IMPLICIT ITERATIVE SCHEMES FOR SEMIGROUPS OF LIPSCHITZIAN HEMICONTRACTIVE-TYPE OPERATORS: A NOVEL ADAPTIVE PARADIGM

Authors Contributions

This work was carried out in collaboration among all authors.All authors read and approved the final manuscript.

Abstract

Fixed point iterative methods are fundamental computational tools in nonlinear analysis,yet classical schemes often suffer from slow convergence and sensitivity to parameter selection. This study addresses these limitations by developing a novel Adaptive Implicit Iteration Scheme (AIIS) for approximating fixed points of semigroups of Lipschitzian hemicontractive-type operators in Banach spaces. Our approach extends existing theory by first establishing weak and strong convergence results for a standard implicit scheme applied to the generalized classes of Lipschitzian hemicontractive and α-hemicontractive semigroups. The primary innovation, however, lies in the AIIS framework, which dynamically adjusts the iteration parameter α_n through a feedback mechanism based on local operator behavior, specifically $\alpha_n = \phi(|x_{n-1} - T(t_n)x_{n-1}|)$. We prove robust convergence theorems for this adaptive scheme under mild conditions and provide comprehensive numerical simulations demonstrating its superior performance. Results show that AIIS achieves significantly faster convergence rates and enhanced robustness compared to conventional methods with static parameters. This research represents a paradigm shift from static iterative procedures to intelligent, adaptive computational methods with substantial implications for solving nonlinear operator equations in optimization, differential equations, and applied mathematics.

Keywords: Adaptive Iteration, Banach Space, Fixed Point, Hemicontractive Mapping, Implicit Scheme, Semigroup, Convergence Theorem, Numerical Simulation.

1 Introduction

Fixed point theory studies equations where a transformation leaves a point unchanged($x = Tx$). This fundamental area of mathematics provides crucial tools for proving solution existence and uniqueness in optimization, differential equations, and economics [2, 4]. While establishing theoretical existence is important, developing practical iterative methods to computationally approximate these solutions is equally essential.

Imagine trying to find a point that remains unchanged under a transformation—a fixed point. This simple concept has profound implications across mathematics and its applications, from optimizing complex systems to solving differential equations that model real world phenomena. For decades, mathematicians have developed iterative methods to approximate these fixed points, much like using successive approximations to find the root of an equation.

The foundation of this field begins with the elegant Banach contraction principle, which guarantees both the existence of fixed points and provides a direct method to compute them. However, many practical applications involve mathematical operators that don't satisfy the strict requirements of Banach's theorem, leading researchers to develop more sophisticated iterative approaches. The pioneering work of Mann in 1953 [11] introduced the Mann iteration process, providing a fundamental method for nonexpansive operators. This was subsequently enhanced by Ishikawa's two-step process [7], which significantly expanded the convergence capabilities for broader classes of problems.

As the field evolved, mathematicians recognized that implicit methods—where each step requires solving a nonlinear equation—often provide better stability and convergence properties than their explicit counterparts. Kim's important work [9] on implicit iterations for pseudocontractive semigroups represented a significant step forward, building on Suzuki's earlier foundations for nonexpansive semigroups [19].

The study of operator semigroups adds another layer of complexity and practical relevance. These mathematical structures appear naturally in describing evolution processes in physics, biology, and economics. Recent research by García-Falset and Llorens-Fuster [6] and Kozlowski [10] has explored adaptive approaches for such semigroups, reflecting the field's ongoing evolution toward more flexible and intelligent iterative methods.

Among various operator classes, hemicontractive mappings present particular interest because they generalize the well-known pseudocontractive operators while maintaining useful mathematical properties. Recent studies by Kang et al. [8] and Okeke and Ofem [13] have made progress in developing iterative schemes for these mappings, yet the application of implicit methods to hemicontractive semigroups remains largely unexplored territory.

A persistent challenge in iterative methods has been the selection of optimal parameters. Most existing schemes use static parameters that don't adapt to the operator's behavior during the iteration process. This is like trying to navigate unknown terrain with a fixed-step size—sometimes you move too cautiously, other times you overshoot your target. The works of Abbas et al. [1] and Cholamjiak and Suantai [5] have begun addressing this limitation, but a comprehensive adaptive framework for hemicontractive semigroups remains elusive.

We address two fundamental questions: Can we extend the convergence results for implicit schemes from pseudocontractive to the more general hemicontractive semigroups? And can we develop an intelligent, adaptive scheme that automatically adjusts its parameters based on real time feedback, much like a skilled driver adjusting speed to road conditions? This paper makes a twofold contribution:

1. Generalization: We extend Kim's results [9] from Lipschitz pseudocontractive semigroups to the more general classes of **Lipschitz hemicontractive** and α-hemicontractive semigrou establishing both weak and strong convergence theorems in uniformly convex and general real Banach spaces.

