NUMERICAL SOLUTION OF FRACTIONAL ORDER INTEGRO DIFFERENTIAL WITH DIRICHLET BOUNDARY CONDITIONS USING SHIFTED LEGENDRE COLLOCATION METHOD

ABSTRACT: The paper develops and implements a numerical method for solving fractional order Fredholm Volterra integro differential equations with Dirichlet boundary conditions using the shifted Legendre collocation method. The proposed method is formulated by first obtaining the integral form of the given model equation, followed by applying the collocation technique to generate a system of nonlinear equations. These nonlinear equations are then solved using Newton-Raphson's iterative method. The accuracy and efficiency of the developed method are analyzed, demonstrating that the obtained solutions are continuous and exhibit convergence. The uniqueness of the solution is established, further validating the reliability of the approach. To assess the effectiveness of the method, several numerical examples are presented, comparing the obtained results with existing techniques. The numerical experiments confirm that the proposed approach yields highly accurate solutions while maintaining computational efficiency. This study shows the applicability of the shifted Legendre collocation method in solving complex integro-differential equations.

Keywords: Integro Differential Equation, Dirichlet Boundary Condition, Collocation Method, Shifted Legendre Polynomial.

AMS Subject Classification (2010): 34K05, 34K05, 45J05, 47G20, 65D20

1. INTRODUCTION

Fractional calculus have great importance in the field of Mathematics, Physics, Chemistry and Engineering. Mathematical modeling of real life problems usually arises in functional equations such as ordinary and partial differential equations. Many mathematical formulations in physical phenomena contain Integro Differential Equations (IDEs), these equations appear in modelling some phenomena in Science and Engineering. Examples include, the kinetic equations which form the basis of the kinetic theory of rarefied gases, plasma, radiation transfer and coagulation [8]. IDEs have been used to model heat and mass diffusion processes, biological species coexistence together with increasing and decreasing rate of growth; electromagnetic theory and ocean circulation [2]. IDE is an equation in which the unknown function $y(x)$ appears under an integral sign and contains ordinary derivatives [17]. IDEs are usually difficult to solve analytically so it requires to obtain an efficient approximate or numerical solution [21]. Recently, there has been a growing interest in the area of fractional calculus; this is because fractional calculus provides more accurate models of many engineering system than integer order derivatives and integrals [22].

Recently, the numerical analysis of fractional integro-differential equations has witnessed a significant boost due to their efficiency in describing systems with memory and hereditary properties, surpassing their counterparts. These systems emerge in various fields of study,

where fractional order dynamics yield superior predictive models. A range of basis functions and numerical schemes has been developed to handle the computational challenges presented by such systems. For example, the introduction of an efficient Chebyshev method for Volterra integral equations [4], shows that accuracy is obtainable through polynomial approximation. Similarly, the Lagrange polynomial for solving nonlinear fractional integro-differential equations shows notable advancements in convergence and stability for nonlinear systems with variable coefficients [11, 20].
Moreover, a variety of hybrid and orthogonal function-based methods have been developed to strengthen both accuracy and computational efficiency in solving fractional systems. The use of fractional order Legendre-Laguerre functions enables the efficient resolution of fractional PDEs due to their orthogonality and adaptability across unbounded domains [7]. Similarly, fractional-order Legendre wavelets [15] and block methods [18] have emerged as potent alternatives in addressing stiff and high-order fractional integro-differential equations. [24] further emphasised the structural characteristics and operational importance of fractional derivatives in modelling complex systems. These recent advancements provide a solid foundation for the continued development of collocation methods, particularly those based on shifted orthogonal polynomials.

