n}\mathcal{U}^* = \mathcal{U}\mathcal{G}\mathcal{B}^{*2m}\mathcal{B}^{2n}\mathcal{U}^*\mathcal{G}^*.$$

Let $\mathcal{M} = \mathcal{U}\mathcal{G}$, where $\mathcal{M}$ is unitary. Thus:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{M}\mathcal{B}^{*2m}\mathcal{B}^{2n}\mathcal{M}^*.$$

Hence, $\mathcal{A} \sim_{m^{2(n,m)}} \mathcal{B}$.

□

Proposition 3.1. *If $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ are (n,m)-Metrically Equivalent Operators, then they are (n,m)-Square-Metrically Equivalent.*

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are (n,m)-square-metrically equivalent, we have:

$$\begin{aligned}\mathcal{A}^{*2m}\mathcal{A}^{2n} &= \mathcal{A}^{*m}\mathcal{A}^{*m}\mathcal{A}^{n}\mathcal{A}^{n} \\ = \mathcal{A}^{*m}\mathcal{A}^{n}\mathcal{A}^{*m}\mathcal{A}^{n} &= \mathcal{B}^{n}\mathcal{B}^{*m}\mathcal{B}^{n}\mathcal{B}^{*m} \\ &= \mathcal{B}^{*m}\mathcal{B}^{n}\mathcal{B}^{*m}\mathcal{B}^{n} \\ &= \mathcal{B}^{*m}\mathcal{B}^{*m}\mathcal{B}^{n}\mathcal{B}^{n} \\ &= \mathcal{B}^{*2m}\mathcal{B}^{2n}\end{aligned}$$

□

Remark 3.1. The converse need not be true. We give an example of (n, m)-Square Metrically Equivalent Operators that are not (n, m)-Metrically Equivalent.

Example 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be defined as :*

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}.$$

such that $\theta \neq \phi$ and $e^{i\theta} \neq e^{i\phi} = 1$

It follows that;

$$\mathcal{A}^*\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{I}$$

$$\mathcal{B}^*\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{I}$$

So $\mathcal{A}$ and $\mathcal{B}$ are metrically equivalent. But;

$$\mathcal{A}^{*2m}\mathcal{A}^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2mi\theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{2ni\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(n-m)\theta} \end{bmatrix}$$

$$\mathcal{B}^{*2m}\mathcal{B}^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2mi\phi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{2ni\phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i(n-m)\phi} \end{bmatrix}$$

since $\theta \neq \phi$, then

$$e^{2i(n-m)\theta} \neq e^{2i(n-m)\phi}$$

hence $\mathcal{A}^{*2m}\mathcal{A}^{2n} \neq \mathcal{B}^{*2m}\mathcal{B}^{2n} \quad \forall n \in \mathbb{R}^+$. Hence, $\mathcal{A}$ and $\mathcal{B}$ are metrically equivalent but not (n, m)-square metrically equivalent.

Theorem 3.7. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$ with*

$$\mathcal{A}^{*\,2m}\mathcal{A}^{2n} = \mathcal{B}^{*\,2m}\mathcal{B}^{2n}.$$

Then:

1. *If $\mathcal{A}$ has Bishop's property (β), then $\mathcal{B}$ has (β).*
2. *If $\mathcal{A}$ is isoloid and $\mathcal{A}^n$ and $\mathcal{B}^m$ share the same isolated spectrum, then $\mathcal{B}$ is isoloid.*
3. *If λ is an isolated eigenvalue of finite multiplicity of $\mathcal{A}$, then both $\mathcal{A}, \mathcal{B}$ are polaroid at λ.*

***(1) Bishop's property** (β). Let $U \subset \mathbb{C}$ be open and $f : U \to X$ analytic such that*

$$(\mathcal{B} - \lambda I)f(\lambda) = 0, \quad \lambda \in U.$$

From the metric equivalence we have

$$\|\mathcal{A}^n x\| = \|\mathcal{B}^m x\| \quad \text{for all } x \in X.$$

Thus the analytic core of $\mathcal{A}^n$ coincides with that of $\mathcal{B}^m$. Since $\mathcal{A}$ has Bishop's property (β), the only analytic solution for $(\mathcal{A} - \lambda I)g(\lambda) = 0$ is the trivial one $g \equiv 0$. Applying this to f via the shared core gives $f \equiv 0$. Hence $\mathcal{B}$ also has (β).

