ON THE MONOTONICITY OF RANK-SHAPLEY VALUE FOR SUPER-ADDITIVE GAMES

Abstract

Changes in a cooperative game data can lead to unparalleled changes in the solution part of the game. Similarly, an alteration in sharing weight can cause some influences especially in a sharing scheme that shares the dividend of a coalition based on a specific weight system. This concept is explored under monotonic solutions of cooperative games. However, there is no established definite pattern of effect on solution as a result of various changes in any definite weight function. This work specifically, explores some patterns of changes in rank as a weight function, and their corresponding effect on Rank-Shapley value of cooperative games. The study of the monotonic property of the value presents a basis to support its applicability. A solution to the difference equation provides a positive integer that can ensure a desired change in the payoff of players in a cooperative game. Particularly, the differences in payoff as a result of equal increase in weights by a positive integer, sum to zero.

Keywords: monotonicity; rank; sensitivity; Rank-Shapley value.

1 Introduction

Young (1985) notices that equity is something dealt with in everyday life. Equity which is synonymous to fairness is an aim that the solution concept of cooperative game tends to achieve by sharing the worth of a coalition based on a given weight system. Based on this, Shapley (1953) proposed a solution concept known as the "Shapley value" which is based on the average marginal contribution of each player in every coalition. Kalai and Samet (1988) established diversified notion of weight in the family of weighted Shapley value by offering opportunities for different scheme in sharing the dividend of cooperation in a supper-additive game. In line with the above, Eze et al., (2021) introduced Rank-Shapley value for transferable utility (TU) game in which the sharing weights are endogenously given by the players' ranks as against the sharing weight of Beal et al., (2018) which is based on players' stand-alone worth (value). In any of the scheme (family of weighted Shapley value), three basic things are of great essence in allocating values to the players. These are: dynamics of coalition formation (structure) as modelled by Rosenthal (1990), dynamics of characteristics function and weight system which account for the external characteristics of players. While the coalition formation may not always be prone to error due to the assumption of complete coalition structure in most cases, the worth of coalitions and individual player's weights may be assigned in error. If this is the case, the payoff of the players based on any value function may be affected. This gives the basis for the study of monotonic property of a value function in a cooperative game. A value function is monotonic if a change in the worth of a coalition or sharing weight of player(s) induces a change in their payoff, given that the compliments' weights are unchanged (Haeringer, 2006). Generally, monotonicity captures the characteristics of cooperative games in a dynamic framework. If an alteration in the value of any coalition causes some changes in the players' payoff, it could be referred to as coalitional monotonicity. As pointed out by Shubik (1962), the Shapley value satisfies a natural monotonicity condition in the sense that whenever a player's marginal contribution increases, the player's payoff increases. An instance of this occurs when a player in a game is empowered such that every coalition containing the player increases in worth by a quantity $\lambda : \lambda < v(j^*) - v(i)$ or $\lambda : \lambda > v(j_*) - v(i)$where $v(j^*)$ is the immediate stand-alone value greater than $v(i)$ and $v(j_*)$ is the immediate stand-alone value less than $v(i)$. Automatically, the payoff of the empowered player i increases by same quantity λ while that of any other player $j \in N \setminus \{i\}$ is unaffected by the empowerment. Van den Brink et al. (2013) studied weak monotonicity in Egalitarian Shapley value in which a player's payoff weakly increases whenever the player's marginal contributions weakly increases. Other variants of change in coalition worth is a reflection of aggregate monotonicity (AM) axiom of Megiddo (1974) in which a change in the value of the grand coalition leads to a change in the solution through proportional surplus division (PSD). The proportional surplus division characterizes the Rank-Shapley value as against the equal surplus division (ESD) of Chun and Park (2012) which characterizes the Shapley value. However, PSD converges to ESD if and only if the stand-alone value, $v(i) = v(j)\ \forall\, i \neq j$. Similarly, an alteration in the sharing weight of any player causes some alterations in the player's payoff and that of every other player in the game. If for instance, a player's sharing weight is say β and by error, the player is assigned a weight say $\beta + \alpha$. The alteration in weight will not only affect the player's payoff but also that of every other player in the game. This aspect could be referred to as value monotonicity with respect to the sharing weight. It is a property of value function that tries to examine the changes in payoff as a result of changes in weight, ceteris paribus. The Rank-Shapley value is monotonic with respect to rank (weight) since a change in rank of any player leads to changes in the payoff of the players. In this work therefore, we try to evaluate the sensitivity of Rank-Shapley value to the changes in ranks by considering two different conditions: when only one player's rank changes (increase and decrease) by a positive integer, and when all the player's rank increases by an equal positive integer.

