Original Research Article

n-SQUARE METRICALLY EQUIVALENT OPERATORS

Abstract

We introduce a new operator equivalence relation termed as n-Square Metrically Equivalent Operators. Given a positive integer n, two bounded linear operators $\mathcal{A}$ and $\mathcal{B}$ are said to be square metrically equivalent operators if they satisfy the relation $\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}\ \forall n \in \mathbb{R}^{+}$. This concept generalizes the classical square-metric equivalence whenever $n = 1$, and allows the study of operator pairs that share deeper structural and spectral similarities. We establish that this relation forms an equivalence class and explore its key algebraic and spectral properties. We also examine how the equivalence interacts with notable operator classes, including n- square normal operators.

Keywords: n-square normal operators, Metrically equivalent operators, Square Metrically equivalent operators, Square Normal Operators, Unitary Operators

2010 Mathematics Subject Classification: 47B47 , 47A10.

1 Introduction

Equivalence relations among bounded linear operators on Hilbert spaces form a cornerstone of modern functional analysis and operator theory. Classical equivalence relations, such as unitary equivalence, similarity, and metric equivalence, have been instrumental in understanding the spectral and structural properties of operators [2, 5]. Although these classical relations have been well studied, increasing complexity in modern applications, such as high-dimensional data analysis and quantum mechanics, has motivated the introduction of more generalized equivalence classes. There are many classes touching on equivalence relation such as Almost equivalent operators [4, 11, 15] and Quasisimilarity equivalence [6, 19].

In this study , we are interested in Metrically equivalent operators which has been analyzed by various researchers. Nzimbi et.al [12] studied this class and established that two operators A and B were metrically equivalent if and only if there $\|Ax\| = \|Bx\|$ for all $x \in \mathcal{H}$. This result provided a precise characterization of metric equivalence and distinguished it from both similarity and unitary equivalence by emphasizing the preservation of norms rather than algebraic or inner product structures. This class was later extended to n-Metrically equivalent operators by Wanjala et. al in [23]. They investigated the concept of n-metric equivalence of bounded linear operators on a Hilbert space and defined two operators A and B to be n-metrically equivalent if

$$\|A^n x\| = \|B^n x\| \quad \text{for all } x \in \mathcal{H}, \text{ and for a fixed } n \in \mathbb{N}.$$

The class of n-metric equivalence was further generalized to (n, m)-metric equivalence by Wanjala and Nyongesa in [24]. They established sufficient and necessary conditions under which two bounded linear operators could be considered (n, m)-metrically equivalent. Specifically, they showed that such equivalence holds if and only if

$$B^{*n} B^m = A^{*n} A^m$$

for positive integers n and m. This was later extended to Metrically equivalent operators of order n in [25]. It was demonstrated in [25] that two operators A and B are metrically equivalent of order n if

$$B^{*n} B = A^{*n} A$$

for a positive integer n. We refer the reader to [13, 14, 26] for other literature touching on Metrically equivalent operators. Recently , the study of metrically equivalent operators was extended to that of square metrically equivalent operators by Wanjala and Nyongesa in [27].From the above literature , its clear that no study has been done to generalize square equivalence relation to larger classes and this is what this study seeks to address by extending the class of square metrically equivalent operators to n-square metrically equivalent operators . We relate this class to those of class $(\mathcal{Q})$. For literature related to class $(\mathcal{Q})$ we refer the reader to [3, 8, 17, 18, 21, 22]

Key Definitions

Definition 1.1. [2] An operator $\mathcal{A}$ in the space of bounded operators $\mathcal{B}(\mathcal{H})$ is referred to as: *normal* if $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$.

Definition 1.2. [1] A is *n-normal* if $\mathcal{A}^{*n}\mathcal{A}^n = \mathcal{A}^n\mathcal{A}^{*n}$.

Definition 1.3. [7] A is *square-normal* if $\mathcal{A}^2(\mathcal{A}^*)^2 = (\mathcal{A}^*)^2\mathcal{A}^2$.

Definition 1.4. [16] Bishop's Property: An operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ has *Bishop's property* if, for every sequence of analytic functions $f_p : U \to H$, where $U \subset \mathbb{C}$ is open, $(\lambda - A)f_p(\lambda) \to 0$ uniformly on compact subsets of U and $f_p(\lambda) \to 0$ locally uniformly in U as $p \to \infty$.

Definition 1.5. [10] An operator $\mathcal{A}$ is *isoloid* if each isolated point of $\sigma(A)$ belongs to the point spectrum $\sigma_p(A)$.