***(2) Isoloid.** Suppose λ is isolated in $\sigma(\mathcal{B})$. Then λ^m is isolated in $\sigma(\mathcal{B}^m)$. By hypothesis, λ^m is also isolated in $\sigma(\mathcal{A}^n)$. Since $\mathcal{A}$ is isoloid, λ is an eigenvalue of $\mathcal{A}$, say $\mathcal{A}x = \lambda x$. By metric equivalence, $\mathcal{B}^m x = \lambda^m x$, so λ is an eigenvalue of $\mathcal{B}$. Thus $\mathcal{B}$ is isoloid.*

***(3) Polaroid.** Let λ be an isolated eigenvalue of finite multiplicity of $\mathcal{A}$. Then λ^n is isolated in $\sigma(\mathcal{A}^n)$ and the resolvent of $\mathcal{A}^n$ has a pole at λ^n. Since $\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{B}^{*2m}\mathcal{B}^{2n}$, the same holds for $\mathcal{B}^m$, hence λ^m is a pole of the resolvent of $\mathcal{B}^m$. It follows that λ is a pole of the resolvent of both $\mathcal{A}$ and $\mathcal{B}$. Therefore both are polaroid at λ.*

Remark 3.2. The following results establish the (n,m)-square metric equivalence relation of operators on n,m-power class $\mathcal{Q}$; $(n,m\mathcal{Q})$.

Theorem 3.8. *Let $\mathcal{A}\in(n,m\mathcal{Q})$ and $\mathcal{B}\in(n,m\mathcal{Q})$. Then $\mathcal{A}\in(n,m\mathcal{Q})$ and $\mathcal{B}\in(n,m\mathcal{Q})$ are said to be (n,m)-square metrically equivalent $(\mathcal{Q})$-operators if and only if $\mathcal{A}$ and $\mathcal{B}$ are isometries for (n,m)=1 .*

Proof. Given that $\mathcal{A}\in(n,m\mathcal{Q})$ and $\mathcal{B}\in(n,m\mathcal{Q})$, then:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=(\mathcal{A}^{*m}\mathcal{A}^{n})^2 \quad \text{and} \quad \mathcal{B}^{*2m}\mathcal{B}^{2n}=(\mathcal{B}^{*m}\mathcal{B}^{n})^2.$$

This implies:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=(\mathcal{A}^{*m}\mathcal{A}^{n})^2=\mathcal{A}^{*2m}\mathcal{A}^{2n},$$

and:

$$\mathcal{B}^{*2m}\mathcal{B}^{2n}=(\mathcal{B}^{*m}\mathcal{B}^{n})^2=(\mathcal{B}^{*})^2\mathcal{B}^{2n}.$$

Since both $\mathcal{A}$ and $\mathcal{B}$ are isometries for (n,m)=1 , the following holds:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=\mathcal{I}=(\mathcal{A}^{*m}\mathcal{A}^{n})^2 \tag{3.3}$$

hence

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=\mathcal{I} \tag{3.4}$$

as well as:

$$\mathcal{B}^{*2m}\mathcal{B}^{2n}=\mathcal{I}=(\mathcal{B}^{*m}\mathcal{B}^{n})^2 \tag{3.5}$$

and equivalently:

$$\mathcal{B}^{2n}\mathcal{B}^{*2m}=\mathcal{I} \tag{3.6}$$

From 3.4 and 3.6, it is observed that:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=\mathcal{I}=\mathcal{B}^{*2m}\mathcal{B}^{2n},$$

and so:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=\mathcal{B}^{*2m}\mathcal{B}^{2n}.$$

For the converse, since $\mathcal{A}\in(n,m\mathcal{Q})$ and $\mathcal{B}\in(n,m\mathcal{Q})$ are isometries, we have:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=(\mathcal{A}^{*m}\mathcal{A}^{n})^2=\mathcal{A}^{*2m}\mathcal{A}^{2n}=\mathcal{I},$$

and:

$$\mathcal{B}^{*2m}\mathcal{B}^{2n}=(\mathcal{B}^{*m}\mathcal{B}^{n})^2=\mathcal{B}^{*2m}\mathcal{B}^{2n}=\mathcal{I}.$$

Hence:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=\mathcal{I}=\mathcal{B}^{*2m}\mathcal{B}^{2n},$$

and so:

$$\mathcal{A}^{*2m}\mathcal{A}^{2n}=\mathcal{B}^{*2m}\mathcal{B}^{2n}.$$

□

Remark 3.3. The above results suggest that many spectral properties of $\mathcal{A}$ can be transferred to $\mathcal{B}$ under (n,m)-square metric equivalence, especially when they share strong isometric behavior and aligned spectral structures.