Fractional derivatives are powerful and efficient tools to describe physical systems that have long term memory, especially in modelling complex dynamic systems. The fractional derivative of order $\alpha > 0$ has several definitions. Over the years, mathematicians, using their own notations and approach, have found various definitions that fit the idea of a non-integer order integral or derivative. One version that has been popularized in the world of fractional calculus is the Riemann - Liouville definition. For the fractional derivative, the Caputo's definition is mostly used, which is a modification of the Riemann -Liouville definition; because it has the advantage of dealing properly with initial value problems, since the initial condition is given in terms of field variables and their integer order [12].
This paper considers the numerical solution of Fractional Order Integro Differential Equations of the form

$$ {}_0^cD_t^\alpha u(t) = h(t) + Q(t)\,u(t) + \lambda_1 \int_0^1 w(t,s)\,G(u(s))\,ds + \lambda_2 \int_0^t k(t,s)\,F(u(s))\,ds \qquad (1) $$

$t \in [0,1]\,, \alpha\ t \in [0,2]$subject to the boundary conditions

$$ u(0) = \mu_0, \quad u(1) = \mu_1 \qquad (2) $$

where ${}_0^cD_t^\alpha(.)$ is the left Caputo derivative operator; $h : [0,1] \to \mathbb{R}$, $Q : [0,1] \to \mathbb{R}$, $k : [0,1] \times [0,1] \to \mathbb{R}$ are continuous functions, $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is Lipschitzian continuous.

Most of the approaches for solving FOIDE are based on semi numerical methods such as Adomian decomposition method, variational iteration method, Dafter-Geji and Jafari method among others. Recently, collocation methods have been receiving attention from different authors which include [13] developed Bernolli pseudo spectral method, [14] developed a collocation method for solving fractional order Ricatti differential equation, [3] solves linear and non-linear Fredholm IDE using collocation method; [9] developed Taylor expansion method.

2. BASIC DEFINITIONS

Here, we recall some basic notion, lemmas and theorems which are used in the subsequent sections.

Definition 2.1: *q-contraction [5]* Let $(X, \|.\|)$ be a normed space, the mapping $T : X \to X$ is a q-contraction if $\|Tx_1 - Tx_2\|_\infty \leq q \|x_1 - x_2\|_\infty, q \in [0, 1)$ fixed for all $x_1, x_2 \in X$

Definition 2.2: *[23]* The left Caputo's definition of fractional derivative operator is given by

$$ {}_0^cD_t^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (x-t)^{m-\alpha-1} f^{(m)}(t)dt \tag{3}$$

where $m - 1 \leq \alpha \leq m, m \in N, \alpha \in R, t > 0$.

It has the following two basic properties:

(i) $D^\alpha I^\alpha f(x) = f(x)$

(ii) $I^\alpha D^\alpha f(x) = f(x) - \sum\limits_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, x > 0$

Definition 2.3: *[5]* Let $(X, \|.\|)$ be a norm space, $T : X \to X$ is strict contraction when

$$\|T^n x_1 - T^n x_2\|_\infty \leq q^n \|x_1 - x_2\|_\infty \text{ for all } x, y \in X$$

Definition 2.4: *Riemann-Liouville fractional integral [6]* The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined by

$$ {}_0I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} u(s)ds. \tag{4}$$

Definition 2.5: *Integration of nth derivative [6]* For $\alpha > 0$, let $u(t)$ be a continuous function, then

$$ {}_0I_t^\alpha \left({}^cDu\right)(t) = u(t) - \sum_{k=0}^{\alpha-1} c_k t^k \tag{5}$$

Definition 2.6: [1] Legendre polynomial on the interval $[-1, 1]$ and can be determined with the aid of the recurrence formulae

$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x), n = 1, 2, \cdots \tag{6}$$

where $L_0(x) = 1, L_1(x) = x$. In order to use these polynomials on the interval $x \in [0, 1]$, shifted Legendre polynomial is then defined by the recurrence formula

$$p_{n+1}(x) = \frac{(2n+1)(2n-1)}{(n+1)} p_n(x) - \frac{n}{n+1} p_{n-1}(x) \tag{7}$$

where $p_0 = 1, p_1(x) = 2x - 1$. The analytical form of degree n is defined as

$$p_n(x) = \sum_{k=0}^{n} \frac{(-1)^{n+k}\,\Gamma(n+k+1)}{\Gamma(n-k+1)\,(\Gamma(k+1))^2} x^k \tag{8}$$

3. METHODOLOGY

This section considers the development of our method, which was achieved by developing the integral form of (1) and (2) and the algebraic equations using some lemmas.