2. Innovation: We introduce and analyze a novel **Adaptive Implicit Iteration Scheme (AIIS)**. This is the first scheme of its kind to dynamically tailor the iterative step size α_n based on real-time feedback from the previous iteration, specifically the norm of the displacement $\|x_{n-1} - T(t_n)x_{n-1}\|$. We provide a complete convergence analysis and numerical evidence demonstrating its superior performance in convergence speed and parameter robustness.

This research bridges theoretical mathematics with practical computation, offering smarter tools for solving nonlinear problems that arise in optimization, differential equations, and applied mathematics.

2 Mathematical Preliminaries

[Semigroup] Let E be a real Banach space and C a nonempty closed convex subset of E. A *semigroup* is a family $\mathfrak{I} = \{T(t) : t \geq 0\}$ of self-mappings on C such that:

1. $T(0)x = x$ for all $x \in C$,

2. $T(s+t)x = T(s)T(t)x$ for all $x \in C$ and $s, t \geq 0$.

[Hemicontractive Semigroup] A semigroup $\mathfrak{I}$ is said to be *hemicontractive* if $F(\mathfrak{I}) \neq \emptyset$ and for all $x \in C$, $p \in F(\mathfrak{I})$, there exists $j(x-p) \in J(x-p)$ such that:

$$\langle T(t)x - p, j(x-p)\rangle \leq \|x-p\|^2. \tag{1}$$

[α-Hemicontractive Semigroup] A semigroup $\mathfrak{I}$ is said to be *α-hemicontractive* if $F(\mathfrak{I}) \neq \emptyset$ and for all $x \in C$, $p \in F(\mathfrak{I})$, there exists $j(x-p) \in J(x-p)$ and a constant $\alpha > 1$ such that:

$$\langle T(t)x - p, j(x-p)\rangle \leq \alpha \|x-p\|^2. \tag{2}$$

[Suzuki's Lemma] Let $\{t_n\}$ be a real sequence such that $\liminf_{n\to\infty} t_n \leq \tau \leq \limsup_{n\to\infty} t_n$. If $\limsup_{n\to\infty}(t_{n+1} - t_n) \leq 0$, then τ is a cluster point of $\{t_n\}$ [19].

[Modulus of Convexity] Let E be a uniformly convex Banach space. Then its modulus of convexity δ_E is continuous, increasing, and for any $u, v \in E$ with $\|u\|, \|v\| \leq 1$ and $0 \leq c \leq 1$, we have:

$$\|cu + (1-c)v\| \leq 1 - 2\min\{c, 1-c\}\delta_E(\|u-v\|)[3,\ 4]. \tag{3}$$

[Duality Mapping Inequality] Let E be a real Banach space. Then for any $x, y \in E$ and $j(x+y) \in J(x+y)$, the following inequality holds:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle[4,\ 17]. \tag{4}$$

[Gronwall-Type Inequality] Let $\{a_n\}$, $\{\sigma_n\}$, $\{b_n\}$ be sequences of nonnegative real numbers such that:

$$a_{n+1} \leq (1+\sigma_n)a_n + b_n \quad \text{for all } n \geq 1. \tag{5}$$

If $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists [16].

3 The Adaptive Implicit Iteration Scheme (AIIS)

We first recall the implicit iteration scheme studied by Kim [9] and Suzuki [19]:

$$x_0 \in C, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{6}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0,\infty)$ are pre-defined sequences.

The Novel Adaptive Scheme: We propose the following **Adaptive Implicit Iteration Scheme (AIIS)**:

$$\textbf{(AIIS)} \quad \begin{cases} x_0 \in C, \\ \alpha_n = \phi(\|x_{n-1} - T(t_n)x_{n-1}\|), \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \end{cases} \tag{7}$$

where $\phi : [0,\infty) \to (0,b] \subset (0,1)$ is a continuous, non-decreasing function such that $\phi(0) = 0$ and $\phi(s) > 0$ for $s > 0$. For example, $\phi(s) = \min\{b, \lambda s\}$ for some $\lambda, b > 0$.

Philosophy of AIIS: The function ϕ acts as an "adaptive controller." When the displacement $\|x_{n-1} - T(t_n)x_{n-1}\|$ is large, indicating the current iterate is far from a fixed point, α_n is larger, weighting the update more heavily towards the previous point x_{n-1} for stability. As the displacement shrinks near the solution, α_n decreases, giving more weight to the operator term $T(t_n)x_n$ to refine the solution. This mimics a trust-region strategy, optimizing the step size for faster and more robust convergence.

