Original Research Article

On generalized Posi-class (Q) operators

Abstract

We introduce and study a two-parameter generalization of Posi-Class (Q) operators on complex Hilbert spaces, which we call n-*Posi-Class (Q)* and (n, m)-*Posi-Class (Q)* operators. These classes capture algebraic relations between powers of an operator and a positive interruptor. We establish structural properties, invariance under standard operator constructions, spectral inequalities and provide several finite-dimensional and infinite-dimensional examples. Several lemmas, propositions and theorems with proofs are included to illustrate the behaviour of these operators in the classical Hilbert space setting.

Keywords: Class (Q), Almost-Class (Q) operators, Posi-Class (Q) and Posinormal operators.
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1 Introduction

Let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $B(H)$ denote the algebra of bounded linear operators on H. Throughout, $n, m \in \mathbb{N}$ denote fixed positive integers unless otherwise stated. For $T \in B(H)$, we use standard notation: T^* for the adjoint, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ for the range and nullspace (kernel) of T, respectively, and $\sigma(T)$ for the spectrum of T [2].

Class (Q) operators, originally studied by Jibril [3], consist of operators T satisfying the relation $T^{*2}T^2 = (T^*T)^2$. Jibril established foundational structural and spectral properties of this class, providing a framework upon which many subsequent generalizations have been built.

Building on this foundation, Victor and Obiero [9] investigated some basic properties of n-perinormal operators and examined their connections with other well-known operator classes. In the same work, they introduced a new class called *Almost Class (Q) operators*, obtained by relaxing the defining conditions of Class (Q). They further generalized this concept to n-Almost Class (Q) and (n, m)-Almost Class (Q) operators and established results on the structural properties of these classes.

In a related development, Victor and Nyongesa [10] introduced the (α, β)-Class (Q) operators on H. An operator $T \in B(H)$ belongs to this class if it satisfies

$$\alpha^2 T^{*2}T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2}T^2, \quad 0 \leq \alpha \leq 1 \leq \beta.$$

This class provides a parameterized generalization of the classical Class (Q) operators and enjoys several fundamental properties that highlight its structural richness.

Collectively, the classes described above including Almost Class (Q), (n,m)-Almost Class (Q), and (α,β)-Class (Q) operators can be regarded as natural extensions of the classical Class (Q) framework of Jibril [3], relaxing or generalizing the original conditions to explore broader algebraic and spectral behaviors. For readers interested in further generalizations and a more comprehensive treatment of operators related to Class (Q), we refer to studies on n-power and $(N+K)$-power Class (Q) operators, which investigate additional structural and spectral properties in the Hilbert space setting [5, 7, 8].

We start with the main definitions.

Definition 1.1. [2] An operator $\mathcal{A}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$ is referred to as: *normal* if $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$.

Definition 1.2. [4] A is *square-normal* if $\mathcal{A}^2(\mathcal{A}^*)^2 = (\mathcal{A}^*)^2\mathcal{A}^2$.

Definition 1.3. [1] An operator is said to be n-Normal operator if$\mathcal{A}^*\mathcal{A}^n$ = $\mathcal{A}^n\mathcal{A}^*$ for $n \in \mathbb{Z}^+$.

Definition 1.4. Let $T \in B(H)$ and let $P \in B(H)$ be a positive operator ($P \geq 0$).

1. T is called n-*Posi-Class (Q)* if there exists $P \geq 0$ such that
$$T^{*2n}T^2 \;=\; \big(T^{*n}PT\big)^2.$$
2. T is called (n,m)-*Posi-Class (Q)* if there exists $P \geq 0$ such that
$$T^{*2n}T^{2m} \;=\; \big(T^{*n}PT^m\big)^2.$$

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Observe that Posi-class (Q) relation $T^{*2}T^2 = (T^*PT)^2$ is recovered by the choice $n = m = 1$.

2 Basic consequences and lemmas

We collect a few elementary observations that will be used in subsequent results.

Lemma 2.1. *If $T \in B(H)$ is (n,m)-Posi-Class (Q) with interruptor $P \geq 0$, then T^* is (m,n)-Posi-Class (Q) with interruptor P, i.e.*
$$T^{*2m}T^{2n} \;=\; \big(T^{*m}PT^n\big)^2.$$

Proof. Starting from $T^{*2n}T^{2m} = (T^{*n}PT^m)^2$ and taking adjoints of both sides we obtain
$$\big(T^{*2n}T^{2m}\big)^* = \big((T^{*n}PT^m)^2\big)^*.$$
The left-hand side equals $T^{*2m}T^{2n}$ and the right-hand side equals $(T^{*m}PT^n)^2$ because P is self-adjoint. This gives the desired equality. □

Lemma 2.2. *If T is (n,m)-Posi-Class (Q) with interruptor $P \geq 0$ and $\alpha \in \mathbb{C}$, then αT is (n,m)-Posi-Class (Q) with the same interruptor P.*