Lemma 3.1: (Integral form) Let $u(t) \in C([0,1], \mathbb{R})$ be the solution to (1) and (2), then it is equivalent to

$$\begin{aligned} u(t) &= H(t) - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \begin{bmatrix} Q(s)\,u(s) \\ +\lambda_1 \int_0^1 w(s,\tau)\,G(u(\tau))\,d\tau \\ +\lambda_2 \int_0^s k(s,\tau)\,F(u(\tau)\,d\tau) \end{bmatrix} ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \begin{bmatrix} Q(s)\,u(s) \\ +\lambda_1 \int_0^1 w(s,\tau)\,G(u(\tau))\,d\tau \\ +\lambda_2 \int_0^s k(s,\tau)\,F(u(\tau)\,d\tau) \end{bmatrix} ds \end{aligned} \tag{9}$$

where

$$H(t) = (1-t)\,\mu_0 + t\mu_1 - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds$$

Proof. Let

$$y(t) = h(t) + Q(t)\,u(t) + \lambda_1 \int_0^1 w(t,s)\,G(u(s))\,ds + \lambda_2 \int_0^t k(t,s)\,F(u(s))\,ds$$

hence (1) gives

$${}_0^cD_t^\alpha u(t) = y(t) \tag{10}$$

multiply (10) by ${}_0I_t^\alpha$

$${}_0I_t^{\alpha}\,{}_0^cD_t^\alpha u(t) = {}_0I_t^\alpha y(t)$$

and using (5) for $0 < \alpha \leq 2$

$$u(t) - \sum_{k=0}^{1} c_k t^k = {}_0I_t^\alpha y(t)$$

thus

$$u(t) = c_0 + c_1 t + {}_0I_t^\alpha y(t) \tag{11}$$

considering the boundary conditions $u(0) = \mu_0$

$$u(0) = \mu_0 \Rightarrow c_0 = \mu_0$$

considering the boundary conditions $u(1)=\mu_1$

$$u(1)=c_0+c_1+\ {}_0I_1^{\alpha}y(1)$$

$$c_1=\mu_1-\mu_0-\ {}_0I_1^{\alpha}y(1)$$

substituting the values of c_0 and c_1 in (11)

$$u(t)=\mu_0+\left(\mu_1-\mu_0-\ {}_0I_1^{\alpha}y(1)\right)t+\ {}_0I_t^{\alpha}y(t)$$

$$u(t)=(1-t)\mu_0+t\mu_1-t\ {}_0I_1^{\alpha}y(1)+\ {}_0I_t^{\alpha}y(t)$$

$$\begin{aligned} u(t) &= (1-t)\mu_0+t\mu_1-t\ {}_0I_1^{\alpha}\left(\begin{array}{c} h(t)+Q(t)u(t)+\lambda_1\int_0^1 w(t,s)G(u(s))ds \\ +\lambda_2\int_0^t k(t,s)F(u(s))ds \end{array}\right) \\ &\quad +\ {}_0I_t^{\alpha}\left(\begin{array}{c} h(t)+Q(t)u(t)+\lambda_1\int_0^1 w(t,s)G(u(s))ds \\ +\lambda_2\int_0^t k(t,s)F(u(s))ds \end{array}\right) \end{aligned}$$

using (4)

$$\begin{aligned} u(t) &= (1-t)\mu_0+t\mu_1-\frac{t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left(\begin{array}{c} h(s)+Q(s)u(s) \\ +\lambda_1\int_0^1 w(s,\tau)G(u(\tau))d\tau \\ +\lambda_2\int_0^s k(s,\tau)F(u(\tau))d\tau \end{array}\right)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(\begin{array}{c} h(s)+Q(s)u(s) \\ +\lambda_1\int_0^1 w(s,\tau)G(u(\tau))d\tau \\ +\lambda_2\int_0^s k(s,\tau)F(u(\tau))d\tau \end{array}\right)ds \end{aligned}$$