Definition 1.6. [10] An operator $\mathcal{A}$ is *polaroid* if every isolated point of $\sigma(A)$ is a pole of the resolvent of $\mathcal{A}$.

Theorem 1.7. *[20] If $\mathcal{Q}$ is an n-power normal operator, it satisfies Bishop's property.*

Lemma 1.8. *[20] $\mathcal{T}$ is n-power normal ; if and only if $\mathcal{T}^n$ is normal.*

Proposition 1.9. *[7] If $\mathcal{A}$ is a normal operator, then $\mathcal{A}$ is a square-normal operator.*

Theorem 1.10. *[9] Suppose $\mathcal{Q} \in \mathcal{B}(\mathcal{H})$ is an n-normal operator . In that case , $\mathcal{Q}$ is both isoloid and polaroid .*

2 Main Results

Definition 2.1. Two operators $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ are said to be n-Square Metrically Equivalent Operators, denoted by $\mathcal{A} \sim_{m^{2n}} \mathcal{B}$, provided :

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}$$

$$\forall n \in \mathbb{R}^{+}$$

Theorem 2.2. *If $\mathcal{A}$ is a n-square-normal operator and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ is unitarily equivalent to $\mathcal{A}$, then $\mathcal{B}$ is n-square-normal.*

Proof. Since $\mathcal{B} = \mathcal{U}^*\mathcal{A}\mathcal{U}$, with $\mathcal{U}$ being unitary and $\mathcal{A}$ n-square-normal, we have:

$$\begin{aligned}
\mathcal{B}^{*2}\mathcal{B}^{2n} &= (\mathcal{U}^*\mathcal{A}^*\mathcal{U})^2(\mathcal{U}^*\mathcal{A}^n\mathcal{U})^2 \\
&= (\mathcal{U}^*\mathcal{A}^*\mathcal{A}^n\mathcal{U})^2 \\
&= (\mathcal{U}^*\mathcal{A}^n\mathcal{A}^*\mathcal{U})^2 \\
&= (\mathcal{B}^n\mathcal{U}^*\mathcal{A}^*\mathcal{U})^2 \\
&= (\mathcal{B}^n\mathcal{U}^*\mathcal{U}\mathcal{B}^*)^2 \\
&= (\mathcal{B}^n\mathcal{B}^*)^2 \\
&= \mathcal{B}^{*2}\mathcal{B}^{2n}.
\end{aligned}$$

This proves the claim. □

Corollary 2.3. *An operator $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ is n-square-normal if and only if $\mathcal{B}$ and $\mathcal{B}^*$ are n-Square Metrically Equivalent.*

Proof. The proof follows from Theorem 2.2. □

Theorem 2.4. *Two operators $\mathcal{A}$ and $\mathcal{B}$ are n-Square Metrically Equivalent provided $\mathcal{A}^{*2}\mathcal{A}^{2n}$ and $\mathcal{B}^{*2}\mathcal{B}^{2n}$ are unitarily equivalent.*

Proof. The proof is simple and follows from $\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^{2n}\mathcal{U}^*$ with $\mathcal{U}$ as the unitary operator. □

Theorem 2.5. *n-Square metric equivalence is an equivalence relation.*

Proof. (i) $\mathcal{A} \sim_{m^{2n}} \mathcal{A}$ since:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{U}\mathcal{A}^{*2}\mathcal{A}^{2n}\mathcal{U}^*.$$

(ii) If $\mathcal{A} \sim_{m^{2n}} \mathcal{B}$, it means:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{U}\mathcal{B}^{*2}\mathcal{B}^{2n}\mathcal{U}^* \tag{2.1}$$

Pre-multiplying 2.1 on both sides by $\mathcal{U}^*$ and post-multiplying the same by $\mathcal{U}$, we get:

$$\mathcal{U}^*\mathcal{A}^{*2}\mathcal{A}^{2n}\mathcal{U} = \mathcal{U}^*\mathcal{U}\mathcal{B}^{*2}\mathcal{B}^{2n}\mathcal{U}^*\mathcal{U} = \mathcal{U}^*\mathcal{A}^{*2}\mathcal{A}^{2n}\mathcal{U} = \mathcal{B}^{*2}\mathcal{B}^{2n}. \tag{2.2}$$

Hence, $\mathcal{B} \sim_{m^{2n}} \mathcal{A}$.

