**Crank-Nicolson Analysis of Black-Scholes Equation and Effects of Covariance Properties for Stock Market Prices**

**ABSTRACT**

This paper analyzes the Black-Scholes partial differential equation for European option pricing using both analytical and numerical approaches. The Crank-Nicolson finite difference method is employed to obtain approximate solutions for call and put options, which are then compared with closed-form Black-Scholes prices. A covariance matrix analysis and quantile-quantile (QQ) normality test are conducted to evaluate the statistical properties of the approximate solutions. The results show a strong correlation between the analytical and numerical outcomes, supporting the robustness of the Crank-Nicolson approach in modeling option prices. These findings provide valuable insights for investors and enhance the reliability of numerical methods in financial decision-making.

**Key words: Black-Scholes, Stock market, Crank-Nicolson, Covariance and Option traders**

**1.1 INTRODUCTION**

In financial market generally stock are seen as the capital rose by a company or corporation through the issue and subscription of shares. It is a security that represents the ownership of a fraction of a corporation. The stock market is made up of buyers and sellers of stocks, which represents ownership claims on businesses. It refers to a number of exchanges and other venues in which shares of publicly held companies are bought and sold. The legal platform upon which the financial activities are done is referred to as stock exchange. They are done through institutionalized formal exchanges or through over-the-counter platforms bound by a set of defined regulations. The stock market is one of the most vital components of a free-market economy. It is among the best options to various companies for expansion or set up a new business venture. The stock market performance and operation has been widely recognized as a significantly viable investment field in financial markets. Investments can be done in stock, bond, mutual funds etc. An increase in price of stocks in the stock market indicates an increase in investment; [2-3].

Therefore, an option is a tool whose worth is derived from the principal asset which is otherwise known as financial derivative. This type of derivative does have anything in common with mathematical meaning of derivative. In other words, an option on underlying asset is a business between parties who come together to agree on either buying or selling an underlying asset at a determined strike price in the future for a fixed price. In the nonappearance of transactions costs, an in-the money option is always exercised on the expiration date if it has not been exercised earlier, [12]. The significance of options valuation was first established by [5] when option challenged complications in valuation of option at expiration. They applied no-arbitrage argument to explain a partial differential equation which governs the growth of the option price with esteem to the expiration and cost of the fundamental asset. The Black-Scholes equation has been used broadly. Also, many researchers have used Black-Scholes Equation in different approaches. For instance; [1] considered Black-Scholes partial differential equation on stock market prices for both analytic and numerical. [18] focused on terminal value problem and provided its solutions through the Laplace transform. More so, [6] proposed a framework based on the celebrated transform of Mellin type (MT) for the analytic solution of the Black-Scholes-Merton European Power Put Option Model (BSMEPPOM). In the same vein, [4] analyzed BS formula for the valuation of European options; using hermit polynomials. In the work of [17], time varying factor were incorporated in the explicit formula for different aspect of options with the aim of providing exact solution for dividend paying equity of option. [12] applied Crank-Nicolson numerical scheme to BS model. The results showed stock prices being stable and its increasing rate of stock shares was obtained. Not quite long, [11] investigated the variation of stock market price using BS PDE. However, [3] studied the perception of European option which is geared towards valuation of financial assets, the application of share prices of Fidelity and Access banks which gave closed form prices of call options.

Nevertheless, [2] modified Black-Scholes model to assume a probability which measures risk-free interest rate of the underlying asset for Call and Put options. The Black-Scholes exact values and Modified Black-Scholes values were obtained and compared to close form prices. Lots of scholars has written extensively to mention but a few: [2],[7],[8], [10],[13], [15] , [16] and[18].

The aim of this paper is to develop a Crank-Nicolson analysis of Black-Scholes partial Differential Equation and effects of Covariance matrix solutions for stock market prices. The main issue of investors is unable to take suitable decisions due unstable nature of stock prices. These issues may have arisen owing to its assumptions and some market uncertainties. This motivated the researchers of this paper to develop a good empirical method that can stand in terms of decisions making. It is reasonable that [3] has studied the Analysis of Black-Scholes of Option pricing with Time-Varying parameters on Share prices for capital market The progress of this paper over [3] is that, this present paper combined analytical and numerical solutions with covariance matrices and the use of eigenvalues for analysis. Our novel idea compliments previous efforts and extends frontiers in this dynamic area of mathematical finance.