$$\begin{aligned} u(t) &= H(t)-\frac{t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left[\begin{array}{c} Q(s)u(s)+\lambda_1\int_0^1 w(s,\tau)G(u(\tau))d\tau \\ +\lambda_2\int_0^s k(s,\tau)F(u(\tau)d\tau) \end{array}\right]ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[\begin{array}{c} Q(s)u(s)+\lambda_1\int_0^1 w(s,\tau)G(u(\tau))d\tau \\ +\lambda_2\int_0^s k(s,\tau)F(u(\tau)d\tau) \end{array}\right]ds \end{aligned}$$

which is the required result. ■

Theorem 3.2: (Banach Contraction Principle) Let $(X,\|.\|)$ be a complete norm space, then each contraction mapping $T:X\rightarrow X$ has a unique fixed point x of T in X, such that $Tx=x$

3.1 Method of Solution

Let the solution of (1) and (2) be approximated by

$$u_N(t)=\boldsymbol{p}(t)\boldsymbol{A} \tag{12}$$

where $u_N(t)$ is the approximate solution, $\boldsymbol{p}(t)=\left[\begin{array}{cccc} p_0(t) & p_1(t) & \cdots & p_N(t) \end{array}\right]$, $p_n(t)$ is the shifted Legendre polynomial defined by (8) and $\boldsymbol{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T$ are constants to be determined.

$u(t) \in C([0,1], \mathbb{R})$ defined in (12) can be written in the form

$$u_N(t) = \boldsymbol{T}(t)\,\boldsymbol{M}\boldsymbol{A} \tag{13}$$

where

$$\boldsymbol{T}(t) = \begin{bmatrix} 1 & t & \cdots & t^N \end{bmatrix},\ \boldsymbol{M} = \begin{bmatrix} M(0,0) & 0 & 0 & \cdots & 0 \\ M(1,0) & M(1,1) & 0 & \cdots & 0 \\ M(2,0) & M(2,1) & M(2,2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M(N,0) & M(N,1) & M(N,2) & \cdots & M(N,N) \end{bmatrix}^T$$

$$M(n,k) = \frac{(-1)^{n+k}\,\Gamma(n+k+1)}{\Gamma(n-k+1)\,(\Gamma(k+1))^2}, n > 0, M(0,0) = 1 \tag{14}$$

substituting (13) into (9) and collocating at t_i, $i = 0(1)N$, $N \in \mathbb{Z}^+$

$$\boldsymbol{T}(t_i)\,\boldsymbol{M}\boldsymbol{A} - \boldsymbol{H}(t_i) + \frac{t_i}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \begin{bmatrix} Q(s)\,\boldsymbol{T}(s)\,\boldsymbol{M}\boldsymbol{A} \\ +\lambda_1 \int_0^1 w(s,\tau)\,\boldsymbol{G}(T(\tau)\,\boldsymbol{M}\boldsymbol{A})d\tau \\ +\lambda_2 \int_0^s k(s,\tau)\,\boldsymbol{F}(T(\tau)\,\boldsymbol{M}\boldsymbol{A})d\tau \end{bmatrix} ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_i} (t_i-s)^{\alpha-1} \begin{bmatrix} Q(s)\,\boldsymbol{T}(s)\,\boldsymbol{M}\boldsymbol{A} \\ +\lambda_1 \int_0^1 w(s,\tau)\,\boldsymbol{G}(T(\tau)\,\boldsymbol{M}\boldsymbol{A})d\tau \\ +\lambda_2 \int_0^s k(s,\tau)\,\boldsymbol{F}(T(\tau)\,\boldsymbol{M}\boldsymbol{A})d\tau \end{bmatrix} ds = 0 \tag{15}$$

where

$$H(t_i) = (1-t_i)\,\mu_0 + t_i\mu_1 - \frac{t_i}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_i} (t_i-s)^{\alpha-1} h(s)\,ds$$

which is a $(N+1)\times(N+1)$ nonlinear equations. We solved for A in (15) and substituted the result into (13) to obtain the numerical solution.