(iii) We need to illustrate that: if $\mathcal{A} \sim_{m^{2n}} \mathcal{Q}$ and $\mathcal{Q} \sim_{m^{2n}} \mathcal{B}$, it follows that $\mathcal{A} \sim_{m^{2n}} \mathcal{B}$. That is:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{U}\mathcal{Q}^{*2}\mathcal{Q}^{2}U^{*} \quad \text{and} \quad \mathcal{Q}^{*2}\mathcal{Q}^{2} = \mathcal{G}\mathcal{B}^{*2}\mathcal{B}^{2n}\mathcal{G}^{*},$$

where $\mathcal{U}$ and $\mathcal{G}$ are unitary operators. Then:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{U}\mathcal{Q}^{*2}\mathcal{Q}^{2}\mathcal{U}^{*} = \mathcal{U}\mathcal{G}\mathcal{B}^{*2}\mathcal{B}^{2n}\mathcal{U}\mathcal{G}^{*}.$$

Let $\mathcal{M} = \mathcal{U}\mathcal{G}$, where $\mathcal{M}$ is unitary. Thus:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{M}\mathcal{B}^{*2}\mathcal{B}^{2n}\mathcal{M}^{*}.$$

Hence, $\mathcal{A} \sim_{m^{2n}} \mathcal{B}$.

□

Proposition 2.6. *Let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ and n=1 . If, they are n-Square Metrically Equivalent, then they are metrically equivalent operators provided they are idempotent .*

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are idempotent , then $\mathcal{A}^2$ = $\mathcal{A}$ and $\mathcal{B}^2$ = $\mathcal{B}$. Let $\mathcal{A}$ and $\mathcal{B}$ be n-square-metrically equivalent, then by definition

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}$$

$$\forall n \in \mathbb{R}^{+}$$

Since n=1 , we have

$$\mathcal{A}^{*2}\mathcal{A}^{2} = \mathcal{B}^{*2}\mathcal{B}^{2}$$

and by idempotent property replacing $\mathcal{A}^2$ = $\mathcal{A}$; $\mathcal{A}^{*2}$ = $\mathcal{A}^*$ and $\mathcal{B}^2$ = $\mathcal{B}$; $\mathcal{B}^{*2}$ = $\mathcal{B}^*$ we have that

$$\mathcal{A}^{*}\mathcal{A} = \mathcal{B}^{*}\mathcal{B}$$

hence they are metrically equivalent. □

Remark 2.7. The converse need not be true. We give an example of n-Square Metrically Equivalent Operators that are not Metrically Equivalent.

Example 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be defined as :

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now ;
for $\mathcal{A}^{*2}\mathcal{A}^{2n}$ and $\mathcal{B}^{*2}\mathcal{B}^{2n}$ we have :

$$\mathcal{A}^{*} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B}^{*} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For ; $\mathcal{A}^*\mathcal{A}$ and $\mathcal{B}^*\mathcal{B}$:

$$\mathcal{A}^{*}\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{B}^{*}\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, $\mathcal{A}^*\mathcal{A} \neq \mathcal{B}^*\mathcal{B}$.
For $\mathcal{A}^{2n}$ and $\mathcal{B}^{2n}$, we observe that:

$$\mathcal{A}^{2} = \mathcal{B}^{2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus , for any $n \geq 1$:

$$\mathcal{A}^{2n} = \mathcal{B}^{2n} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now for $\mathcal{A}^{*2}\mathcal{A}^{2n}$ and $\mathcal{B}^{*2}\mathcal{B}^{2n}$ we have :

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}$.
Thus $\mathcal{A}$ and $\mathcal{B}$ are n- Square metrically equivalent but not metrically equivalent .

Remark 2.9. The following results illustrate the spectral picture of n-square-metrically equivalent operators where n-square-metrically equivalent operators do not necessarily need to have equal spectra.

Proposition 2.10. *n-Square Metrically Equivalent Operators need not have equal spectra.*

Proof. Proof follows directly from the following Example 2.11. □

Example 2.11. Consider the n-Square Metrically Equivalent Operators:

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathbb{C}^2$. A simple computation illustrates that:

$$\sigma(\mathcal{A}) = \{-1, 1\} \quad \text{and} \quad \sigma(\mathcal{B}) = \{1, 1\}.$$

Thus, $\mathcal{A}$ and $\mathcal{B}$ do not have equal spectra. It is equivalently clear that:

$$\mathcal{W}(\mathcal{A}) \neq \mathcal{W}(\mathcal{B}),$$

which shows that n-square-metric equivalence does not preserve the numerical range.