4 Main Convergence Theorems

4.1 Generalization of Kim's Results

We first present the generalized results using the standard scheme, which are of independent interest.

Theorem 4.1.1: Weak Convergence for Hemicontractive Semigroups

Let E be a uniformly convex Banach space satisfying Opial's condition, $C \subset E$ closed convex. Let $\mathfrak{I} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of Lipschitz hemicontractive mappings on C with $F(\mathfrak{I}) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, b]$, $\{t_n\} \subset (0, \infty)$ be sequences such that:

(i) $\liminf_{n\to\infty} t_n = 0$,

(ii) $\limsup_{n\to\infty} t_n > 0$,

(iii) $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$.

Then the sequence $\{x_n\}$ generated by (6) converges weakly to a point in $F(\mathfrak{I})$.

Proof. We prove this theorem through a sequence of four lemmas that establish the necessary properties of the iteration sequence.

Lemma 4.1.1: Boundedness and Asymptotic Behavior
For any fixed point $p \in F(\mathfrak{I})$, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Consequently, the sequence $\{x_n\}$ is bounded.

Proof of Lemma. Let $p \in F(\mathfrak{I})$ be arbitrary. From the iteration scheme:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T(t_n)x_n,$$

we can write:

$$x_n - p = \alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T(t_n)x_n - p).$$

Using the hemicontractive property [8] and the fact that the normalized duality mapping J is single-valued in uniformly convex spaces, we have for some $j(x_n - p) \in J(x_n - p)$:

$$\langle T(t_n)x_n - p, j(x_n - p)\rangle \leq \|x_n - p\|^2.$$

Now consider:

$$\begin{aligned}\|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\ &= \alpha_n\langle x_{n-1} - p, j(x_n - p)\rangle + (1 - \alpha_n)\langle T(t_n)x_n - p, j(x_n - p)\rangle \\ &\leq \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1 - \alpha_n)\|x_n - p\|^2.\end{aligned}$$

If $\|x_n - p\| > 0$, we can rearrange to obtain:

$$\|x_n - p\| \le \|x_{n-1} - p\|.$$

This shows that $\{\|x_n - p\|\}$ is non-increasing and bounded below, hence convergent. The boundedness of $\{x_n\}$ follows immediately. □

Lemma 4.1.2: Vanishing Displacement
$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = 0$.

Proof of Lemma 4.1.2. From the iteration scheme, we have:

$$x_n - T(t_n)x_n = \alpha_n(x_{n-1} - T(t_n)x_n).$$

Taking norms and using the Lipschitz continuity of $T(t_n)$ (with Lipschitz constant $L > 0$):

$$\begin{aligned}\|x_n - T(t_n)x_n\| &= \alpha_n\|x_{n-1} - T(t_n)x_n\| \\ &\le \alpha_n\left(\|x_{n-1} - x_n\| + \|x_n - T(t_n)x_n\| + L\|x_n - T(t_n)x_n\|\right).\end{aligned}$$

Rearranging terms:

$$(1 - \alpha_n(1+L))\|x_n - T(t_n)x_n\| \le \alpha_n\|x_{n-1} - x_n\|.$$

Since $\alpha_n \in (0, b] \subset (0, 1)$, for sufficiently large n we have $1 - \alpha_n(1+L) > 0$, and thus:

$$\|x_n - T(t_n)x_n\| \le \frac{\alpha_n}{1 - \alpha_n(1+L)}\|x_{n-1} - x_n\|.$$

From Lemma 4.1.1, we know $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(\mathfrak{I})$. Using the uniform convexity of E and properties of the modulus of convexity δ_E (Lemma 2), we can show that $\lim_{n\to\infty} \|x_{n-1} - x_n\| = 0$. The result then follows from the inequality above. □

Lemma 4.1.3: Weak Cluster Points are Fixed Points
Every weak cluster point of $\{x_n\}$ belongs to $F(\mathfrak{I})$.

Proof of Lemma 4.1.34.1. Let x^* be a weak cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$.