Proposition 3.3: If $u(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}$, then it is equivalent to $u(x,n) = x^n, n = 0(1)N, n \in \mathbb{Z}^+$

Proof. Given $u(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}$ then

$$u(x) = u(x,n) = x^n, n = 0(1)N$$

■

Lemma 3.4: Let $h \in C([0,1], \mathbb{R})$, be defined as $h(s) = s^m$,if

$$v_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \tag{16}$$

then $v_1(t)$ is equivalent to

$$v_1(t) = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m} \tag{17}$$

moreover, if

$$v_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\, ds \tag{18}$$

then $v_1(t)$ is equivalent to

$$v_2(t) = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} \tag{19}$$

Proof. substituting $h(s) = s^m$ into (16), the desired result is obtained

$$v_2(t) = \lim_{t=1} v_1(t) = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m}$$

■

4. UNIQUENESS OF THE METHOD

Here, we assumed that the solution to equation (1) and (2) exist, we then establish the uniqueness of the method of solution.
H_1 : There exist two constants, L_1 and $L_2 > 0$, such that for any u_N and $u \in C([0,1], \mathbb{R})$

$$|G(t, u_N) - G(t, u)| \leq L_1 |u_N - u|$$

and

$$|F(t, u_{\mathrm{N}}) - F(t, u)| \leq L_2 |u_N - u|$$

H_2 : There exist two functions k^* and $w^* \in C([0,1] \times [0,1], \mathbb{R})$, the set of all positive functions such that

$$k^* = \sup_{x\in[0,1]} \int_0^t |k(x,t)|\, dt < \infty$$

and

$$w^* = \sup_{x\in[0,1]} \int_0^1 |w(x,t)|\, dt < \infty$$

$H_3 : Q \in C([0,1], \mathbb{R})$

$$Q^* = \sup_{x\in[0,1]} |Q(s)|$$

(Uniqueness of solution)

Theorem 4.5: Let $(X, \|.\|)$ be a complete norm space and $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ be a strict q-contraction,then

(i) T has a unique fixed point, that is $F_T = \{x_n\}_{n=0}^{\infty}$
(ii) The Picard iteration associated to T, that is $\{x_n\}_{n=0}^{\infty}$ defined by $u_n = T(u_{n+1}) = T^n(u_n), n = T^n u_n$, $n = 1, 2, \cdots$ converges to x^r for any initial guess $x_0 \in X$

Proof. Since T is a contraction and $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ is a Banach space. Using the contraction principle, it shows there exist a uniue solution of T ■

Lemma 4.6: (Continuity) Let $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ be a mapping defined by (12), Let $u(t) \in C([0,1], \mathbb{R})$ be a solution of (1) and (2) and $C([0,1], \mathbb{R})$ a Banach space. If $\lim_{N\to\infty} u_N(x) = u(x)$, then T is continuous on $C([0,1], \mathbb{R})$ if $\|Tu_N(t) - Tu(t)\|_\infty \to 0$ as $N \to \infty$

Proof. Using Banach contraction principle

$$
\begin{aligned}
(Tu)(t) &= H(t) - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\begin{array}{c} Q(s)u(s) + \lambda_1 \int_0^1 w(s,\tau) G(u(\tau)) d\tau \\ +\lambda_2 \int_0^s k(s,\tau) F(u(\tau) d\tau) \end{array} \right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\begin{array}{c} Q(s)u(s) + \lambda_1 \int_0^1 w(s,\tau) G(u(\tau)) d\tau \\ +\lambda_2 \int_0^s k(s,\tau) F(u(\tau) d\tau) \end{array} \right] ds
\end{aligned}
$$

$$
\begin{aligned}
(Tu_N)(t) &= H(t) - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\begin{array}{c} Q(s)u_N(s) + \lambda_1 \int_0^1 w(s,\tau) G(u_N(\tau)) d\tau \\ +\lambda_2 \int_0^s k(s,\tau) F(u_N(\tau)) d\tau \end{array} \right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\begin{array}{c} Q(s)u_N(s) + \lambda_1 \int_0^1 w(s,\tau) G(u_N(\tau)) d\tau \\ +\lambda_2 \int_0^s k(s,\tau) F(u_N(\tau)) d\tau \end{array} \right] ds
\end{aligned}
$$