This paper is prescribed as follows: Section 2.1 is mathematical preliminaries, Section 3.1, Results and Discussion while the paper is concluded in 4.1.

**2.1 Mathematical Preliminaries**

Here we present some basic definitions that cut across the area of study:

**Definition 1**.A stochastic process whose finite dimensional probability distributions are all Gaussian.(Normal distribution). [2] and [12].

**Definition 2.** Random Walk: There are different methods to which we can state a stochastic process. Then relating the process in terms of movement of a particle which moves in discrete steps with probabilities from a point  to a point  . A random walk is a stochastic sequence with , defined by

  (1)

where are independent and identically distributed random variables, [9-12].

**Definition 3**: A Stochastic Differential Equation (SDE) is integration of differential equation with stochastic terms. So ,in considering the Geometric Brownian Motion (GBM) which govern price dynamics of a non-dividend paying stock as:

  , (2)

Where S denotes the asset value,  is the stock rate of return (drift) which is also known as the average rate of the growth of asset price and  denotes the volatility otherwise called standard deviation of the returns. The  is a Brownian motion or Wiener process which is defined on probability space , [12]

However our interest in this paper is the parabolic financial PDE which is governed with the dynamics of option pricing; hence we have the following:

  (3)

Where  represents interest rate,  represents volatility of the underlying assets and  represents time of maturity. The details of the above option model can be expressly found in the following papers:[1],[7-8], [19-20].

**2.1.1 Black-Scholes (B-S) Model Assumptions**

However, Black-Scholes model is based on seven assumptions: The asset price follows a Brownian motion with and  as constants, there are no transaction costs or taxes, all securities are perfectly divisible, there is no dividend during the life of the derivatives, there are no riskless arbitrage opportunities, the security trading is continuous.

The analytic formula for the prices of European call option is given as:

  (4)

Also the analytic formula for the prices of European put option is given as:

  (5)

where  is Price of a put option, is price of underlying asset,is the strike price,is the riskless rate,is time to maturity,is variance of underlying asset,is standard deviation of the( generally referred to as volatility) underlying asset, andis the cumulative normal distribution. The concepts can be seen in the books of [2], [9-11],[22-23].

**3.1 European Options**

In the work of [5] they obtained a mathematical structure for finding the reasonable price of European options by the use of no-arbitrage principle to describe a PDE which governs the growth of the option price that evolves time to expiration. The details of this options can be found in the following books:[5] ,[9-10].

**2.1.1 European Call Option**

The BS PDE for European Call and Put Options with value  and  is given in the following equations:

  (6)

With the following initial and boundary conditions

  (7)

**2.1.2 European Put Option**

The BS PDE for European Put Options with value  is given in the following equations:

  (8)

With the following initial and boundary conditions:

  (9)

**2.2 The Numerical Scheme and Analysis**

 Here to implement Crank-Nicolson approximation scheme on Black-Scholes partial differential equation, there must be a price time mesh in order to enhance efficiency as solution exits, the vertical axis in the mesh denotes the stock prices, while the horizontal axis denotes time. Therefore, every grid point in the mesh denotes a horizontal index  and a vertical index  such that every point in the mesh is the option price for a distinct time and a distinct stock price. At every time in the mesh  is equivalent to the stock price, and is equivalent to the time. There exist boundary conditions which help in the numerical calculations; by means of the pay-off function. The maturity period,  and the option are well computed for all the different initial stock prices using a boundary conditions for uniqueness of solution. To get the prices at , the model solves backwards for every time step from ,[20-21].

**2.2.1 Formulation of the Scheme**

One of the normal ways of approximating the solution of partial differential equations is applying Crank-Nicolson finite difference method which we shall use our proposed model to transform into the scheme. Hence , we have the price time mesh below.

