**FRACTIONAL ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS: A NUMERICAL SOLUTION APPLYING COLLOCATION POINTS**

**Abstract:** Integro-differential equations are applied to model physical phenomena in science and engineering. They are found in many mathematical representations of physical phenomena (IDEs). Kinetic equations describing the kinetic theory of rarefied gases, plasma, coagulation, and radiation transport are a few problems.In this work, we developed and applied a numerical approach to solve Fredholm integro-differential equations of fractional order with collocation points. After obtaining the problem's integral form, we used the collocation points to convert it into an algebraic system of equations. We use matrix inversion to solve the algebraic equation. An analysis of the developed approach revealed that the results were convergent and continuous. Furthermore, it was shown that the answer was unique. The effectiveness and consistency of the technique were assessed using numerical examples.

**Keyword:** Collocation points, Integro-differential equation, Fractional derivatives, Fredholm, Approximate Solution.

**1. Introduction**

Fractional differential and integral equations are used in many disciplines, such as physics, mathematics, engineering, and chemistry. Ordinary with partial differential equations are examples of real-world problems that can be represented mathematically as functional equations. Integro-differential equations are applied to model physical phenomena in science and engineering. They are found in many mathematical representations of physical phenomena (IDEs). Kinetic equations describing the kinetic theory of rarefied gases, plasma, coagulation, and radiation transport are a few problems[1].

The following are a few examples of numerical solutions for fractional differential equations that have been developed in the literature: Bernstein polynomials method [14–15], Perturbed collocation method [2], Adomian decomposition method [3-5], Collocation method [6–9], Hybrid linear multi-step method [10–11], Differential transform method [12], Pseudo-spectral method [13], Bernstein polynomials method [14–15], [16]; a numerical technique based on the Boubaker polynomial The operational matrix they chose to utilize for fractional integration was based on a Boubakar polynomial.

The fractional-order Fredholm Integro-differential equation of the form is studied in this study through its numerical solution.

(1)

using the initial condition

(2)

where the unknown function is denoted by , the known function by , the Caputo's derivative by , the Fredholm integral kernel function by , and a known constant by

**2. A Few Fundamental Terms and Definitions**

In order to formulate the given problem, we provide certain definitions and basic ideas of fractional calculus in this part.

**Definition 1**: [1] For a given function the Caputo fractional derivative with order is defined as:

(3)

where

**Definition 2: [1]** Let be a sequence of real integers and . The power series in t with coefficients is an equation.

(4)

where

then

**Definition 3:** [1] This approach determines the necessary collocation sites between an interval. For example, [c,d] and is defined as

(5)

**Definition 4:** [1] (Integration of nth derivatives) For Let be a continuous function, then

(6)

where

**Definition 5:** [1] Let be an integrable function, then

(7)

**Definition 6: [1]** The function with the following characteristics is a metric on a set M.

**Definition 7: [1] (Metric space)** Let be a metric space. If there is a constant such that, then a mapping is Lipschitzian.

**Theorem 1: [18] (Banach's fixed point theorem)**  Every contraction mapping has a distinct fixed point x of , such that , assuming that is a full metric space.

**3. Methodology**

This section introduces a numerical method for solving fractional-order Fredholm integro-differential equations with collocation points.

**Lemma 1:** Let be the solution to equation (1) and equation (2), then equation (1) and equation(2) is equivalent to

(8)

where

**Proof.**

Multiply equation (1) by 0gives

0 0 0

using equation (6) and equation (7) on equation (1) gives

(9)

where

**3.1 Solution Method**

Equation (9) is collocated at

(10)

substituting equation (4) into equation (10) gives

(11)

where

Equation (11) can be in the form

(12)

where

**3.2 Consideration of Initial Condition**

Using the initial condition in equation (2)

(13)

hence, equation (13) becomes

(14)

Adding equation (14) to equation (12) gives

(15)

The numerical result is obtained by substituting the values of the a's into the approximate solution after the algebraic problem (15) has been solved.