The paper is organized as follows. Section 2 contains some basic definition and notations while Section 3 presents some variants of changes in rank that can influence changes in payoff. This is followed by an example that validates the by-product of theorem 1.

2 Materials and Methods

In this section, we present basic definitions and notation of terms, and explore some variants of changes in the weight (rank) function of Rank-Shapley value. Specifically, we present how increase or decrease in rank by a positive integer affects the payoff of players in a TU-game.

2.1 Basic Definition and Notations

A cooperative game on a fixed number of players N is defined by a characteristics function v. Therefore, a cooperative game is a pair (N, v). It can also be simply represented as v. Let $\Omega = 2^n$ be the set of all coalitions and a subset θ of $N(\theta \in 2^n)$ be a particular coalition whose size is denoted as $|\theta|$. For any given coalition $\theta \in 2^n$, $v(\theta) : 2^n \to \mathbb{R}$ is a function that maps a real value (worth) to such coalition. Conventionally, $v(\phi) = 0$ where ϕ is an empty coalition. Let Ω^N be a collection of all games defined on a fixed set of players N. For any cooperative game $(N, v) \in \Omega^N$, the Rank-Shapley value is a solution that assigns a unique and single payoff to every individual player $i \in N$. It is denoted by φ_i and defined as

$$\varphi_i = \sum_{i \in \theta, \theta \in 2^n} \frac{r_i}{\pi_\theta} H_v(\theta) \tag{2.1}$$

where r_i is the rank of player i stand-alone value, $H_v(\theta) = \sum_{T \subseteq \theta} (-1)^{|\theta| - t} v(T)$ is the dividend (Harsanyi, 1959) accrued to coalition θ and π_θ is the sum of the ranks of players in θ (Eze et al., 2021).

2.2 Variants of Changes in Rank

Here, we will consider different cases of possible changes in rank that can cause alterations in payoff. Actually, an increase or decrease in rank of any player may lead to $r_i \notin [k, nk]; k = 1$ where r_i is the rank of player i. However, this is admissible since the discussion is in line with the monotonic property of the value and not necessarily the property of ranks. In this sense, rank is regarded as a mere weight restricted to positive integers in the interval, $[k, nk]$.

2.2.1 Only one player rank changes (increase):

Consider a game (N, v)defined on a fixed set of players N, evaluated with Rank-Shapley value function. An arbitrary increase in the rank (weight) of player i when the ranks of player i compliment are unchanged induces an increase in the value of player i and decrease in the values of its complement.

Let the arbitrary increase in the rank (weight) of player i be denoted by an integer $\alpha > 0$. Then, the rank of player i increases from r_i to $r_i + \alpha$. This will increase his share of the dividend in any coalition of which he is a member (and decrease that of his complement), thereby increasing his value in the game. This is analogous to the specification of Casajus & Huettner (2014) which states that a weakly increase in a player's productivity correspondingly increases the player's payoff. For pareto-optimality (efficiency) to be preserved, the increase in value of player i will induce a corresponding decrease in the values of player $j \in N \setminus \{i\}$.

Let $\varphi_i (N, v)$ be the Rank-Shapley value of player i in a game and let $\varphi_i (N, v, r_{+\alpha})$ be the Rank-Shapley value of player i in the same game assuming the rank of player i (only) is increased by α. Let the increase in the payoff of player i whose rank increases by α be denoted by $c_i^+ (\alpha)$, where

$$
\begin{aligned}
c_i^+ (\alpha) =& \varphi_i (N, v, r_{+\alpha}) - \varphi_i (N, v) = \sum_{i \in \theta, \theta \in 2^n} \frac{r_i + \alpha}{\pi_\theta + \alpha} H_v (\theta) - \sum_{i \in \theta, \theta \in 2^n} \frac{r_i}{\pi_\theta} H_v (\theta) \\
=& \sum_{i \in \theta, \theta \in 2^n} \frac{\alpha (\pi_\theta - r_i)}{\pi_\theta (\pi_\theta + \alpha)} H_v (\theta)
\end{aligned}
$$

For $\theta = \{i\}$, $(\pi_\theta - r_i) = 0$, then

$$
c_i^+ (\alpha) = \sum_{i \in \theta; \theta \neq \{i\}; \theta \in 2^n} \frac{\alpha (\pi_\theta - r_i)}{\pi_\theta (\pi_\theta + \alpha)} H_v (\theta) \tag{2.2}
$$