Theorem 3.9. *Let $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_r$ and $\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_r\in\mathcal{B}(\mathcal{H})$ be (n,m)-Square Metrically Equivalent Operators.; then :*

1. *$\mathcal{A}_1\oplus\mathcal{A}_2\oplus\cdots\oplus\mathcal{A}_r$ and $\mathcal{B}_1\oplus\mathcal{B}_2\oplus\cdots\oplus\mathcal{B}_r$ are (n,m)-Square Metrically Equivalent Operators.*
2. *$\mathcal{A}_1\otimes\mathcal{A}_2\otimes\cdots\otimes\mathcal{A}_r$ and $\mathcal{B}_1\otimes\mathcal{B}_2\otimes\cdots\otimes\mathcal{B}_r$ are (n,m)-Square Metrically Equivalent Operators.*

Proof. (i) We show that :

$$\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{*2m}\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{*m}\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{*m}\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{n}\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{n}$$

using the facts that the adjoint of a direct sum is the direct sum of adjoints,

$$\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{*}=\bigoplus_{i=1}^{r}\mathcal{A}_i^{*},$$

and that powers of a block–diagonal operator act on each block independently,

$$\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{*}\right)^{k}=\bigoplus_{i=1}^{r}(\mathcal{A}_i^{*})^{k}.$$

Hence,

$$\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{*2m}=\bigoplus_{i=1}^{r}\mathcal{A}_i^{*2m}=\bigoplus_{i=1}^{r}(\mathcal{A}_i^{*m}\mathcal{A}_i^{*m})=\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{*m}\right)\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{*m}\right).$$

Similarly,

$$\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\bigoplus_{i=1}^{r}\mathcal{A}_i^{2n}=\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{n}\right)\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{n}\right).$$

Combining these results gives

$$\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{*2m}\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{*m}\right)\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{*m}\right)\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{n}\right)\left(\bigoplus_{i=1}^{r}\mathcal{A}_i^{n}\right),$$

Since each $\mathcal{A}_i \sim_{m^{2(n,m)}} \mathcal{B}_i$, there exists a unitary operator $\mathcal{U}$ such that:

$$\begin{aligned}&\mathcal{U}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{*m}\right)\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{*m}\right)\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{n}\right)\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{n}\right)\mathcal{U}^*\\&=\mathcal{U}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{*m}\mathcal{B}_i^{*m}\right)\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{n}\mathcal{B}_i^{n}\right)\mathcal{U}^*\\&=\mathcal{U}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{*m}\right)^{2}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i^{n}\right)^{2}\mathcal{U}^*\\&=\mathcal{U}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i\right)^{*2m}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i\right)^{2n}\mathcal{U}^*\end{aligned}$$

Thus

$$\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{*2m}\left(\bigoplus_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\mathcal{U}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i\right)^{*2m}\left(\bigoplus_{i=1}^{r}\mathcal{B}_i\right)^{2n}\mathcal{U}^*$$

Hence,

$$\bigoplus_{i=1}^{r}\mathcal{A}_i \sim_{m^{2(n,m)}} \bigoplus_{i=1}^{r}\mathcal{B}_i$$

.

(ii) Similarly , we show :

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*2m}\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*m}\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*m}\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{n}\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{n}$$

using the facts that the adjoint of a tensor product is the tensor product of adjoints,

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*}=\bigotimes_{i=1}^{r}\mathcal{A}_i^{*},$$

and that powers of a block–diagonal operator act on each block independently,

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{*}\right)^{k}=\bigotimes_{i=1}^{r}\left(\mathcal{A}_i^{*}\right)^{k}.$$

Hence,

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*2m}=\bigotimes_{i=1}^{r}\mathcal{A}_i^{*2m}=\bigotimes_{i=1}^{r}\left(\mathcal{A}_i^{*m}\mathcal{A}_i^{*m}\right)=\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{*m}\right)\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{*m}\right).$$

Similarly,

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\bigoplus_{i=1}^{r}\mathcal{A}_i^{2n}=\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{n}\right)\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{n}\right).$$

Combining these results gives

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*2m}\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{*m}\right)\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{*m}\right)\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{n}\right)\left(\bigotimes_{i=1}^{r}\mathcal{A}_i^{n}\right),$$

Since each $\mathcal{A}_i \sim_{m^2(n,m)} \mathcal{B}_i$, there exists a unitary operator $\mathcal{U}$ such that:

$$\begin{aligned}
&\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{*m}\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{*m}\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{n}\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{n}\right)\mathcal{U}^*\\
&=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{*m}\mathcal{B}_i^{*m}\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{n}\mathcal{B}_i^{n}\right)\mathcal{U}^*\\
&=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{*m}\right)^{2}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^{n}\right)^{2}\mathcal{U}^*\\
&=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{*2m}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{2n}\mathcal{U}^*
\end{aligned}$$

Thus

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*2m}\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{*2m}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{2n}\mathcal{U}^*$$

Therefore,

$$\bigotimes_{i=1}^{r}\mathcal{A}_i\sim_{m^2(n,m)}\bigotimes_{i=1}^{r}\mathcal{B}_i$$

. ☐

4 Results and Discussion

In this section, we explain the meaning of the results presented earlier. We focus on what the findings tell us about the structure and behavior of (n, m)-Square Metrically Equivalent Operators. We also compare this new class with other well-known operator classes.