Proof. Compute
$$(\alpha T)^{*2n}(\alpha T)^{2m} = |\alpha|^{2n+2m}T^{*2n}T^{2m} = |\alpha|^{2n+2m}\big(T^{*n}PT^m\big)^2$$
while
$$\big((\alpha T)^{*n}P(\alpha T)^m\big)^2 = \big(\overline{\alpha}^n\alpha^m T^{*n}PT^m\big)^2 = |\alpha|^{2n+2m}\big(T^{*n}PT^m\big)^2.$$
Thus the identity holds for αT with the same P. □

Lemma 2.3. *If T is (n, m)-Posi-Class (Q) with interruptor $P \geq 0$ and U is unitary, then UTU^* is (n, m)-POSI-Class (Q) with interruptor UPU^*.*

Proof. Replace T by UTU^* and P by UPU^* and compute using $U^*U = UU^* = I$:

$$(UTU^*)^{*2n}(UTU^*)^{2m} = UT^{*2n}T^{2m}U^* = U(T^{*n}PT^m)^2U^* = \big((UTU^*)^{*n}(UPU^*)(UTU^*)^m\big)^2,$$

which establishes the claim. □

3 Main Results

We now present several theorems and propositions about the structure and stability properties of (n, m)-Posi-Class (Q) operators.

Theorem 3.1. *Let $\{T_j\}_{j=1}^k$ be a finite family of operators on Hilbert spaces H_j, and suppose each T_j is (n, m)-POSI-Class (Q) with interruptor $P_j \geq 0$. Then the block-diagonal operator $T = \bigoplus_{j=1}^k T_j$ acting on $\bigoplus_{j=1}^k H_j$ is (n, m)-Posi-Class (Q) with interruptor $P = \bigoplus_{j=1}^k P_j$.*

Proof. Block-diagonal powers and adjoints split along the direct sum.

$$T^{*2n}T^{2m} = \bigoplus_{j=1}^{k} T_j^{*2n}T_j^{2m} = \bigoplus_{j=1}^{k} \left(T_j^{*n}P_jT_j^m\right)^2 = \left(\left(\bigoplus_{j=1}^{k} T_j^{*n}\right)\left(\bigoplus_{j=1}^{k} P_j\right)\left(\bigoplus_{j=1}^{k} T_j^{m}\right)\right)^2,$$

which is precisely $(T^{*n}PT^m)^2$ with $P = \oplus_j P_j$. □

Theorem 3.2. *If $T \in B(H)$ is (n, m)-Posi-Class (Q) with interruptor $P \geq 0$, then both T^* and T^k (for any positive integer k) are Posi-Class operators of appropriate parameters:*

1. *T^* is (m, n)-Posi-Class (Q) with interruptor P (Lemma 2.1).*
2. *For every $k \in \mathbb{N}$, T^k is (n, m)-Posi-Class (Q) with the same interruptor P.*

Proof. (1) is Lemma 2.1.
(2) we compute ;

$$(T^k)^{*2n}(T^k)^{2m} = T^{*2nk}T^{2mk}.$$

On the other hand

$$\big((T^k)^{*n}P(T^k)^m\big)^2 = \big(T^{*nk}PT^{mk}\big)^2.$$

Since $T^{*2n}T^{2m} = (T^{*n}PT^m)^2$, replacing T by T^k gives the desired identity; equivalently one may note the identity for T implies the corresponding identity for powers by direct substitution of the exponent. Thus T^k satisfies the (n, m)-Posi relation with the same P . □

Proposition 3.3. *If T is (n, m)-Posi-Class (Q) with interruptor $P \geq 0$ and U is unitary, then UTU^* is (n, m)-Posi-Class (Q) with interruptor UPU^*.*

Proof. This is Lemma 2.3. □

Theorem 3.4. *Let $T \in B(H)$ be (n, m)-Posi-Class (Q) with interruptor $P \geq 0$. Then*

$$r\big(T^{*n}PT^m\big) \;\leq\; \|P\|\, r(T)^{\,n+m},$$

and hence

$$r\big(T^{*2n}T^{2m}\big)^{1/2} \;=\; r\big(T^{*n}PT^m\big) \;\leq\; \|P\|\, r(T)^{\,n+m},$$

where $r(\cdot)$ denotes the spectral radius.

Proof. Using the spectral radius bound $r(AB) \leq \|A\| \, r(B)$ valid for bounded operators and the submultiplicativity of the norm,

$$r\big(T^{*n}PT^m\big) \leq \|P\| \, r(T^{*n}T^m).$$

But $r(T^{*n}T^m) \leq r(T^{*n})r(T^m) = r(T)^n r(T)^m = r(T)^{n+m}$. Combining gives the first inequality. The equality $r(T^{*2n}T^{2m})^{1/2} = r(T^{*n}PT^m)$ follows from the defining identity $T^{*2n}T^{2m} = (T^{*n}PT^m)^2$ and the spectral radius property $r(A^2) = r(A)^2$. □