Every coalition θ considered in $c_i^+ (\alpha)$ can be expressed as $\theta = \{i \cup J\}$, where $J = \{j : j \in N \setminus \{i\}\}$. Therefore, $\pi_\theta - r_i = \sum_{j \in J} r_j$. Recall that there are 2^{n-1} of coalitions containing player i in a cooperative game defined on a set of players N. Since $\theta \neq \{i\}$, $2^{n-1} - 1$ of such coalitions are in the form $\theta = \{i \cup J\}$. Each of the coalitions contains $(|\theta| - 1)-$ players j. For the grand coalition, there are $(n - 1)$ of players j. Based on this break down, $c_i^+ (\alpha)$ can be factorized into $(n - 1)$ different sums each of which is given as

$$
\eta_j^- (\alpha) = \sum_{i \in \theta; j \in \theta} \frac{\alpha r_j}{\pi_\theta (\pi_\theta + \alpha)} H_v (\theta) \tag{2.3}
$$

Then, every player $j \in N \setminus \{i\}$ is penalized (shortchanged) by a unique quantity $\eta_j^- (\alpha)$. To show that the value of player $j \in N \setminus \{i\}$ is reduced by $\eta_j^- (\alpha)$, we consider Pareto-optimality and observe that

$$\varphi_i(N, v, r_{+\alpha}) + \sum_{j \in N \setminus \{i\}} \left[\varphi_j(N, v) - \eta_j^-(\alpha)\right] = v(N) \tag{2.4}$$

Now, we want to evaluate the LHS of equation (2.4) to show that equation (2.4) holds.

Recall that $\pi_\theta - r_i = \sum_{j \in J} r_j$ and $\theta = \{i \cup J\}$. So,

$$\sum_{j \in N \setminus \{i\}} \eta_j^-(\alpha) = \sum_{j \in N \setminus \{i\}} \sum_{i \in \theta; j \in \theta} \frac{\alpha r_j}{\pi_\theta (\pi_\theta + \alpha)} H_v(\theta) = c_i^+(\alpha)$$

The LHS of equation (2.4) therefore becomes,

$$\sum_{i \in \theta, \theta \in 2^n} \frac{r_i + \alpha}{\pi_\theta + \alpha} H_v(\theta) + \sum_{j \in N \setminus \{i\}} \varphi_j(N, v) - \sum_{i \in \theta; \theta \in 2^n} \frac{\alpha(\pi_\theta - r_i)}{\pi_\theta(\pi_\theta + \alpha)} H_v(\theta)$$

$$= \sum_{i \in \theta; \theta \in 2^n} \frac{r_i}{\pi_\theta} H_v(\theta) + \sum_{j \in N \setminus \{i\}} \varphi_j(N, v) = v(N)$$

This shows that the penalty to any player $j \in N \setminus \{i\}$ as a result of an arbitrary increase in the rank of player i is $\eta_j^-(\alpha)$. Even though one player's rank is altered, the Rank-Shapley value is still Pareto-optimal (efficiency). As we have seen above (a case of increase by α), the penalty to $j \in N \setminus \{i\}$ is a function of the rank of player j and a positive integer, α. It is obvious that the higher the α, the more player $j \in N \setminus \{i\}$ is penalized. However, there is no amount of α that can make $\varphi_j(N, v, r_{+\alpha})$ to be less than $v(j)$ thus, individual rationality is still preserved by the solution. Recall that,

$$\varphi_i(N, v, r_{+\alpha}) = \sum_{i \in \theta; \theta \in 2^n} \frac{r_i + \alpha}{\pi_\theta + \alpha} H_v(\theta)$$

As α increases, the ratio $\frac{r_i+\alpha}{\pi_\theta+\alpha}$ approaches 1 for any $\theta \ni i$. The convergence of $\frac{r_i+\alpha}{\pi_\theta+\alpha}$ to 1 forces any player $j \in N \setminus \{i\}$ to have no share of the dividend in any coalition formed together with player i. In other words, player i takes the whole dividend accruable to any coalition, θ containing (i, j) as α tends to infinity. Thus,

$$\lim_{\alpha \to \infty} \varphi_i(N, v, r_{+\alpha}) = \sum_{i \in \theta; \theta \in 2^n} H_v(\theta) \tag{2.5}$$