4.1 Spectral Properties

Spectral properties play a central role in the study of (n, m)-Square Metrically Equivalent Operators. Several key results demonstrate that this equivalence preserves important spectral features. By Theorem 2.9, if $\mathcal{A}$ has Bishop's property (β), then so does any (n, m)-square metrically equivalent operator $\mathcal{B}$. Theorem 2.5 further shows that if $\mathcal{A}^n$ and $\mathcal{B}^m$ share an isolated eigenvalue of finite multiplicity, then both $\mathcal{A}$ and $\mathcal{B}$ are polaroid at that point. Likewise, Corollary 2.3 and Theorem 2.11 confirm that isoloid behavior is preserved, while Theorem 2.13 extends these properties to direct sums and tensor products.

However, equivalence does not guarantee full spectral invariance. Example 2.7 illustrates that two operators may be (n, m)-square metrically equivalent yet still have different spectra. Thus, this relation preserves Bishop's property, isoloid, and polaroid behavior, but not the entire spectrum or numerical range.

4.2 Relation to Other Operator Classes

The notion of (n, m)-square metric equivalence connects naturally with other operator classes. When $(n, m) = 1$, it reduces to familiar cases of metric equivalence. For instance, Theorem 2.8 shows that if $\mathcal{A}$ and $\mathcal{B}$ are quasi-isometries and (n, m)-square metrically equivalent with $(n, m) = 1$, then they are also metrically equivalent.

Further links appear in preservation results: Corollary 2.3 characterizes (n, m)-square-normality, Theorem 2.9 ensures transfer of isoloid behavior, and Theorem 2.11 extends polaroid behavior. These connections show that (n, m)-square metric equivalence not only generalizes earlier equivalence relations but also aligns with classical spectral properties.

4.3 Examples and Warnings

Although (n, m)-square metric equivalence preserves many important properties, it does not guarantee full spectral invariance. For instance, Example 2.7 shows that two operators $\mathcal{A}$ and $\mathcal{B}$ may be (n, m)-square metrically equivalent yet still have different spectra.

This case highlights that while the equivalence preserves Bishop's property, isoloid, and polaroid behaviors, caution is needed when extending conclusions to the entire spectrum or numerical range.

4.4 Behavior under Direct Sums and Tensor Products

The stability of (n, m)-square metric equivalence extends to operator constructions. Theorem 2.13 establishes that if operators A_i and B_i are (n, m)-square metrically equivalent for each i, then their direct sums and tensor products are also (n, m)-square metrically equivalent. This result highlights the robustness of the relation when analyzing systems of operators rather than individual ones.

4.5 Background and Problem Statement

In operator theory, understanding the structure and relationships between different classes of operators is very important. Traditional equivalence relations, like unitary and metric equivalence, have helped in this study. However, they are sometimes not flexible enough to capture deeper similarities between operators.

To address this, we introduced a new relation called n-Square Metrically Equivalent Operators. This new class extends the usual square-metric equivalence by including a power n, which allows us to study more general relationships between operators.

This work focuses on understanding the properties of this new class. We also compare it with known operator classes, such as normal, hyponormal, isoloid, and polaroid operators. The main goal is to find out what properties are preserved under this new equivalence and how it can be useful in operator theory.

5 CONCLUSIONS

The main goal of this study was to introduce and explore the concept of (n, m)-Square Metrically Equivalent Operators. Based on the results and discussion, we can make the following conclusions:

a) The relation $\sim_{m^{2(n,m)}}$ defines a new class of equivalence for bounded linear operators. This generalizes classical metric and square-metric equivalence.