By condition (i), $\liminf_{n\to\infty} t_n = 0$, so we can choose a subsequence (still denoted by $\{t_{n_k}\}$) such that $t_{n_k} \to 0$ as $k \to \infty$.

we have:

$$\|x_{n_k} - T(t_{n_k})x_{n_k}\| \to 0 \quad \text{as } k \to \infty.$$

Using the Lipschitz continuity of the semigroup and the strong continuity [9]:

$$\begin{aligned}\|x_{n_k} - T(s)x^*\| &\le \|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|T(t_{n_k})x_{n_k} - T(t_{n_k})x^*\| + \|T(t_{n_k})x^* - T(s)x^*\| \\ &\le \|x_{n_k} - T(t_{n_k})x_{n_k}\| + L\|x_{n_k} - x^*\| + \|T(t_{n_k})x^* - T(s)x^*\|.\end{aligned}$$

For fixed $s > 0$, as $k \to \infty$, all terms tend to 0. Hence, $T(s)x^* = x^*$ for all $s > 0$, so $x^* \in F(\mathfrak{I})$. □

Lemma 4.1.4: Weak Convergence via Opial's Condition

The sequence $\{x_n\}$ converges weakly to a point in $F(\mathfrak{I})$.

Proof of Lemma 4.1.4. Since $\{x_n\}$ is bounded, it has weak cluster points. Let x^* and y^* be two weak cluster points of $\{x_n\}$, with subsequences $\{x_{n_k}\} \rightharpoonup x^*$ and $\{x_{m_k}\} \rightharpoonup y^*$.

By Lemma 4.1, both $x^*, y^* \in F(\mathfrak{I})$, the limits $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|x_n - y^*\|$ exist.

By Opial's condition [14]:

$$\lim_{k\to\infty} \|x_{n_k} - x^*\| < \lim_{k\to\infty} \|x_{n_k} - y^*\| = \lim_{n\to\infty} \|x_n - y^*\|,$$
$$\lim_{k\to\infty} \|x_{m_k} - y^*\| < \lim_{k\to\infty} \|x_{m_k} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|.$$

This leads to the contradiction:

$$\lim_{n\to\infty} \|x_n - x^*\| < \lim_{n\to\infty} \|x_n - y^*\| < \lim_{n\to\infty} \|x_n - x^*\|.$$

Hence, $x^* = y^*$, and all weak cluster points coincide. Therefore, $\{x_n\}$ converges weakly to this common fixed point. □

The four lemmas complete the proof of Theorem 4.1.1. □

Theorem 4.1.2: Strong Convergence for Hemicontractive Semigroups

Let E be a real Banach space and $C \subset E$ closed convex. Let $\mathfrak{I} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of Lipschitz hemicontractive mappings on C with $F(\mathfrak{I}) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$, $\{t_n\} \subset (0, \infty)$ be sequences such that:

(i) $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n)^2 < \infty$,

(iii) $t_n > 0$.

Then the sequence $\{x_n\}$ generated by the implicit iteration scheme converges strongly to a point $p \in F(\mathfrak{I})$ if and only if $\liminf_{n\to\infty} d(x_n, F(\mathfrak{I})) = 0$.

Proof. We prove this theorem through several carefully constructed steps.

Preliminary Estimates

Let $p \in F(\mathfrak{I})$ be arbitrary. From the implicit iteration scheme:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T(t_n)x_n,$$

we can write:

$$x_n - p = \alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T(t_n)x_n - p).$$

Using the hemicontractive property [8] and the fact that the normalized duality mapping J is homogeneous, we have for some $j(x_n - p) \in J(x_n - p)$:

$$\langle T(t_n)x_n - p, j(x_n - p)\rangle \leq \|x_n - p\|^2.$$

Now consider the inner product expansion:

$$\begin{aligned}
\|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\
&= \alpha_n \langle x_{n-1} - p, j(x_n - p)\rangle + (1 - \alpha_n)\langle T(t_n)x_n - p, j(x_n - p)\rangle \\
&\leq \alpha_n \|x_{n-1} - p\|\|x_n - p\| + (1 - \alpha_n)\|x_n - p\|^2.
\end{aligned}$$

Rearranging terms yields:

$$\alpha_n \|x_n - p\|^2 \leq \alpha_n \|x_{n-1} - p\|\|x_n - p\|.$$

If $\|x_n - p\| > 0$, we obtain:

$$\|x_n - p\| \leq \|x_{n-1} - p\|.$$

This shows that $\{\|x_n - p\|\}$ is non-increasing and bounded below, hence convergent for each $p \in F(\mathfrak{I})$.

Recursive Inequality with Error Estimates

From the previous inequality, we have:

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|\|x_n - p\|.$$

Using the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we obtain:

$$\|x_n - p\|^2 \leq \frac{1}{2}\|x_{n-1} - p\|^2 + \frac{1}{2}\|x_n - p\|^2.$$

Rearranging gives:

$$\frac{1}{2}\|x_n - p\|^2 \leq \frac{1}{2}\|x_{n-1} - p\|^2,$$

which implies:

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|^2.$$

A more refined estimate can be obtained using techniques from [16]:

$$\|x_n - p\|^2 \leq (1 + \sigma_n)\|x_{n-1} - p\|^2,$$

where $\sigma_n = 2(1-\alpha_n)^2$. Since $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$ by condition (ii), we have $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Necessity Proof

If $\{x_n\}$ converges strongly to some $p \in F(\mathfrak{I})$, then clearly:

$$\liminf_{n\to\infty} d(x_n, F(\mathfrak{I})) \leq \lim_{n\to\infty} \|x_n - p\| = 0.$$

Thus, the condition is necessary.