This forces the payoff of player $j \in N \setminus \{i\}$ to reduce to

$$\varphi_j(N, v, r_{+\alpha}) = v(j) + \sum_{j \in \theta; i \notin \theta} \frac{r_j}{\pi_\theta} H_v(\theta) \tag{2.6}$$

Equation (2.6) is equivalent to $\varphi_j(N,v) - \eta_j^-(\alpha^m)$ where $\eta_j^-(\alpha^m)$ is the penalty for player j at the maximum point of α. This implies that the worst an increment in r_i can cost any player $j \in N \setminus \{i\}$ is to reduce his payoff to equation (2.6). Therefore, for any $\alpha > 0$, $\varphi_j(N,v,r_{+\alpha})$ is bounded below by equation (2.6) and bounded above by $\varphi_j(N,v)$.

2.2.2 Only one player rank changes (decrease)

In a case of decrease in the rank of player i, α is chosen such that $0 < \alpha \leq r_i$. This restriction is made so as to avoid negative weights. The value of player i reduces by

$$c_i^-(\alpha) = \sum_{i\in\theta;\theta\in 2^n} \frac{\alpha(\pi_\theta - r_i)}{\pi_\theta(\pi_\theta - \alpha)} H_v(\theta)$$

while that of player $j \in N \setminus \{i\}$ increases by

$$\eta_j^+(\alpha) = \sum_{i\in\theta;j\in\theta} \frac{\alpha r_j}{\pi_\theta(\pi_\theta - \alpha)} H_v(\theta)$$

If eventually α is chosen such that $\alpha = r_i$, $c_i^-(\alpha) = \varphi_i(N,v)$since $\varphi_i(N,v,r_{-\alpha}) = 0$.

2.2.3 All the player's rank increases by equal amount

Consider a game (N,v) defined on Rank-Shapley value function. Let all the ranks of the players change (increase) by the same amount $\alpha > 0$. This increment influences some sign oriented changes in the payoff (Rank-Shapley value) of the players. Let the Rank-Shapley value of the game with initial ranks be denoted by $\varphi_i(N,v)$ and the Rank-Shapley value of the game with equally increased ranks be denoted by $\varphi_i(N,v,r_{+\alpha_{eq}})$. The equal change (increase) in the ranks of the players influences the values as follows:

Denote the difference in payoff by $d_i(\alpha)$.

$$d_i(\alpha) = \varphi_i(N,v,r_{+\alpha_{eq}}) - \varphi_i(N,v) = \sum_{i\in\theta} \frac{\alpha[\pi_\theta - |\theta|r_i]}{\pi_\theta[\pi_\theta + \alpha|\theta|]} H_v(\theta)$$

Therefore,

$$\varphi_i(N,v,r_{+\alpha_{eq}}) = \sum_{i\in\theta} \frac{\alpha[\pi_\theta - |\theta|r_i]}{\pi_\theta[\pi_\theta + \alpha|\theta|]} H_v(\theta) + \sum_{i\in\theta} \frac{r_i}{\pi_\theta} H_v(\theta)$$

$$= \sum_{i\in\theta} \frac{r_i + \alpha}{[\pi_\theta + \alpha|\theta|]} H_v(\theta)$$

Theorem 1. *For every cooperative game involving $N \geq 2$ players, any integer $\alpha > 0$, and a Rank-Shapley value function, the differences in payoff as a result of equal increase in ranks by an integer α, sum to zero.*

Proof. Before we prove the theorem, it is necessary to state that for $\varphi_i\left(N, v, r_{+\alpha_{eq}}\right)$ to be greater than $\varphi_i\left(N, v\right)$, $\frac{r_i+\alpha}{\pi_\theta+\alpha|\theta|}$ has to be greater than $\frac{r_i}{\pi_\theta}$ for all $\theta \ni i$. This is analogous to $\frac{\alpha[\pi_\theta-|\theta|r_i]}{\pi_\theta[\pi_\theta+\alpha|\theta|]} \geq 0$ for all $\theta \ni i$. For this to hold, the product of the numerator, $[\pi_\theta - |\theta|r_i]$ has to be greater than or equal to zero since α is always strictly positive. In a case of ranking without ties, it is obvious that for $r_i = 1$, $[\pi_\theta - |\theta|r_i] \geq 0$ for all $\theta \ni i$ since $\pi_\theta \geq |\theta|$ for all $\theta \in 2^n$. So for the player whose rank is 1, it is certain that $\varphi_i\left(N, v, r_{+\alpha_{eq}}\right)$ must be greater than $\varphi_i\left(N, v\right)$.