Applied Example (Novel Grouping): Consider two operators on $\mathbb{C}^2$:

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/3} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/6} \end{bmatrix}.$$

They are classically metrically equivalent ($\mathcal{A}^*\mathcal{A} = \mathcal{B}^*\mathcal{B} = I$). However, for $(n, m) = (1, 1)$-square metric equivalence:

$$\mathcal{A}^{*2}\mathcal{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i2\pi/3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/3} \end{bmatrix} = I, \quad \mathcal{B}^{*2}\mathcal{B}^2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/3} \end{bmatrix} = I.$$

Thus, $\mathcal{A} \sim_{m^{2(1,1)}} \mathcal{B}$. This new relation groups operators with different phase evolution properties that are considered identical under the classical metric equivalence, demonstrating its broader and more nuanced classification power.

b) (n, m)-Square metric equivalence preserves many important properties of operators, including Bishop's property (β), isoloid, and polaroid properties under certain conditions.

Applied Example (Robust Quantum Gates): Let an ideal quantum gate be represented by a normal, polaroid operator $\mathcal{A}$ (e.g., a phase gate, $\mathcal{A} = \text{diag}(1, i)$). A physical implementation might have a slight distortion, represented by the operator $\mathcal{B} = U^*\mathcal{A}U$, where U is a unitary matrix (e.g., a small rotation). By Theorem 3.2, $\mathcal{B}$ is also $(1, 1)$-square-normal. Furthermore, since unitary equivalence is a special case of (n, m)-square metric equivalence, Theorem 3.11 implies that if $\mathcal{A}$ is polaroid, then $\mathcal{B}$ is also polaroid. This means the distorted gate's errors remain isolated and correctable, which is a fundamental requirement for fault-tolerant quantum computation.

c) This equivalence does not always preserve the spectrum or the numerical range, as shown in the examples.

Applied Example (Signal Processing Filters): Consider two digital filters, $\mathcal{A}$ and $\mathcal{B}$, acting on a two-dimensional signal. Let:

$$\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

Their spectra are different: $\sigma(\mathcal{A}) = \{2, 0.5\}$ and $\sigma(\mathcal{B}) = \{2, 1\}$. However, one can verify that for a specific choice of n and m (e.g., $n = 2, m = 1$), the condition $\mathcal{A}^{*2m}\mathcal{A}^{2n} = \mathcal{B}^{*2m}\mathcal{B}^{2n}$

could potentially hold, depending on the constructed example. This illustrates a crucial point: two systems can process "energy" (the square of the norm) in the same way for specific power operations ($2n$ and $2m$) while still having fundamentally different amplification factors (eigenvalues) for individual signal components.

d) The class is compatible with operations like direct sums and tensor products, which makes it useful in studying systems of operators.

Applied Example (Distributed Sensor Networks): Consider a network of r sensors. The data processing of each sensor i is modeled by an operator $\mathcal{A}_i$. If each local operator $\mathcal{A}_i$ is (n, m)-square metrically equivalent to a corresponding ideal operator $\mathcal{B}_i$ (i.e., $\mathcal{A}_i^{*2m}\mathcal{A}_i^{2n} = \mathcal{B}_i^{*2m}\mathcal{B}_i^{2n}$), then Theorem 3.13 guarantees that the global processing operator of the entire network–whether modeled as a direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \oplus \mathcal{A}_r$ (independent processing) or a tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \ldots \otimes \mathcal{A}_r$ (correlated processing)–is also (n, m)-square metrically equivalent to the ideal global system $\mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \ldots \oplus \mathcal{B}_r$. This allows system-wide properties (like stability) to be inferred from the properties of the individual components, simplifying the analysis of large-scale systems.

e) When $(n, m) = 1$, the new equivalence reduces to known cases, such as metric equivalence for idempotent or quasi-isometric operators.

Applied Example (Projections in Computer Graphics): The operation of projecting a 3D point onto a 2D screen is represented by a projection operator $\mathcal{P}$, which is idempotent ($\mathcal{P}^2 = \mathcal{P}$). Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two different projection matrices that both project onto the exact same screen plane. They are metrically equivalent ($\mathcal{P}_1^*\mathcal{P}_1 = \mathcal{P}_2^*\mathcal{P}_2$). For $(n, m) = (1, 1)$-square metric equivalence:

$$\mathcal{P}_1^{*2}\mathcal{P}_1^2 = (\mathcal{P}_1^*\mathcal{P}_1)^2 = (\mathcal{P}_1)^2 = \mathcal{P}_1.$$

Similarly, $\mathcal{P}_2^{*2}\mathcal{P}_2^2 = \mathcal{P}_2$. Therefore, if $\mathcal{P}_1 = \mathcal{P}_2$ (they produce the same image), they are $(1, 1)$-square metrically equivalent. This shows that for projections, this sophisticated new equivalence relation correctly reduces to the classical notion of them doing the same job.

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Competing Interests

Authors have declared that no competing interests exist.

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