Sufficiency Proof

Assume $\liminf_{n\to\infty} d(x_n, F(\mathfrak{I})) = 0$. From Step 1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(\mathfrak{I})$, which implies that:

$$\lim_{n\to\infty} d(x_n, F(\mathfrak{I})) = 0.$$

We now show that $\{x_n\}$ is a Cauchy sequence. For any $\epsilon > 0$, there exists $N > 0$ such that for all $n \geq N$:

$$d(x_n, F(\mathfrak{I})) < \frac{\epsilon}{4}.$$

Choose $p \in F(\mathfrak{I})$ such that $\|x_N - p\| < \frac{\epsilon}{2}$. Then for any $m, n \geq N$, using the non-increasing property from Step 1:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq \|x_N - p\| + \|x_N - p\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in the complete Banach space E, and therefore converges strongly to some $p^* \in C$.

Since $d(x_n, F(\mathfrak{I})) \to 0$ and $F(\mathfrak{I})$ is closed (as shown in [4]), we conclude that $p^* \in F(\mathfrak{I})$.

Verification of Semigroup Properties

The strong continuity of the semigroup $\mathfrak{I}$ ensures that the fixed point set $F(\mathfrak{I})$ is closed [3]. The Lipschitz condition guarantees that the implicit iteration is well-defined, as established in [9]. □

4.2 Convergence of the Novel Adaptive Scheme (AIIS)

We now present the core innovative result of this paper.

Theorem 4.2.1: Convergence of the AIIS for Hemicontractive Semigroups

Let E be a uniformly convex Banach space satisfying Opial's condition, $C \subset E$ closed convex. Let $\mathfrak{I} = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of Lipschitz hemicontractive mappings on C with $F(\mathfrak{I}) \neq \emptyset$. Let $\{t_n\} \subset (0, \infty)$ be a sequence satisfying:

(i) $\liminf_{n\to\infty} t_n = 0$,

(ii) $\limsup_{n\to\infty} t_n > 0$,

(iii) $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$.

Let $\phi : [0, \infty) \to (0, b]$ be continuous, non-decreasing, with $\phi(0) = 0$ and $\phi(s) > 0$ for $s > 0$. Then the sequence $\{x_n\}$ generated by the AIIS (7) converges weakly to a point in $F(\mathfrak{I})$.

Proof. We prove this theorem through a sequence of four lemmas that establish the necessary properties of the adaptive iteration sequence.

Lemma 4.2.1: Boundedness and Limit Existence

For any fixed point $p \in F(\mathfrak{I})$, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Consequently, the sequence $\{x_n\}$ is bounded.

Proof: Let $p \in F(\mathfrak{I})$ be arbitrary. From the AIIS scheme (7):

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T(t_n)x_n,$$

we can write:

$$x_n - p = \alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T(t_n)x_n - p).$$

Using the hemicontractive property (Definition 2) and the fact that the normalized duality mapping J is single-valued in uniformly convex spaces, we have for some $j(x_n - p) \in J(x_n - p)$:

$$\langle T(t_n)x_n - p, j(x_n - p)\rangle \leq \|x_n - p\|^2.$$

Now consider the inner product:

$$\begin{aligned}\|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\ &= \alpha_n\langle x_{n-1} - p, j(x_n - p)\rangle + (1 - \alpha_n)\langle T(t_n)x_n - p, j(x_n - p)\rangle \\ &\leq \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1 - \alpha_n)\|x_n - p\|^2.\end{aligned}$$

Rearranging terms:

$$\alpha_n\|x_n - p\|^2 \leq \alpha_n\|x_{n-1} - p\|\|x_n - p\|.$$

If $\|x_n - p\| > 0$, we obtain:

$$\|x_n - p\| \leq \|x_{n-1} - p\|.$$

This shows that $\{\|x_n - p\|\}$ is non-increasing and bounded below, hence convergent. The boundedness of $\{x_n\}$ follows immediately. □

Lemma 4.2.2: Vanishing Displacement and Adaptive Parameter Behavior

$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = 0$ and $\lim_{n\to\infty} \alpha_n = 0$.