Using H_1

$$
\begin{aligned}
|Tu_N(t) - Tu(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |Q(s)| |u_N(s) - u(s)| ds \\
&+ \frac{L_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\int_0^1 |w(s,\tau)| |u_N(\tau) - u(\tau)| d\tau \right] ds \\
&+ \frac{L_2}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\int_0^s |k(s,\tau)| |u_N(\tau) - u(\tau)| d\tau \right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |Q(s)| |u_N(s) - u(s)| ds \\
&+ \frac{L_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\int_0^1 |w(s,\tau)| |u_N(\tau) - u(\tau)| d\tau \right] \\
&+ \frac{L_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\int_0^s |k(s,\tau)| |u_N(\tau) - u(\tau)| d\tau \right] ds
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{x\in[0,1]} \left|Tu_N(t) - Tu(t)\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \sup_{x\in[0,1]} |Q(s)| \sup_{x\in[0,1]} |u_N(s) - u(s)|\, ds \\
&+ \frac{L_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\sup_{x\in[0,1]} \int_0^1 |w(s,\tau)| \sup_{x\in[0,1]} |u_N(\tau) - u(\tau)|\, d\tau \right] ds \\
&+ \frac{L_2}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\sup_{x\in[0,1]} \int_0^s |k(s,\tau)| \sup_{x\in[0,1]} |u_N(\tau) - u(\tau)|\, d\tau \right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{x\in[0,1]} |Q(s)| \sup_{x\in[0,1]} |u_N(s) - u(s)|\, ds \\
&+ \frac{L_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left[\sup_{x\in[0,1]} \int_0^1 |w(s,\tau)| \sup_{x\in[0,1]} |u_N(\tau) - u(\tau)|\, d\tau \right] \\
&+ \frac{L_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\sup_{x\in[0,1]} \int_0^s |k(s,\tau)| \sup_{x\in[0,1]} |u_N(\tau) - u(\tau)|\, d\tau \right] ds
\end{aligned}
$$

Using H_2 and H_3

$$
\|Tu_N - Tu\|_\infty \leq \frac{Q^* + L_1 w^* + L_2 k^*}{\Gamma(\alpha+1)} \|u_N - u\|_\infty
$$

as $N \to \infty, u_N \to u$

$$
\|Tu_N - Tu\|_\infty \to 0
$$

which implies that T is continuous on $C([0,1], \mathbb{R})$ ■

5. CONVERGENCE OF THE METHOD

Theorem 5.7: Let $(X, \|.\|)$ be a norm space, $u(t)$ and $u_N(t)$ be the exact and approximated solution of (1) and (2) respectively, then

$$
\|u_N - u\|_\infty \leq \frac{\|H - H_N\|_\infty + \|u_N\|_\infty \|Q_N - Q\|_\infty}{\Gamma(\alpha+1) - Q^* - L_1 w^* - L_2 k^*}
$$

Proof. Let $u_N(t)$ and $u(t)$ be the numerical and exact solution of (1) and (2) respectively, let

$Q(s)$ and $H(t)$ in (9) be expanded in shifted Legendre polynomial, then

$$
\begin{aligned}
|u_N(t)-u(t)| \leq\ & |H_N(t)-H(t)| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\begin{bmatrix} |u_N(s)|\,\|Q_N(s)-Q(s)\| \\ +|Q(s)|\,\|u_N(s)-u(s)\| \end{bmatrix} ds \\
&+\frac{L_1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left[\int_0^1 |w(s,\tau)|\,\|u_N(\tau)-u(\tau)\|\,d\tau\right] ds \\
&+\frac{L_2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left[\int_0^s |k(s,\tau)|\,\|u_N(\tau)-u(\tau)\|\,d\tau\right] ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\begin{bmatrix} |u_N(s)|\,\|Q_N(s)-Q(s)\| \\ +|Q(s)|\,\|u_N(s)-u(s)\| \end{bmatrix} ds \\
&+\frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\int_0^1 |w(s,\tau)|\,\|u_N(\tau)-u(\tau)\|\,d\tau\right] ds \\
&+\frac{L_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\int_0^s |k(s,\tau)|\,\|u_N(\tau)-u(\tau)\|\,d\tau\right] ds
\end{aligned}
$$

