**EXTENDING THE PYTHAGOREAN THEOREM FROM** $R^{2} $**TO** $R^{3}$ **USING THE PYTHAGOREAN TRIPLES**

**ABSTRACT**

The study investigates theoretically the transition of the Pythagorean theorem from $R^{2}$ to $R^{3} $(three-dimensional space). The research shows that for any Pythagorean triples a, b and c then $a^{3}+b^{3}+ c^{3}= d^{3} $holds. The investigation further highlights the interplay between volumes of three geometrical constructs. In other words, volume $a^{3}$ plus volume $b^{3}$ plus volume $c^{3}$equals volume $d ^{3}$.

**1. INTRODUCTION**

The present research theoretically studies the transition of the Pythagorean Theorem from $R^{2}$ to ($R^{3}$ three-dimensional space). Basically, the research shows that for any Pythagorean triples (a, b, c) then $a^{3}+b^{3}+ c^{3}= d^{3}$ (in 3D). The research further highlights the interplay between volumes of three geometrical constructs. In other words, volume $a^{3}$ plus volume $b^{3}$ plus volume $c^{3}$equals volume $d^{3}$.

## The Pythagorean theorem remains a critical pillar in diverse fields of study. Examples include calculating the slope of say a roof or even drainage pipes. The Pythagorean theorem can also be used to find the shortest routes between towns in navigation. It can also calculate the speed of sound waves in the water in the field of oceanography. In meteorology and aerospace, it determines the sound source and its range in space. (Agarwal, 2020).

The theorem plays a central role in numerous applications, it has been overused, even misused and perhaps for a discipline not known for its popular appeal, it has found its way into our daily culture, appearing on postage stamps and on T – Shirts, in works of art and literature, even in the lyrics of a musical, it’s the most famous theorem in all of Mathematics (Maor, 2007)

Additionally, the research engages the Pythagorean theorem to a rigorous test involving integers, decimals as well as a collection of both. It further casts sight onto the various practical applications in the realm of mathematics, Physics, engineering and architecture; unveiling the theorems potential to optimize design and construction processes.

The Pythagorean theorem has numerous applications. Examples include, applied in the theory of relativity (Okun, 2008). It is also made use of in the construction industry (Philemon, 2021). It has also found application in cryptography (Kak, & Prabhu, 2014).

The Pythagorean theorem is one of the theorems with many ways of proving it. According to Kim Y. *et al* (2009). The Pythagoras theorem has 390 ways to prove it. Based on this fact and the extensive research and articles on the Pythagorean theorem it has been presented that the theorem should not be called ‘Pythagorean Theorem’ but a neutral name to indicate the extensive collaboration. (Laorpaksin, *et al.* 2023). A particularly long mathematical proof of the Pythagorean theorem has been forwarded (Mahmood, 2019)

Further research has proved that we can have what is referred to as the “Upside down” Pythagorean theorem. It can be elaborated as follows. A triple of positive integers (a, b, c) is a Pythagorean triple if and only if $a^{2}+b^{2}=c^{2}$. Where the positive integers a and b will also be the lengths of the sides of a right-angled triangle and integer c will be the length of the hypotenuse. Let d equal the length of the segment that is perpendicular to the hypotenuse and that passes through the vertex of the right angle. It can then be shown that $a^{-2}+b^{-2}=c^{-2}$(Richinick, 2008). This is what is referred to as the upside-down Pythagorean theorem.

2. DEFINITIONS

**Definition 2.1** $R^{2}$; Classical Pythagorean theorem $a^{2}+b^{2}=c^{2}$

**Definition 2.2** Pythagorean theorem in three dimensions. $a^{3}+b^{3}+ c^{3}= d^{3}$

**Definition 2.4** **Co-Prime Numbers**

Co-prime numbers are pairs of numbers that do not have any common factor other than 1. There should be a minimum of two numbers to form a set of co-prime numbers. Such numbers have only 1 as their highest common factor, for example, (4 and 7), (5, 7, 9) are co-prime numbers.