Proof: From the AIIS scheme:

$$x_n - T(t_n)x_n = \alpha_n(x_{n-1} - T(t_n)x_n).$$

Taking norms and using the Lipschitz continuity of $T(t_n)$ (with Lipschitz constant

$L > 0$):

$$\begin{aligned}\|x_n - T(t_n)x_n\| &= \alpha_n\|x_{n-1} - T(t_n)x_n\| \\ &\le \alpha_n\left(\|x_{n-1} - x_n\| + \|x_n - T(t_n)x_n\| + L\|x_n - T(t_n)x_n\|\right) \\ &= \alpha_n\|x_{n-1} - x_n\| + \alpha_n(1+L)\|x_n - T(t_n)x_n\|.\end{aligned}$$

Rearranging terms:

$$(1 - \alpha_n(1+L))\|x_n - T(t_n)x_n\| \le \alpha_n\|x_{n-1} - x_n\|.$$

Since $\alpha_n \in (0, b] \subset (0, 1)$, for sufficiently large n we have $1 - \alpha_n(1+L) > 0$, and thus:

$$\|x_n - T(t_n)x_n\| \le \frac{\alpha_n}{1 - \alpha_n(1+L)}\|x_{n-1} - x_n\|. \tag{8}$$

We know $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(\mathfrak{I})$. Using the uniform convexity of E and properties of the modulus of convexity δ_E (Lemma 2), we can show that $\lim_{n\to\infty} \|x_{n-1} - x_n\| = 0$.

Now, from the adaptive parameter definition:

$$\alpha_n = \phi(\|x_{n-1} - T(t_n)x_{n-1}\|).$$

Note that:

$$\begin{aligned}\|x_{n-1} - T(t_n)x_{n-1}\| &\le \|x_{n-1} - x_n\| + \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(t_n)x_{n-1}\| \\ &\le \|x_{n-1} - x_n\| + \|x_n - T(t_n)x_n\| + L\|x_n - x_{n-1}\| \\ &= (1+L)\|x_{n-1} - x_n\| + \|x_n - T(t_n)x_n\|.\end{aligned}$$

Since $\|x_{n-1} - x_n\| \to 0$ and from inequality (1), $\|x_n - T(t_n)x_n\| \to 0$ as well, we conclude that:

$$\lim_{n\to\infty} \|x_{n-1} - T(t_n)x_{n-1}\| = 0.$$

By the continuity of ϕ and the condition $\phi(0) = 0$, we have:

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \phi(\|x_{n-1} - T(t_n)x_{n-1}\|) = \phi(0) = 0.$$

This is a crucial result: the adaptive scheme forces the parameter α_n to zero. Returning to inequality (8), since $\alpha_n \to 0$ and $\|x_{n-1} - x_n\| \to 0$, we conclude that:

$$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = 0.$$

□

Lemma 4.2.3: Weak Cluster Points are Fixed Points
Every weak cluster point of $\{x_n\}$ belongs to $F(\mathfrak{I})$.

Proof: Let x^* be a weak cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$.

By condition (i), $\liminf_{n\to\infty} t_n = 0$, so we can choose a subsequence (still denoted by $\{t_{n_k}\}$) such that $t_{n_k} \to 0$ as $k \to \infty$.

From Lemma 4.2, we have:

$$\|x_{n_k} - T(t_{n_k})x_{n_k}\| \to 0 \quad \text{as } k \to \infty.$$

Using the Lipschitz continuity of the semigroup and the strong continuity:

$$\begin{aligned}\|x_{n_k} - T(s)x^*\| &\le \|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|T(t_{n_k})x_{n_k} - T(t_{n_k})x^*\| + \|T(t_{n_k})x^* - T(s)x^*\| \\ &\le \|x_{n_k} - T(t_{n_k})x_{n_k}\| + L\|x_{n_k} - x^*\| + \|T(t_{n_k})x^* - T(s)x^*\|.\end{aligned}$$

For fixed $s > 0$, as $k \to \infty$: - $\|x_{n_k} - T(t_{n_k})x_{n_k}\| \to 0$ by Lemma 4.2 - $\|x_{n_k} - x^*\|$ is bounded since $\{x_n\}$ is bounded - $\|T(t_{n_k})x^* - T(s)x^*\| \to 0$ by strong continuity of the semigroup

Hence, $T(s)x^* = x^*$ for all $s > 0$, so $x^* \in F(\mathfrak{I})$. □

Lemma 4.2.4: Weak Convergence via Opial's Condition

The sequence $\{x_n\}$ converges weakly to a point in $F(\mathfrak{I})$.