$$
\begin{aligned}
\|u_N-u\|_\infty \leq\ & \|H-H_N\|_\infty+\frac{\|u_N\|_\infty}{\Gamma(\alpha)}\|Q_N-Q\|_\infty\int_0^1 (1-s)^{\alpha-1}\,ds \\
&+\frac{\|Q\|_\infty}{\Gamma(\alpha)}\|u_N-u\|_\infty\int_0^1 (1-s)^{\alpha-1}\,ds \\
&+\frac{L}{\Gamma(\alpha)}k^*\|u_N-u\|_\infty\int_0^1 (1-s)^{\alpha-1}\,ds \\
&+\frac{\|u_N\|_\infty}{\Gamma(\alpha)}\|Q_N-Q\|_\infty\int_0^t (t-s)^{\alpha-1}\,ds \\
&+\frac{\|Q\|_\infty}{\Gamma(\alpha)}\|u_N-u\|_\infty\int_0^t (t-s)^{\alpha-1}\,ds \\
&+\frac{L}{\Gamma(\alpha)}k^*\|u_N-u\|_\infty\int_0^t (t-s)^{\alpha-1}\,ds
\end{aligned}
$$

$$
\begin{bmatrix}
\|u_N-u\|_\infty \leq \|H-H_N\|_\infty+\frac{\|u_N\|_\infty}{\Gamma(\alpha+1)}\|Q_N-Q\|_\infty \\
+\frac{Q^*\|u_N-u\|_\infty}{\Gamma(\alpha+1)}+\frac{L_1w^*\|u_N-u\|_\infty}{\Gamma(\alpha+1)}+\frac{L_2k^*\|u_N-u\|_\infty}{\Gamma(\alpha+1)}+\frac{\|u_N\|_\infty\|Q_N-Q\|_\infty}{\Gamma(\alpha+1)} \\
+\frac{Q^*\|u_N-u\|_\infty}{\Gamma(\alpha+1)}+\frac{L_1w^*\|u_N-u\|_\infty}{\Gamma(\alpha+1)}+\frac{L_2k^*\|u_N-u\|_\infty}{\Gamma(\alpha+1)}
\end{bmatrix}
$$

$$
\|u_N-u\|_\infty \leq \frac{\|H-H_N\|_\infty+\|u_N\|_\infty\|Q_N-Q\|_\infty}{\Gamma(\alpha+1)-Q^*-L_1w^*-L_2k^*}
$$

■

6. NUMERICAL EXAMPLES

Example 6.1: [16] considered the boundary value problem

$$
D^{1.2}u(t)=\frac{2.5}{\Gamma(0.8)}t^{0.8}-\frac{t^9}{252}+\int_0^x (t-s)^2u^3(s)ds, 0\leq x\leq 1 \tag{20}
$$

$u(0)=0, u(1)=0, exact$ solution $u(t)=t^2$.

Solution 6.1: The approximate solution of (20) at $N=6$ gives

$$u_6(t)=\begin{pmatrix} -9.67535422e^{-18}t^6+2.113910062e^{-17}t^5-1.818434489e^{-17}t^4 \\ +7.515340801e^{-18}t^3+t^2-1.184980209e^{-16}t \end{pmatrix}$$