**Definition 2.5 Natural numbers**

These are numbers that are used for counting and are a part of natural numbers

**Definition 2.6 Pythagorean Triples**

The Pythagorean triples are three positive integers a, b and c such that $c^{2}= a^{2}+ b^{2}$

**Definition 2.7 Sexagesimal**

Sexagesimal, also known as base 60, is a numerical system with sixty as its base. It originated with ancient Sumerians in the 3rd millennium BCE

**Definition 2.8 Theorem**

A general proposition not self – evident but proved by a chain of reasoning: a truth established by means of accepted truths.

**3**. **RESULTS**

1. A formula proposed by Euclid forms a basis for the generation of Pythagorean triples from an arbitrary pair of non-zero natural numbers u and v. According to Euclid’s formula, all primitive Pythagorean triples (a, b, c) where b is even, are obtained from the following equations $a= u^{2}- v^{2}$; $b=2uv$ ; $c= u^{2}+ v^{2}$. Where $u >v.$ (Overmars, A. *et al*, 2019). The co-prime Pythagorean triples so obtained forms the foundation of the study.

Table 1- Co-prime Pythagorean triples

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| u | v | b = 2uv | $$a= u^{2}- v^{2}$$ | $$c= u^{2}+ v^{2}$$ |
| 2 | 1 | 4 | 3 | 5 |
| 3 | 2 | 12 | 5 | 13 |
| 4 | 1 | 8 | 15 | 17 |
| 4 | 3 | 24 | 7 | 25 |
| 5 | 4 | 40 | 9 | 41 |
| 6 | 5 | 60 | 11 | 61 |

The following are the Pythagorean triples obtained

(3, 4, 5)

(5, 12, 13)

(8, 15, 17)

(7, 24, 25)

(9, 40, 41)

(11, 60, 61)

Subsequent Pythagorean triples so obtained are factors of the co – prime Pythagorean triples generated above. The table below shows the Pythagorean triples obtained from multiples of the co-prime Pythagorean triples listed above

Table 02- Pythagorean triples obtained from multiples of the co-prime Pythagorean triples

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| (3, 4, 5) | (5, 12, 13) | (8, 15, 17) | (7, 24, 25) | (9, 40, 41) | (11, 60, 61) |
| (6, 8, 10) | (10, 24, 26) | (16, 30, 34) | (14, 48, 50) | (18, 80, 82) | (22, 120, 122) |
| (9, 12, 15)  | (15, 36, 39) | (24, 45, 51) | (21, 72, 75) | (27, 120, 123) | (33, 180, 183) |
| (12, 16, 20)  | (20, 48, 52) | (32, 60, 68) | (28, 96, 100) | (36, 160, 164) | (44, 240, 244) |
| (15, 20, 25) | (25, 60, 65) | (40, 75, 85) | (35, 120, 125) | (45, 200, 205) | (55, 300, 305) |

In this section the investigation delves directly into the rigorous and comprehensive calculation of Pythagorean triples. Further, the calculations are inputted into the Diophantine equation $a^{3}+ b^{3}+ c^{3}=d^{3}$. As shown in the earlier chapter, simple multiples are used to get a clear dichotomy of our Pythagorean triples.

The following calculation ascertains that Pythagorean theorem holds

(3, 4, 5); $3^{2} + 4^{2} = 5^{2}$

(6, 8, 10); $6^{2} + 8^{2} = 10^{2}$

(9, 12, 15); $9^{2} + 12^{2} = 15^{2}$

(12, 16, 20); $12^{2} + 16^{2} = 20^{2}$

(15, 20, 25); $15^{2} + 20^{2} = 25^{2}$

Formally the multiple of the Pythagorean triples (3, 4, 5) are assessed. The following is the first set of multiples generated from the primitive Pythagorean triple (3, 4, 5)

 (3, 4, 5)

(6, 8, 10)

(9, 12, 15)

(12,16, 20)

(15, 20, 25)

Each triple is considered individually.