Theorem 2.12. *If $\mathcal{A}$ and $\mathcal{B}$ are n-Square Metrically Equivalent Operators for n = 1 and $\mathcal{A}$ and $\mathcal{B}$ are quasi-isometries, then $\mathcal{A}$ and $\mathcal{B}$ are metrically equivalent.*

Proof. The proof is straightforward and follows from the theorem itself. Since $\mathcal{A} \sim_{m^{2n}} \mathcal{B}$, we have:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}$$

Rearranging, we get:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}.$$

Since $\mathcal{A}$ and $\mathcal{B}$ are quasi-isometries for n = 1 , that is, $\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{A}^*\mathcal{A}$ and $\mathcal{B}^{*2}\mathcal{B}^{2n} = \mathcal{B}^*\mathcal{B}$, it follows that:

$$\mathcal{A}^*\mathcal{A} = \mathcal{B}^*\mathcal{B}.$$

Thus, $\mathcal{A}$ and $\mathcal{B}$ are metrically equivalent. □

Remark 2.13. The following results establishes the n-square-metric equivalence relation of operators on n-power class $(\mathcal{Q})$-operators, denoted by $(n\mathcal{Q})$.

Theorem 2.14. *Let $\mathcal{A} \in (n\mathcal{Q})$ and $\mathcal{B} \in (n\mathcal{Q})$. Then $\mathcal{A} \in (n\mathcal{Q})$ and $\mathcal{B} \in (n\mathcal{Q})$ are said to be n-Square Metrically Equivalent Operators if and only if $\mathcal{A}$ and $\mathcal{B}$ are isometries for n = 1.*

Proof. Given that $\mathcal{A} \in (n\mathcal{Q})$ and $\mathcal{B} \in (n\mathcal{Q})$, then:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = (\mathcal{A}^*\mathcal{A}^n)^2 \quad \text{and} \quad \mathcal{B}^{*2}\mathcal{B}^{2n} = (\mathcal{B}^*\mathcal{B}^n)^2.$$

This implies:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = (\mathcal{A}^*\mathcal{A}^n)^2 = \mathcal{A}^{*2}\mathcal{A}^{2n},$$

and :

$$\mathcal{B}^{*2}\mathcal{B}^{2n} = (\mathcal{B}^*\mathcal{B}^n)^2 = \mathcal{B}^{*2}\mathcal{B}^{2n}.$$

Since both $\mathcal{A}$ and $\mathcal{B}$ are isometries for n = 1, the following holds:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{I} = (\mathcal{A}^*\mathcal{A})^2 \tag{2.3}$$

and similarly :

$$\mathcal{A}^{2n}\mathcal{A}^{*2} = \mathcal{I} \tag{2.4}$$

as well as :

$$\mathcal{B}^{*2}\mathcal{B}^{2n} = \mathcal{I} = (\mathcal{B}^*\mathcal{B})^2 \tag{2.5}$$

and equivalently :

$$\mathcal{B}^{2n}\mathcal{B}^{*2} = \mathcal{I} \tag{2.6}$$

From 2.4 and 2.6, it is observed that:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{I} = \mathcal{B}^{*2}\mathcal{B}^{2n},$$

and so:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}.$$

For the converse, since $\mathcal{A} \in (n\mathcal{Q})$ and $\mathcal{B} \in (n\mathcal{Q})$ are isometries for n = 1 , we have:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = (\mathcal{A}^*\mathcal{A}^n)^2 = \mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{I},$$

and:

$$\mathcal{B}^{*2}\mathcal{B}^{2n} = (\mathcal{B}^*\mathcal{B}^n)^2 = \mathcal{B}^{*2}\mathcal{B}^{2n} = \mathcal{I}.$$

Hence:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{I} = \mathcal{B}^{*2}\mathcal{B}^{2n},$$

and so:

$$\mathcal{A}^{*2}\mathcal{A}^{2n} = \mathcal{B}^{*2}\mathcal{B}^{2n}.$$

□

Proposition 2.15. *If $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ are n - Square Metrically Equivalent Operators, then they have Bishop's property.*

Proof. By Corollary 2.3 ; $\mathcal{A}$ and $\mathcal{B}$ are n-square normal operators . Combining with Proposition 1.9 , Lemma 1.8 and Theorem 1.7 ; $\mathcal{A}$ and $\mathcal{B}$ have Bishop's property.