S







**Figure 1: An illustration of Price time mesh.**

Recall that the Black-Scholes partial differential equation (3). Let a function  in two dimensional grid points, that is to sayand  stands for the index for stock price,  and time,respectively. The function can be stated as follows in the subsequent difference scheme.

 (10)

where

, for, for

 (11)

 (12)

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below. 

If we use a forward difference approximation to the time partial derivative we obtain explicit scheme

 (13)

and similarly we obtain the implicit scheme

 (14)

The averages of equations (13) and (14) yields Crank-Nicolson method of approximation

 (15)

 From equation (15)

 (16)



 (17)

Substituting (8) in (17) gives in view of (16) and (17) we obtain after collecting like term in ,



That is



Collecting like terms in of  and simplifying gives

 

and



Using (8) in (15) solving simultaneously and taking the average of these two equations we obtain 

 (18)

The expressions inside the square brackets will be replaced with the coefficients a, b, c. The following equations obtained.

 (19)

Where



 .

Equation (18) can now be represented in matrix form as follows





where

.

.

**2.3. Pricing European Options with Covariance Matrices from Approximate Solutions**

This Section offers predictions of covariance matrices of CN numerical solutions on European options as follows: Firstly, a variance-covariance matrix: is the covariance between variable  and variable  the diagonal entries are variance: the covariance in respect to Call and Put options and are basically defined as.

**2.3.1 Covariance matrix for Call options through Approximate Solutions:**

  . (20)

**2.3.2 Covariance matrix for Put options through Approximate Solutions:**

  . (21)

**3.1 Results and Discussions**

Here we present simulation results obtained using Sections 2.1 and 2.3 through matlab codes for Black-Scholes exact values and Crank-Nicolson approximate solutions for European options

**Table 1:The Black-Scholes exact values and Crank-Nicolson Approximate solutions for European Call Option with the following parameter values initial stock prices 40 ,E = 25, r = 0.2 and T = 1**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Volatility** | **B-S Exact values** | **Approximate****Solutions** | **Relative****Error** | **Mean options** |
| 0.25 | 19.5398  | 19.5378 | 1.0236E-04 | 19.5388 |
| 0.35 | 19.6371 | 19.5926 | 2.266E-04 | 19.6149 |
| 0.45 | 19.9117 | 19.7185 | 9.7028E-03 | 19.8151 |
| 0.55 | 20.3607 | 19.9121 | 0.0220 | 20.1364 |
| 0.65 | 20.9441 | 20.1704 | 0.0369 | 20.5572 |
| 0.75 | 21.6219 | 20.4873 | 0.0525 | 21.0546 |
| 0.85 | 22.3630 | 20.8506 | 0.06763 | 21.6068 |
| 0.9 | 22.7497 | 21.0445 | 0.07495 | 21.8971 |
| 0.95 | 23.1442 | 21.2436 | 0.08212 | 21.1939 |



**Figure 2: Quantiles-Quantiles (QQ) normality test of BS Exact values and Crank-Nicolson Approximate values for Call option.**

**Table 2:The Black-Scholes exact values and Crank-Nicolson Approximate solutions for European Put Option with the following parameter values initial stock prices 40 ,E = 100, r = 0.2 and T = 1**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Volatility** | **B-S Exact values** | **Approximate****Solutions** | **Relative****Error** | **Mean options** |
| 0.25 | 41.8817  | 41.8778 | 9.3119E-05 | 41.8798 |
| 0.35 | 42.0213 | 41.9458 | 1.7967E-03 | 41.9835 |
| 0.45 | 42.4721 | 42.1446 | 7.7109E-03 | 42.3083 |
| 0.55 | 43.2622 | 42.4578 | 0.01859 | 42.8600 |
| 0.65 | 44.3292 | 42.8374 | 0.03365 | 43.5833 |
| 0.75 | 45.5988 | 43.2432 | 0.05166 | 44.4210 |
| 0.85 | 47.0083 | 43.6494 | 0.07145 | 45.3289 |
| 0.9 | 47.7501 | 43.8478 | 0.08172 | 45.7990 |
| 0.95 | 48.5099 | 44.0415 | 0.09211 | 46.2757 |