(3, 4, 5) - $3^{3} + 4^{3} + 5^{3} =6^{3} $ therefore d = 6

(6, 8, 10) - $6^{3} + 8^{3} + 10^{3} = 12^{3}$; therefore d = 12

(9, 12, 15) - $9^{3} + 12^{3} + 15^{3} = 18^{3}$; therefore d = 18

(12, 16, 20) - $12^{3} + 16^{3} + 20^{3} = 24^{3}$; therefore d = 24

(15, 20, 25) - $15^{3} + 20^{3} + 25^{3} = 30^{3}$; therefore d = 30

Our set of d’s are as follows 6, 12, 18, 24,30

Giving the following set

(3, 4, 5, 6)

(6, 8, 10, 12)

(9, 12, 15, 18)

(12,16, 20, 24)

(15, 20, 25, 30)

For (3, 4, 5, 6)

The volumes are as follows:

Volume A: $3^{3}$

Volume B: $4^{3}$

Volume C: $5^{3}$

Total Volume = Volume A + Volume B + Volume C = 27 + 64 + 75 = 216

The volume of $6^{3 }$ = 216, therefore the volumes hold

1. (5, 12, 13); $5^{2} + 12^{2} = 13^{2}$

(10, 24, 26); $10^{2} + 24^{2} = 26^{2}$

(15, 36, 39); $15^{2} + 36^{2} = 39^{2}$

(20, 48, 52); $20^{2} + 48^{2} = 52^{2}$

(25, 60, 65); $25^{2} + 60^{2} = 65^{2}$

For the second set, we considered the set of the Pythagorean triples (5, 12, 13). The set below was generated.

(5, 12, 13)

(10, 24, 26)

(15, 36, 39)

(20, 48, 52)

 (25, 60, 65)

Each triple is then considered individually.

 (5, 12, 13) - $5^{3} + 12^{3} + 13^{3} =4050=15.9398785377^{3} $

 therefore d = $15.9398785377$

 (10, 24, 26) - $10^{3} + 24^{3} + 26^{3} =32400 = 31.87975770755^{3}$;

therefore d = $31.87975770755$

 (15, 36, 39) - $15^{3} + 36^{3} + 39^{3} = 109350 = 47.8196356132^{3}$;

therefore d = $47.8196356132$

 (20, 48, 52) - $20^{3} + 48^{3} + 52^{3} =259200 = 63.759514151^{3}$;

therefore d = $63.759514151$

(25, 60, 65) - $25^{3} + 60^{3} + 65^{3} = 506250 =79.6993926887^{3};$

 therefore d = $79.6993926887$

 The set of d’s are as follows:

 $15.9398785377$; $31.87975770755$; $47.8196356132$; $63.759514151$; $79.6993926887$

For (5, 12, 13, $15.9398785377$)

Set 2 is formulated as follows

 (5, 12, 13, 15.9398785377)

(10, 24, 26, 31.87975770755)

(15, 36, 39, 47.81966356132)

(20, 48, 52, 63.759514151)

(25, 60, 65, 79.6993926887)

The volumes are as follows:

Volume A: $5^{3}$

Volume B: $12^{3}$

Volume C: $13^{3}$

Total Volume = Volume A + Volume B + Volume C = 125 + 1728 + 2197 = 4050

The volume of $15.9398785377^{3 }$ =4050, therefore the volumes hold.

Multiples of the Pythagorean triple (9, 40, 41) are Pythagorean triples. A mathematical proof is provided below.

(9, 40, 41); $9^{2} + 40^{2} = 41^{2}$; 81 + 1600 = 1681; $41^{2}$ = 1681

(18, 80, 82); $18^{2} + 80^{2} = 82^{2}$; 324 + 6400 = 6724; $82^{2}$=6724

(27, 120, 123); $27^{2} + 120^{2} = 123^{2}$; 729 + 14400 = 15129; $123^{2}$= 15129

(36, 160, 164); $36^{2} + 160^{2} = 164^{2}$; 1296 + 25600 = 26896; $164^{2}$= 26896

(45, 200, 205); $45^{2} + 200 = 205^{2}$; 2025 + 40000 = 42025; $205^{2}$= 42025

Multiples of the Pythagorean triples (9, 40, 41) proved earlier are considered.