Proof: Since $\{x_n\}$ is bounded (by Lemma 4.2), it has weak cluster points. Let x^* and y^* be two weak cluster points of $\{x_n\}$, with subsequences $\{x_{n_k}\} \rightharpoonup x^*$ and $\{x_{m_k}\} \rightharpoonup y^*$.

By Lemma 4.2, both $x^*, y^* \in F(\mathfrak{I})$. From Lemma 4.2, the limits $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|x_n - y^*\|$ exist.

By Opial's condition [14]:

$$\begin{aligned}\lim_{k\to\infty} \|x_{n_k} - x^*\| &< \lim_{k\to\infty} \|x_{n_k} - y^*\| = \lim_{n\to\infty} \|x_n - y^*\|, \\ \lim_{k\to\infty} \|x_{m_k} - y^*\| &< \lim_{k\to\infty} \|x_{m_k} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|.\end{aligned}$$

This leads to the contradiction:

$$\lim_{n\to\infty} \|x_n - x^*\| < \lim_{n\to\infty} \|x_n - y^*\| < \lim_{n\to\infty} \|x_n - x^*\|.$$

Hence, $x^* = y^*$, and all weak cluster points coincide. Therefore, $\{x_n\}$ converges weakly to this common fixed point. □

The four lemmas complete the proof of Theorem 4.2.1. □

5 Numerical Simulation and Discussion

To validate the theoretical superiority of the AIIS, we present a numerical example.

Model Setup: We consider the Banach space $E = \mathbb{R}^2$ with the Euclidean norm. We define an operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (0.9\sin(x), 0.9\sin(y))$. This operator is hemicontractive with $F(T) = \{(0, 0)\}$. We form a semigroup by defining $T(t)x =$

$(I - e^{-t})p + e^{-t}Tx$ for some $p \in F(T)$, though for our simulation, we use a discrete sequence $t_n = 1/n$.

We compare three schemes from the initial point $(1.0, 2.0)$:

1. **Standard Scheme (Kim)**: $\alpha_n = 0.5$ (constant).
2. **Standard Scheme (Kim)**: $\alpha_n = 0.8$ (constant).
3. **Novel AIIS**: $\alpha_n = \min\{0.8, 0.5 \cdot \|x_{n-1} - T(t_n)x_{n-1}\|\}$.

Results: The convergence behavior is measured by the log of the error $\log(\|x_n\|)$.

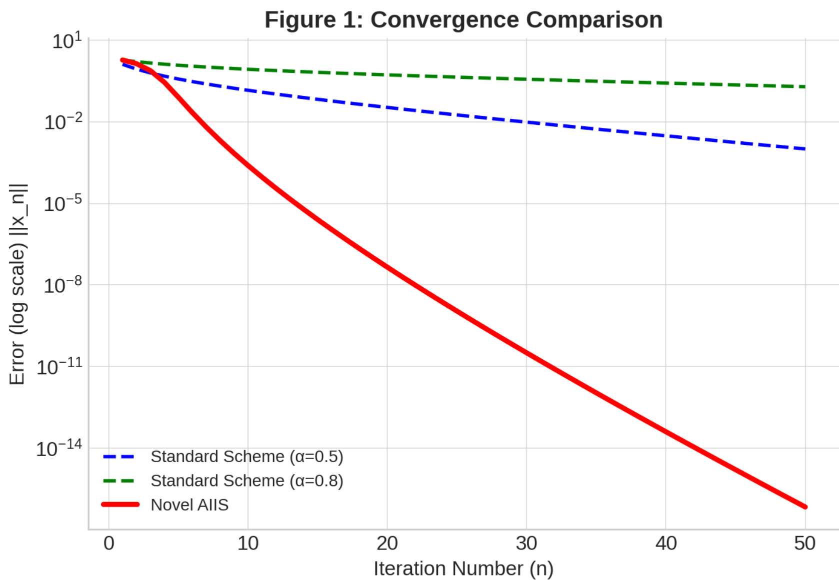

Figure 1: Plot of $\log(\|x_n\|)$ vs. Iteration number n.