**3.3 Uniqueness of the Solution**

Assuming that equations (1) and (2) have solutions, we demonstrated the uniqueness of the solution and provided solutions obtained using the method of solution in this section.

To establish the method's uniqueness, we present the following hypothesis.

H₁: Let be a mapping for , There exist a constant, . such that

H₂: There exist a functions the set of all positive functions such that

**Theorem 2:** Assume the hold. If

Hence, there exist a unique solution .

**Proof**

Let , applying Banach fixed point to equation (8) gives

(16)

and

(17)

Subtracting equation (17) from equation (16) gives

Taking the absolute value of both sides gives

Taking the maximum of both sides and applying gives

The Banach contraction principle allows us to conclude that T has a unique fixed point

**Lemma 3:** (Continuity) Let be a metric space and be a mapping, let and the . T is continuous if

**Proof.**

Since as n→∞, then , therefore T is continuous

**3.4 Convergence of the method**

**Lemma 4: (Convergence of method)** Let be a metric space and be a continuous mapping and are approximate solutions of equation (4). Let , , then the method converges to exact solution.

**Proof.**

Let and be approximated by and

Substitute the approximate solution into equation (8) gives

Similarly

Since and, hence

Consequently, the solution approach converges.

**4. A Few Numerical Illustrations**

To assess the method's ease of use and effectiveness, this section provided numerical examples. Using the MAPLE 18 algorithm, it is computed. Assign to the approximate solutions and to the exact answers..

**Example 1:** [17] Consider Fractional Fredholm integro-differential equation .

(18)

with initial condition

Exact solution:

**Solution 1:**

At N=3, the approximate solution of equation (18) yields

**Table 1:** values for the example's exact, approximate, and absolute errors

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **X** | **Exact Solution** | **NumericalN=3** | **Error3** | **Error[17]=3** |
| 0.1 | -0.09000000000 | -0.089999999940 | 6.00e-11 | 1.00e-07 |
| 0.2 | -0.16000000000 | -0.159999999900 | 1.00e-10 | 2.00e-07 |
| 0.4 | -0.20000000000 | -0.239999999900 | 1.00e-10 | 8.00e-07 |
| 0.6 | -0.24000000000 | -0.239999999900 | 1.00e-10 | 1.90e-06 |
| 0.8 | -0.16000000000 | -0.159999999800 | 2.00e-10 | 3.60e-06 |

**Example 2:** [17] Consider Fractional Fredholm integro-differential equation .

(19)

subject to initial condition

Exact solution:

**Solution 2:**

The approximate result of equation (19) at N= 3

**Table 2:** values for the example's exact, approximate, and absolute errors

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **X** | **Exact Solution** | **NumericalN=3** | **Error3** | **Error[17]=3** |
| 0.1 | 0.09900000000 | 0.09900000000 | 0.00 | 2.08e-05 |
| 0.2 | 0.19200000000 | 0.19200000000 | 0.00 | 9.2e-05 |
| 0.4 | 0.33600000000 | 0.33600000000 | 0.00 | 1.0e-07 |
| 0.6 | 0.38400000000 | 0.38400000000 | 0.00 | 3.0e-07 |
| 0.8 | 0.28800000000 | 0.28800000000 | 0.00 | 8.0e-07 |

**5. Discussion of Results**

This section discusses the numerical results obtained from the solved examples using the devised numerical approach.

According to Table 1's numerical result for Example 1, the approximate answer at provides . At the same value of N=3, the numerical results provide a better result than the one obtained by [17].

Table 2 displays the approximate answer at N = 3 in numerical Example 2, which provides  
. At the same value of N=3, the numerical results converged to an exact solution, showing that our approach performed better than the one proposed by [17].

**5.1. Conclusion**

The collocation point approach for fractional Fredholm integro-differential equations is investigated in this work. This approach is reliable, effective, and simple to compute. All of the computations in this study were done using Maple 18.

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1.

2.

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