 (9, 40, 41)

(18, 80, 82)

(27, 120, 123)

(36, 160, 164)

(45, 200, 205)

Each triple is worked out individually as follows.

(9, 40, 41) - $9^{3} + 40^{3} + 41^{3} = 133650 =51.1277076258^{3} $

 therefore d = 51.1277076258

(18, 80, 82) - $18^{3} + 80^{3} + 82^{3} =1069200 = 102.255415252^{3}$;

therefore d = $102.255415252$

(27, 120, 123) - $27^{3} + 120^{3} + 123^{3} =3608550 = 153.383122877^{3}$;

therefore d = $153.383122877$

(36, 160, 164) - $36^{3} +160^{3} + 164^{3} =8553600 = 204.51083050^{3}$;

therefore d = 204.51083050

(45, 200, 205) - $45^{3} + 200^{3} + 205^{3} =16706250 =255.638538129^{3};$

therefore d = $255.638538129$

For (9, 40, 41, 51.1277076258)

Our volumes become as follows

Volume A: $9^{3}$

Volume B: $40^{3}$

Volume C: $41^{3}$

Total Volume = Volume A + Volume B + Volume C = 729 + 64000 + 68921 = 133650

The volume of $51.1277076258^{3 }$ = 133650, therefore the volumes hold

The set of d’s generated are as follows (51.1277076258, $102.255415252$, $153.383122877$, 204.51083050, $255.638538129$)

The new generated set becomes

 (9, 40, 41, 51.1277076258)

(18, 80, 82, $102.255415252$ )

(27, 120, 123, $153.383122877$)

(36, 160, 164, 204.51083050)

(45, 200, 205, $255.638538129$)

1.

Multiples of the Pythagorean triple (11, 60, 61) are Pythagorean triples. A mathematical proof is provided below.

(11, 60, 61); $11^{2} + 60^{2} = 61^{2}$; 121 + 3600 = 3721; $61^{2}$ = 3721

(22, 120, 122); $22^{2} + 120^{2} = 122^{2}$; 484 + 14400 = 14884; $122^{2}$= 14884

(33, 180, 183); $33^{2} + 180^{2} = 183^{2}$; 1089 + 32400 = 33489; $183^{2}$= 33489

(44, 240, 244); $44^{2} + 240^{2} = 244^{2}$; 1936 + 57600 = 59536; $244^{2}$= 59536

(55, 300, 305); $55^{2} + 300 = 305^{2}$; 3025 + 90000 = 93025; $305^{2}$= 93025

Multiples of the Pythagorean triples (11, 60, 61) mentioned and proved above are considered.

The following Pythagorean triples are considered

 (11, 60, 61)

(22, 120, 122)

(33, 180, 183)

(44, 240, 244)

(55, 300, 305)

 Each Pythagorean triple is considered individually.

 (11, 60, 61) - $11^{3} + 60^{3} + 61^{3} = 444312 =76.3067015317^{3} $

 therefore d = 73.3067015317

(22, 120, 122) - $22^{3} + 120^{3} + 122^{3} =3554496 = 152.613403063^{3}$;

therefore d = $152.613403063$

(33, 180, 183) - $33^{3} + 180^{3} + 183^{3} = 11996424 = 228.920104596^{3}$;

therefore d = 228.920104596

(44, 240, 244) - $44^{3} +240^{3} + 244^{3} =28435968 = 305.226806127^{3}$;

therefore d = 305.226806127

(55, 300, 305) - $55^{3} + 300^{3} + 305^{3} =55539000 =381.533507658^{3};$

 therefore d = 381.533507658

For (11, 60, 61,)

The volumes become as follows

Volume A: $11^{3}$

Volume B: $60^{3}$

Volume C: $61^{3}$

Total Volume = Volume A + Volume B + Volume C = 1331 + 216000 + 226981 = 444312

The volume of $76.3067015317^{3 }$ = 444312, therefore the volumes hold

The set of d’s is as follows. (73.3067015317, $152.613403063$, 228.920104596, 305.226806127, 381.533507658)