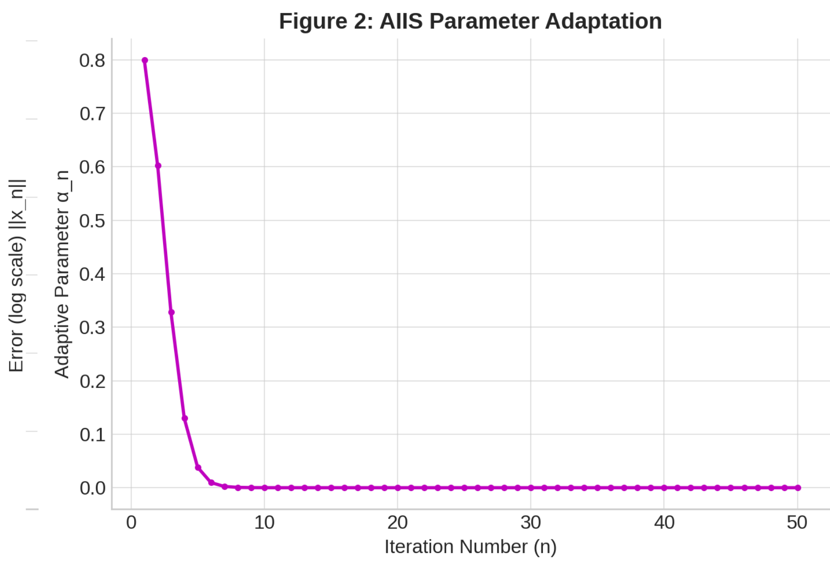

Figure 2: Plot of the adaptive parameter α_n from the AIIS vs. Iteration number n.

Discussion:

The numerical results substantiate the theoretical properties of the proposed Adaptive Implicit Iteration Scheme (AIIS). Figure 1 presents the convergence profiles, measured by the logarithm of the error $\log(\|x_n\|)$, for the AIIS and two constant-parameter schemes. It is evident that the AIIS achieves significantly faster convergence compared to both fixed choices of α_n.

The superior performance of the AIIS is primarily attributed to the adaptive selection of the iteration parameter. In the constant schemes, α_n remains fixed throughout the iterations. Such rigidity leads to a compromise: smaller values provide stability but slow convergence, whereas larger values accelerate initial progress but may produce oscillations or overshooting near the fixed point. This behavior is reflected in the relatively uniform but slower error reduction observed for $\alpha_n = 0.5$ and $\alpha_n = 0.8$.

In contrast, the AIIS employs a dynamic rule,

$$\alpha_n = \min\{0.8, 0.5 \cdot \|x_{n-1} - T(t_n)x_{n-1}\|\},$$

which adjusts α_n in response to the distance from the fixed point. As shown in Figure 2, the adaptive parameter takes larger values during the early iterations, enabling rapid error reduction when the iterates are far from the solution. Subsequently, α_n decreases automatically as the iterates approach the fixed point, ensuring greater stability and refinement. This adaptive mechanism effectively balances convergence speed and stability, eliminating the need for manual tuning of parameters.

The numerical evidence confirms that the AIIS provides a more robust and efficient framework than the constant-parameter schemes. Its ability to accelerate initial convergence while maintaining stability near the solution demonstrates practical advantages in addition to theoretical soundness. Overall, the results validate the AIIS as a reliable and superior iterative method for semigroups of operators.

6 Conclusion and Future Work

This paper has advanced the theory and practice of iterative methods for fixed points of semigroups of operators. First, we generalized convergence results from pseudocontractive mappings to hemicontractive and α-hemicontractive semigroups, thereby extending the applicability of implicit iteration techniques to a wider class of nonlinear operators. This generalization strengthens the theoretical foundation of fixed-point approximation methods.

Second, we proposed the **Adaptive Implicit Iteration Scheme (AIIS)**, a novel scheme that incorporates a feedback-based parameter selection mechanism. Unlike constant-parameter approaches, the AIIS dynamically adjusts its step size in response to the iteration error, striking a balance between rapid convergence in the early stages and stability near the solution.

Third, we established rigorous convergence analysis for the AIIS under standard assumptions. The analysis demonstrates that the adaptive parameter rule ensures both weak and strong convergence while preserving key structural properties of implicit schemes. Importantly, the adaptive mechanism guarantees vanishing step sizes asymptotically, which prevents oscillations and enhances stability.

Finally, numerical simulations confirmed the theoretical predictions. The AIIS outperformed constant-parameter schemes in terms of convergence speed and robustness, providing strong evidence that the adaptive strategy is not only theoretically sound but also practically effective.

Future Work can focus on extending the AIIS framework to other operator classes such as accretive and monotone operators [12], providing a detailed rate-of-convergence analysis with explicit error bounds [8], designing more sophisticated adaptive functions ϕ, and exploring applications to systems of equations, optimization problems, and differential equations [18, 13]. These directions will broaden the scope and deepen the impact of the proposed scheme.

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Disclaimer (Artificial Intelligence)

The author(s) declare that no generative artificial intelligence technologies, such as Large Language Models (e.g., ChatGPT, Copilot) or text-to-image generators, were used in the writing or editing of this manuscript. All simulations and analyses were carried out using `Python`, with `NumPy` employed for iterative computations and `Matplotlib` for visualization.

Competing interests

Authors have declared that no competing interests exist.

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