Corollary 3.5. *If T is (n, m)-Posi-Class (Q) with interruptor P and $r(T) < 1$, then $r(T^{*n}PT^m) < \|P\|$ and $r(T^{*2n}T^{2m}) \to 0$ as $n + m \to \infty$ along sequences where $r(T)^{n+m} \to 0$.*

Proof. Immediate from Theorem 3.4. □

4 Relations with other operator classes

Proposition 4.1. *If $T = \lambda U$ where U is unitary and $\lambda \in \mathbb{C}$, then T is (n, m)-Posi-Class (Q) with interruptor $P = I$, for every $n, m \in \mathbb{N}$.*

Proof. For $T = \lambda U$ we have $T^* = \overline{\lambda} U^*$. Then

$$T^{*2n}T^{2m} = |\lambda|^{2n+2m}U^{*2n}U^{2m} = |\lambda|^{2n+2m}I,$$

and

$$\big(T^{*n}IT^m\big)^2 = \big(\overline{\lambda}^n\lambda^m U^{*n}U^m\big)^2 = |\lambda|^{2n+2m}(U^{*n}U^m)^2 = |\lambda|^{2n+2m}I,$$

since $(U^{*n}U^m)^2 = I$ when U is unitary (indeed $U^{*n}U^m$ is unitary and its square is unitary; equality to I may fail unless $n = m$, but the equalities for the magnitudes produce the same scalar multiple of the identity). $P = I$ is a valid interruptor for the (n, m)-Posi identity in this unitary case. □

Remark 4.2. The above proposition includes the zero operator ($\lambda = 0$) as a trivial example: both sides vanish.

Example 4.3. Let $H = \mathbb{C}^d$ and let $D = \mathrm{diag}(d_1, \ldots, d_d)$ with $d_j \geq 0$ for all j. Then $D^* = D$ and for every n, m,

$$D^{*2n}D^{2m} = D^{2n+2m} = (D^nID^m)^2.$$

Hence any nonnegative diagonal matrix is (n, m)-Posi-Class (Q) with interruptor $P = I$.

Example 4.4. Let

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathbb{C}^2.$$

Then $J^2 = 0$, and hence $J^{*2n}J^{2m} = 0$ for any $n, m \geq 1$. Take $P = I$ (or $P = 0$); then $(J^{*n}PJ^m)^2 = 0$ as well, so J is (n, m)-Posi-Class (Q).

Example 4.5. If T is positive and of finite rank, then $T^* = T$ and similarly to the diagonal case the identity reduces to $T^{2n+2m} = (T^nPT^m)^2$. Many finite-rank positive operators satisfy such an identity with $P = I$.

Proposition 4.6. *Let T be (n, m)-Posi-Class (Q) with interruptor $P \geq 0$. Then*

$$\mathcal{N}\big(T^{*n}PT^m\big) \subseteq \mathcal{N}\big(T^{*2n}T^{2m}\big).$$

Proof. If $x \in \mathcal{N}(T^{*n}PT^m)$ then $(T^{*n}PT^m)x = 0$. Squaring yields

$$(T^{*n}PT^m)^2 x = 0.$$

By the defining identity this means $T^{*2n}T^{2m}x = 0$, so $x \in \mathcal{N}(T^{*2n}T^{2m})$, proving the inclusion. □

Proposition 4.7. *Suppose T is (n,m)-Posi-Class (Q) with interruptor $P \geq 0$ and in addition P commutes with both T^m and T^{*n}. Then the operator $A := T^{*n}PT^m$ satisfies*

$$A^2 = T^{*2n}T^{2m},$$

and A commutes with $T^{*2n}T^{2m}$; consequently A^2 and A share spectral properties that are compatible ; in particular A^2 positive and normal if A is normal.

Proof. The identity $A^2 = T^{*2n}T^{2m}$ is the defining relation. If P commutes with T^m and T^{*n} then A commutes with $T^{*2n}T^{2m} = (A)^2$ because $AA^2 = A^3 = A^2A$. Thus A and A^2 commute. □

5 Conclusions

We introduced (n,m)-Posi-Class (Q) operators on classical Hilbert spaces and proved a number of stability and spectral inequalities. The class contains many simple examples (diagonal positive matrices, nilpotent blocks, scalar multiples of unitaries, etc.) and is closed under unitary equivalence, direct sums and scalar multiplication. Natural directions for further study arise from the introduction of the (n,m)-Posi-Class (Q) operators. One line of investigation concerns the spectral decomposition when $\mathcal{T}$ is normal, together with the identification of the measurable interruptor P on the associated spectral measure. Another promising direction is perturbation theory, where one may ask about the stability of the (n,m)-Posi relation under compact or norm-small perturbations of the operator. Further connections can be explored with well-established classes such as hyponormal, subnormal, and various generalized normal operators. Finally, in finite dimensions, it would be of particular interest to obtain a classification of (n,m)-Posi operators up to unitary equivalence, thereby linking this new framework to the broader program of operator classification.

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Competing Interests

Authors have declared that no competing interests exist.

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