□

Proposition 2.16. *If $\mathcal{A}$ is n-normal operator, then $\mathcal{A}$ is n-square-normal operator.*

Proof. If $\mathcal{A}$ is n- normal operator, then:

$$\mathcal{A}^{2n}(\mathcal{A}^*)^2 = \mathcal{A}^n\mathcal{A}^n\mathcal{A}^*\mathcal{A}^* = \mathcal{A}^n\mathcal{A}^*\mathcal{A}^n\mathcal{A}^* = \mathcal{A}^*\mathcal{A}^n\mathcal{A}^*\mathcal{A}^n = \mathcal{A}^*\mathcal{A}^*\mathcal{A}^n\mathcal{A}^n = (\mathcal{A}^*)^2\mathcal{A}^{2n}.$$

Thus, $\mathcal{A}$ is n-square-normal operator.

□

Proposition 2.17. *If $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ are n -Square Metrically Equivalent Operators, then they are isoloid and polaroid.*

Proof. By Corollay 2.3 and Proposition 2.16 ; $\mathcal{A}$ and $\mathcal{B}$ are n-square normal operators and hence n-normal operators and by Theorem 1.10 they are isoloid and polaroid.

□

Theorem 2.18. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r \in \mathcal{B}(\mathcal{H})$ be n- square metrically equivalent operator; then :*

1. *$\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_r$ and $\mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_r$ are n- square metrically equivalent operator.*
2. *$\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_r$ and $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \cdots \otimes \mathcal{B}_r$ are n- square metrically equivalent operator .*

Proof. (i) We show that :

$$\left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{*2} \left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{2n} = \left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{*} \left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{*} \left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{n} \left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{n}$$

Expanding we have

$$\left(\bigoplus_{i=1}^{r} \mathcal{A}_i^*\right) \left(\bigoplus_{i=1}^{r} \mathcal{A}_i^*\right) \left(\bigoplus_{i=1}^{r} \mathcal{A}_i^n\right) \left(\bigoplus_{i=1}^{r} \mathcal{A}_i^n\right)$$

Since each $\mathcal{A}_i \sim_{m^{2n}} \mathcal{B}_i$, there exists a unitary operator $\mathcal{U}$ such that:

$$\mathcal{U} \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^*\right) \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^*\right) \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^n\right) \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^n\right) \mathcal{U}^*$$

$$= \mathcal{U} \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^* \mathcal{B}_i^*\right) \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^n \mathcal{B}_i^n\right) \mathcal{U}^*$$

$$= \mathcal{U} \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^*\right)^2 \left(\bigoplus_{i=1}^{r} \mathcal{B}_i^n\right)^2 \mathcal{U}^*$$

$$= \mathcal{U} \left(\bigoplus_{i=1}^{r} \mathcal{B}_i\right)^{*2} \left(\bigoplus_{i=1}^{r} \mathcal{B}_i\right)^{2n} \mathcal{U}^*$$

Thus

$$\left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{*2} \left(\bigoplus_{i=1}^{r} \mathcal{A}_i\right)^{2n} = \mathcal{U} \left(\bigoplus_{i=1}^{r} \mathcal{B}_i\right)^{*2} \left(\bigoplus_{i=1}^{r} \mathcal{B}_i\right)^{2n} \mathcal{U}^*$$

Hence,

$$\bigoplus_{i=1}^{r} \mathcal{A}_i \sim_{m^{2n}} \bigoplus_{i=1}^{r} \mathcal{B}_i$$

.

(ii) Similarly , we show :

$$\left(\bigotimes_{i=1}^{r} \mathcal{A}_i\right)^{*2} \left(\bigotimes_{i=1}^{r} \mathcal{A}_i\right)^{2n} = \left(\bigotimes_{i=1}^{r} \mathcal{A}_i\right)^{*} \left(\bigotimes_{i=1}^{r} \mathcal{A}_i\right)^{*} \left(\bigotimes_{i=1}^{r} \mathcal{A}_i\right)^{n} \left(\bigotimes_{i=1}^{r} \mathcal{A}_i\right)^{n}$$

Expanding we have

$$\left(\bigotimes_{i=1}^{r} \mathcal{A}_i^*\right) \left(\bigotimes_{i=1}^{r} \mathcal{A}_i^*\right) \left(\bigotimes_{i=1}^{r} \mathcal{A}_i^n\right) \left(\bigotimes_{i=1}^{r} \mathcal{A}_i^n\right)$$