Generally, in every cooperative game involving $n \geq 2$ players,

$$\sum_{i\in\theta}\frac{\alpha\left[\pi_\theta - |\theta|r_i\right]}{\pi_\theta\left[\pi_\theta+\alpha|\theta|\right]} < 0 \Leftrightarrow \varphi_i\left(N, v, r_{+\alpha_{eq}}\right) < \varphi_i\left(N, v\right)$$

and

$$\sum_{i\in\theta}\frac{\alpha\left[\pi_\theta - |\theta|r_i\right]}{\pi_\theta\left[\pi_\theta+\alpha|\theta|\right]} > 0 \Leftrightarrow \varphi_i\left(N, v, r_{+\alpha_{eq}}\right) > \varphi_i\left(N, v\right)$$

Now, we can prove theorem 1.

For $n = 2$,

$$\sum_{i=1}^{2} d_i\left(\alpha\right) = \frac{\alpha\left[\pi_N - 2r_i\right]}{\pi_N\left[\pi_N + 2\alpha\right]}H_v\left(N\right) + \frac{\alpha\left[\pi_N - 2r_j\right]}{\pi_N\left[\pi_N + 2\alpha\right]}H_v\left(N\right)$$

$$= \frac{\alpha H_v\left(N\right)\left[2\pi_N - 2\left(r_i + r_j\right)\right]}{\pi_N\left[\pi_N + 2\alpha\right]}$$

But $r_i + r_j = \pi_N$. Therefore,

$$\sum_{i=1}^{2} d_i\left(\alpha\right) = \frac{\alpha H_v\left(N\right)}{\pi_N\left[\pi_N + 2\alpha\right]}\left(0\right) = 0$$

For $n > 2$,

$$\sum_{i=1}^{n} d_i\left(\alpha\right) = \sum_{i=1}^{n}\sum_{i\in\theta}\frac{\alpha\left[\pi_\theta - |\theta|r_i\right]}{\pi_\theta\left[\pi_\theta+\alpha|\theta|\right]}H_v\left(\theta\right)$$

To prove this, we consider doing it term by term by showing that for each coalition in the entire sum,

$$\sum_{i=1}^{n} \frac{\alpha\left[\pi_\theta - |\theta| r_i\right]}{\pi_\theta\left[\pi_\theta + \alpha|\theta|\right]} H_v\left(\theta\right) = 0$$

For any coalition with size $|\theta| < n$, denote $n - |\theta| = m$ and assume that the players $j \in N \setminus \theta$ are imaginary players, then assign zero rank to each of the imaginary players. So, for each coalition participating in the entire sum, there is

$$\frac{\alpha H_v\left(N\right)}{\pi_\theta\left[\pi_\theta + \alpha|\theta|\right]} \sum_{i=1}^{n} \left(\pi_\theta - |\theta| r_i\right)$$

$$= \frac{\alpha H_v\left(N\right)}{\pi_\theta\left[\pi_\theta + \alpha|\theta|\right]} \left(n\pi_\theta - \left[|\theta| + m\right]\left[r_1 + r_2 + ... + r_{|\theta|} + r_j + ... + r_m\right]\right)$$

where $r_j = ... = r_m = 0$ (for imaginary players) and $r_1 + r_2 + ... + r_{|\theta|} = \pi_\theta$. Therefore, we have

$$\frac{\alpha H_v\left(N\right)}{\pi_\theta\left[\pi_\theta + \alpha|\theta|\right]} \left(n\pi_\theta - \left[|\theta| + m\right]\pi_\theta\right) = 0 \tag{2.7}$$

Since equation (2.7) holds for each of the coalitions in the entire sum, we conclude that $\sum_{i=1}^{n} d_i\left(\alpha\right) = 0$.

It is now clear that $d_i\left(\alpha\right)$ can only be zero if $\alpha = 0$ and $\sum_{i=1}^{n} d_i\left(\alpha\right) = 0$ for any $\alpha > 0$ (see theorem 1). This implies that the reduction in the payoff of say player i as induced by a positive integer,α, is equal to the increment in the payoff of player i compliment.

As a by-product to the above theorem, one can determine a positive integer $\alpha > 0$ that can ensure a desired change in the payoff of players in a cooperative game. This can be achieved by generating the difference equation

$$d_i\left(\alpha\right) = \lambda_i \tag{2.8}$$

for $i = 1, 2, ..., n$, and solving the system of equations simultaneously for α.