In Tables 1 and 2 , a little increase in the volatility of stock also increases the close form prices of BS and CN for both options. The BS and CN has little differences in comparisons. This remark is reasonable in the aspect of an investor whose primary aim is to maximize profit. The significant changes are due to stock volatility and stochastic formation in the price history of stock market. The strike price of call and put is 25 and 100 respectively. This is because a call option trader whose strike price is 25 wants to buy goods at cheaper rate in order to make more profit while a put option trader whose strike price is 100 want to sell in order to maximize profit throughout the trading days; this is because the investor is only obliged to sell; besides it is reasonable when the strike price is greater than the initial prices for put options.

However, the large mean option price is seen as a positive symbol for investors in stock market trading, as it shows that the stock is doing well and is probably to continue to grow in value. They are good investment opportunity for investors who are expecting for stocks with robust growth potential. A small mean option price, on the other hand, could be seen as a mark of weakness or under-performance; see column 5 of both Tables.



**Figure 3: Quantiles-Quantiles (QQ) normality test of BS Exact values and Crank-Nicolson Approximate values for Put option.**

 It can be observed clearly in Figures 2 and 3 respectively QQ plot designates that the Black-Scholes exact values and Crank-Nicolson approximate solutions for Call and Put option prices come from a common a distribution. They are statistically significant, correlated and have heaps of financial remunerations therefore it waves around the normal distribution which informs investors on a way forward based on the levels of their investments.

**3.1.1: Pricing European Options with Covariance Matrices from Approximate Solutions**

* **Covariance Matrix of Approximate Solutions for Call Option Prices with their respective eigenvalues**



* **Covariance Matrix Approximate Solutions for Put Option Prices with their respective eigenvalues**



The Covariance matrices of call and put options prices of Crank-Nicolson approximate solutions in subsection 3.1.1 processes the joint volatility of the underlying stocks. It really apprehensions the statistical association among stock call and put options, and determines changes in the price of the stock touch the price of the options. The two results also amount the correlation amongst the stock and options, which helps investors achieve their risk by branch out their portfolio of investments.

The values seen in covariance matrix of call options show that the stock market is averagely volatile. They are averagely correlated, which means that changes in the stock price are probable to have a significant influence on the option prices. This does not in any way generate risk for investors, as they are not been exposed to big fluctuations in the value of their investments. This average covariance is well beneficial to call option traders because someone who wants to buy would want to buy at a cheaper rate or at average in order to make good profit margins. This situation is informs investors in terms of decision making.

Though, the high values seen in covariance matrix of put option shows that the stock market is volatile and that investors should be careful based on their decision. Also, high covariance implies that the stock and the options are highly correlated, which means that changes in the stock price are probable to have a significant influence on the option prices. This generates risk for investors, as they are been exposed to large fluctuations in the value of their investments throughout the trading days. This higher covariance is highly beneficial to put option traders because someone who wants to sell will like to sell higher and make more profits. This scenario is informative to investors in terms of decision making.

The corresponding eigenvalues indicates the level of economic stability and instability or volatility in stock data. A high eigenvalue indicate that the data is very volatile, while a low eigenvalues indicates that the data is relatively volatile and negative eigenvalues shows stability in stock options. The eigenvalues helps to identify the major trends in the data such as periods of expansion or contractions in stock options, see subsection 3.1.1.

**4.1.CONCLUSION**

This paper considered European options for both analytic exact values and numerical approximate prices. The simulations of analytical and numerical were effectively carried out .The results showed as follows: increase in volatility increases the values of option for both BS and CN prices, there are no significant difference between BS and CN they are indistinquable; which were further confirmed by probability normality test, an increase in strike prices increases the call and put options, a high eigenvalue indicate that the data is very volatile while a low eigenvalues portrays that the data is relatively volatile and negative eigenvalues shows stability in stock options .To this end, combining delay and control parameters into the BS PDE will be an interesting study to explore.

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Details of the AI usage are given below:

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2.

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