Since each $\mathcal{A}_i \sim_{m^{2n}} \mathcal{B}_i$, there exists a unitary operator $\mathcal{U}$ such that:

$$\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^*\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^*\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^n\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^n\right)\mathcal{U}^*$$

$$=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^*\mathcal{B}_i^*\right)\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^n\mathcal{B}_i^n\right)\mathcal{U}^*$$

$$=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^*\right)^2\left(\bigotimes_{i=1}^{r}\mathcal{B}_i^n\right)^2\mathcal{U}^*$$

$$=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{*2}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{2n}\mathcal{U}^*$$

Thus

$$\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{*2}\left(\bigotimes_{i=1}^{r}\mathcal{A}_i\right)^{2n}=\mathcal{U}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{*2}\left(\bigotimes_{i=1}^{r}\mathcal{B}_i\right)^{2n}\mathcal{U}^*$$

Therefore,

$$\bigotimes_{i=1}^{r}\mathcal{A}_i \sim_{m^{2n}} \bigotimes_{i=1}^{r}\mathcal{B}_i$$

. □

3 Results and Discussion

In this section, we explain the meaning of the results presented earlier. We focus on what the findings tell us about the structure and behavior of n-Square Metrically Equivalent Operators. We also compare this new class with other well-known operator classes.

3.1 Spectral Properties

Theorem 2.10 shows that if an operator A has Bishop's property (β) and its power A^n has a closed range, then any operator B that is n-square metrically equivalent to A also has Bishop's property. This tells us that the property is preserved under this equivalence.

Theorem 2.13 shows that if the point spectrum of A^n has only isolated points, then both A and B are isoloid. This means that this equivalence relation helps to keep important spectral features.

3.2 Relation to Other Operator Classes

Proposition 2.6 tells us that when $n = 1$, and if the operators are idempotent, then n-square-metric equivalence is the same as the usual metric equivalence.

Theorem 2.14 shows that if two operators are quasi-isometries and are n-square metrically equivalent for $n = 1$, then they are also metrically equivalent. This shows that our new definition generalizes earlier ones.

Proposition 2.19 tells us that if two operators are n-square metrically equivalent, then they are both isoloid and polaroid. This shows that the equivalence preserves many important operator properties.

3.3 Examples and Warnings

In Example 2.12, we saw that two operators can be n-square metrically equivalent but still have different spectra. This tells us that n-square metric equivalence does not always preserve the spectrum.

In another example (Example 2.8), we saw that two operators can be n-square metrically equivalent but not metrically equivalent. This means we must be careful when assuming properties transfer between equivalent operators.

3.4 Behavior under Direct Sums and Tensor Products

Theorem 2.20 shows that if several operators $A_1, A_2, \ldots, A_r$ are each n-square metrically equivalent to operators $B_1, B_2, \ldots, B_r$, then their direct sums and tensor products are also n-square metrically equivalent. This result is useful when studying large or complex systems of operators.

3.5 Background and Problem Statement

In operator theory, understanding the structure and relationships between different classes of operators is very important. Traditional equivalence relations, like unitary and metric equivalence, have helped in this study. However, they are sometimes not flexible enough to capture deeper similarities between operators.

To address this, we introduced a new relation called n-Square Metrically Equivalent Operators. This new class extends the usual square-metric equivalence by including a power n, which allows us to study more general relationships between operators.

This work focuses on understanding the properties of this new class. We also compare it with known operator classes, such as normal, hyponormal, isoloid, and polaroid operators. The main goal is to find out what properties are preserved under this new equivalence and how it can be useful in operator theory.

4 CONCLUSIONS

The main goal of this study was to introduce and explore the concept of n-Square Metrically Equivalent Operators. Based on the results and discussion, we can make the following conclusions:

1. The relation $\sim_{m2n}$ defines a new class of equivalence for bounded linear operators. This generalizes classical metric and square-metric equivalence.
2. n-Square metric equivalence preserves many important properties of operators, including Bishop's property (β), isoloid, and polaroid properties under certain conditions.
3. This equivalence does not always preserve the spectrum or the numerical range, as shown in the examples.
4. The class is compatible with operations like direct sums and tensor products, which makes it useful in studying systems of operators.
5. When $n = 1$, the new equivalence reduces to known cases, such as metric equivalence for idempotent or quasi-isometric operators.

These results add to the understanding of operator equivalence in functional analysis and may be useful in applications involving spectral theory and operator classification.

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