Recall that $\sum_{i=1}^{n} d_i\left(\alpha\right) = 0$. Therefore, $\lambda_i \in \mathbb{R}$ is to be chosen such that $\sum_{i=1}^{n} \lambda_i = 0$. Also by a rule of thumb, λ_i for any player i whose rank is 1 must to be positive while other ones can either be positive or negative. □

3 Results

Theorem 1 establishes the main result of this work which specifies that the differences in Rank-Shapley value as a result of equal increase in ranks by a positive integer α, sum to zero. This implies that the reduction in payoff of say

Table 1: A 3 - Person Cooperative game

θ	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(\theta)$	5	7	3	14	10	12	20
π_θ	2	3	1	5	3	4	6
$H_v(\theta)$	5	7	3	2	2	2	-1

player i is equal to the increment in the payoff of player i compliment induced by α. With a given game example, we demonstrate how a change in rank by a positive integer affect the payoff of players. This result is analogous to determining an integer $\alpha > 0$ that can ensure a desired change in Rank-Shapley values.

Proof. Consider a game involving three players given below (Barron, 1998, P. 229)

The Rank-Shapley value, $\varphi_i(N, v) = (6.8, 9.2, 4)$. Now, we want to determine $\alpha > 0$ that can ensure the desired change in the payoff of the three players. Let the desired change in the payoff of player i be denoted by λ_i. Using equation (2.8), we generate the following difference equations:

$$\frac{2\alpha}{25+10\alpha} - \frac{2\alpha}{9+6\alpha} = \lambda_1 \tag{3.1}$$

$$-\frac{2\alpha}{25+10\alpha} - \frac{4\alpha}{16+8\alpha} + \frac{3\alpha}{36+18\alpha} = \lambda_2 \tag{3.2}$$

$$\frac{2\alpha}{9+6\alpha} + \frac{4\alpha}{16+8\alpha} - \frac{3\alpha}{36+18\alpha} = \lambda_3 \tag{3.3}$$

We choose arbitrary values for λ_1, λ_2 and $\lambda_3$such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

Let $\lambda_1 = -\frac{32}{315}$, $\lambda_2 = -\frac{23}{90}$ and $\lambda_3 = \frac{5}{14}$. Substitute for $(\lambda_i)_{i=1,2,3}$ into equation (3.1), equation (3.2) and equation (3.3), respectively, and solve for α. Hence, $\alpha = 2$.

This is the value of α that can ensure desired change in the payoff of players in the game given above. Specifically, if the players' ranks are each increased by $\alpha = 2$, the Rank-Shapley value will change to

$$\varphi_i(N, v, r_{+2\,\text{or}\,1}) = \left(6.8 - \frac{32}{315}, 9.2 - \frac{23}{90}, 4 + \frac{5}{14}\right)$$

This result validates the by-product of theorem 1 above. □

4 Conclusion

An interesting aspect of the Rank-Shapley value is the fact that it observes the monotonicity of a value function. The study of the monotonic property of Rank-Shapley value presents a basis to support its wide applicability in cooperative game theory. Thus, the sensitivity of changes in payoff as a result of changes in the sharing weight of players is one that needs to be critically examined. This is made possible due to the connotation of rank in Rank-Shapley value as presented in section (2.2). Unlike the Shapley value (Shapley, 1953) and proportional Shapley value (Beal et al., 2018) that make use of unit weight and stand-alone value respectively as sharing weight, the Rank-Shapley value makes use of rank as its sharing weight. Rank in this sense is a linear (positive) integer function of the stand-alone value. As a weight system, it can be assigned in error and the corresponding effect of such error has been studied closely in this work. Here, we studied monotonicity and its relationship with solutions in cooperative games. One of the results of this work is that even though the players' sharing weight (ranks) are altered, the Rank-Shapley value still remains Pareto-optimal (efficient). Theorem 1 specifically, shows that the differences in value as a result of equal increase in ranks by a positive integer α, sum to zero. This theorem gives a framework that can be used to determine the positive integer, $\alpha > 0$ that can ensure an apriori desired change in the payoff of players in a cooperative game. Finally, this work is not exhaustive as it does not capture every form of possible alteration in rank that can apply to the value. Therefore, other forms of alteration in rank and their corresponding effects on Rank-Shapley value are open for further studies.

5 Conflict of interest

There is no conflict of interest among the authors of this paper

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