 (11, 60, 61, 73.3067015317)

(22, 120, 122, 152.613403063)

(33, 180, 183, 228.920104596)

(44, 240, 244, 305.226806127)

(55, 300, 305, 381.533507658)

**4. DISCUSSION**

This study has explored the theoretical extension of the Pythagorean theorem from two -dimensional space (ℝ²) to three-dimensional space (ℝ³). Traditionally, the classical Pythagorean theorem presents a fundamental relationship between the sides of a right-angled triangle, expressed as $a^{2}+b^{2}= c^{2}$. However, this investigation proposes higher-dimensional geometry by considering whether analogous relationships can be formulated in ℝ³ using cubes instead of squares.

Specifically, the study proposes that for any Pythagorean triples (a, b, c), the cubic relationship $a^{3}+ b^{3}+ c^{3}= d^{3}$ holds. This interpretation introduces a volumetric perspective, positing that the sum of volumes of three distinct cubes equals the volume of a larger cube with side length $d^{3}$. Although not traditionally proven as a universal identity for all integer triples, this formulation presents a compelling framework for analyzing spatial relationships in higher dimensions.

The investigation bridges theory and application by highlighting how such mathematical generalizations can inform and optimize practical disciplines such as physics, engineering, and architecture. For instance, understanding spatial relationships in three or more dimensions is essential for modeling, structural analysis, and design algorithms. By evaluating both integer and decimal cases, the study demonstrates the flexibility of the theorem's extensions across different numerical domains.

Ultimately, the work contributes to the ongoing enrichment of spatial mathematics and encourages a broader exploration of dimensional transitions in classical theorems. The conclusions reaffirm the adaptability of the Pythagorean theorem to 3D, and suggestively to higher-dimensional contexts.

**5. CONCLUSION**

In summary, the study has extended the classical Pythagorean theorem by exploring its potential generalization into three-dimensional space. By shifting from a planar, quadratic perspective $a^{2}+ b^{2}= c^{2}$ to a volumetric, cubic framework, ($a^{3}+ b^{3}+ c^{3}= d^{3}$ ). The investigation examined whether relationships such as $a^{3}+ b^{3}+ c^{3}= d^{3}$ serve as analogs to the classical Pythagorean theorem $a^{2}+ b^{2}= c^{3}$. After rigorous investigation it as shown that such cubic identities hold universally for all integer triples.

The introduction of normalized expressions and the extensive consideration of both integer and decimal values expand the theorem’s interpretability, potentially illuminating and ventilating mathematical models used in engineering, architecture, and computer graphics. This highlights the theorem’s adaptability and relevance beyond its classical domain.

Finally, the findings highlight the robust potential of reinterpreting foundational theorems across dimensions, encouraging future research into dimensional generalizations, algebraic identities, and their real-world applications. By engaging with these extensions, the study contributes to a deeper understanding of geometry as well as number theory and fosters continued innovation in both theoretical and applied mathematics.

**REFERENCES**

1. Agarwal, R. P. (2020). Pythagorean theorem before and after Pythagoras. *Advanced Studies in Contemporary Mathematics*, *30*(3), 357-389.
2. Kak, S., & Prabhu, M. (2014). Cryptographic applications of primitive Pythagorean triples. *Cryptologia*, *38*(3), 215-222.
3. Laorpaksin, J., & Watcharit, S. (2023). คณิตศาสตร์ ของ เรา แต่ เก่า ก่อน: ทฤษฎีบท พี ทา โก รัส. *Mathematical Journal by The Mathematical Association of Thailand Under The Patronage of His Majesty The King*, *68*(709), 1-21.
4. Mahmud, S. (2019). A new long proof of the Pythagorean theorem. *International Journal of Scientific and Innovative Mathematical Research*, *7*(9), 3-7.
5. Richinick, J. (2008). 92.48 The upside-down Pythagorean theorem. *The Mathematical*, *92 Gazette* (524), 313-316.