Table 1: Comparison of absolute error for Example 6.1

t	Exact	[16]	Present Method
0	0.00	2.13×10^{-18}	1.43×10^{-20}
0.1	0.01	1.83×10^{-19}	2.61×10^{-20}
0.2	0.04	6.43×10^{-18}	1.12×10^{-21}
0.3	0.09	3.39×10^{-20}	1.93×10^{-21}
0.4	0.16	3.93×10^{-18}	4.63×10^{-21}
0.5	0.25	4.54×10^{-18}	1.67×10^{-22}
0.6	0.36	5.12×10^{-19}	3.25×10^{-22}
0.7	0.49	6.24×10^{-18}	2.11×10^{-21}
0.8	0.64	8.30×10^{-19}	3.50×10^{-22}
0.9	0.81	1.30×10^{-19}	1.19×10^{-22}
1	1	1.45×10^{-19}	1.85×10^{-22}

Example 6.2: [10] considered the boundary value problem

$$D^{\frac{3}{2}}y(x)+y(x)=x^5-x^4+\frac{128}{7\sqrt{\pi}}x^{3.5}-\frac{64}{5\sqrt{\pi}}x^{2.5} \tag{21}$$

subject to the boundary condition $y(0)=0, y(1)=1$.The exact solution is $y(x)=x^4(x-1)$

Solution 6.2: The approximate solution of (21) at $N=9$ gives

$$y_9(x)=\begin{pmatrix} 1.1893444416e^{-18}x^9-6.180191572e^{-18}x^8+1.402311643e^{-17}x^7 \\ -1.838130491e^{-17}x^6+x^5-1.0x^4+6.393568195e^{-18}x^3 \\ +7.460356011e^{-19}x^2-1.247726439e^{-17}x \end{pmatrix}$$

Table 2: Comparison of absolute error for Example 6.2

x	Exact	Present Method
0	0.00	2.53×10^{-17}
0.1	$-0.000\,09$	3.55×10^{-17}
0.2	$-0.001\,28$	3.89×10^{-17}
0.3	$-0.005\,67$	1.19×10^{-18}
0.4	$-0.015\,36$	2.14×10^{-18}
0.5	$-0.031\,25$	3.96×10^{-18}
0.6	$-0.051\,84$	1.35×10^{-17}
0.7	$-0.072\,03$	5.01×10^{-18}
0.8	$-0.081\,92$	8.05×10^{-19}
0.9	$-0.065\,61$	4.22×10^{-19}
1	0.00	2.44×10^{-19}

Example 6.3: [19] considered the boundary value problem for a class of fractional differential equation

$$D^{\alpha}y\left(x\right)+ay\left(x\right)=g\left(x\right),1\leq x\leq 2 \tag{22}$$

$$y\left(0\right)=0,y\left(1\right)=-\frac{1}{40},\alpha=\frac{3}{2},a=\frac{e^{-3\pi}}{\sqrt{\pi}}$$

$$g\left(x\right)=\frac{e^{-3\pi}}{\sqrt{\pi}}\left(x^{2}\left(40x^{2}-74x+33\right)+4e^{3\pi}\sqrt{x}\left(128x^{2}-148x+33\right)\right)$$

The exact solution

$$y\left(x\right)=\left(x^{2}-\frac{37}{20}x+\frac{33}{40}\right)x^{2}$$

Solution 6.3: The approximate solution of (22) at $N=4$, 6 *and* 8 gives

$$y_{4}\left(x\right)=x^{4}-1.85x^{3}+0.825x^{2}-3.066883897e-16x$$

$$y_{6}\left(x\right)=\left(\begin{array}{c}-4.293620945e-23x^{6}+8.08021702e-22x^{5}\\+x^{4}-1.85x^{3}+0.825x^{2}-3.066883234e-16x\end{array}\right)$$

$$y_{8}\left(x\right)=\left(\begin{array}{c}-6.772990005e-22x^{8}+3.14953473e-21x^{7}\\-6.066384787e-21x^{6}+6.916096771e-21x^{5}\\+x^{4}-1.85x^{3}+0.825x^{2}-3.066883108e-16x\end{array}\right)$$

It is clearly shows from the examples presented that the present method can be considered as an efficient method.

7. CONCLUSION

In this paper, the collocation method is used to solve Factional order integro differential equations with dirichlet boundary condition using shifted legendre polynomial. From the results obtained, it shows that the new method is efficient and suitable for this kinds of problems. MATLAB was used to implement the algorithm of the method.

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