**Completion Problem of Weakly Sign Symmetric PO-Matrices for Digraphs of Order 5 with 4 or 5 Arcs and a Positionally Symmetric Cycle**

**ABSTRACT**  
A square matrix is a Wss Po-matrix if the off-diagonal elements have the property that if the element at position (i, j) is not zero (i ≠ j), then the element at position (j, i) must either have the same sign or be zero. A Wss Po -matrix is considered to have positionally symmetric cycle if its entries are symmetric with respect to their positions in the matrix. A digraph D has a Wss Po-matrix completion if every partial weakly sign symmetric Po -matrix that describes D can be extended to a complete weakly sign symmetric Po -matrix. This work extends our earlier investigation into the completion of weakly sign-symmetric Po-matrices, focusing on digraphs of order 5 with up to 5 arcs. In that study, we proved that every acyclic or cyclic digraph of order 5 with up to 5 directed edges can be completed into Wss Po-matrix. Our research therefore, advances our previous findings by examining digraphs that contain positionally symmetric cycles. Our goal is to further identify and characterize the structural properties of such digraphs that leads to completion or non-completion. Our study established that directed graphs with 5 vertices and either 4 or 5 directed edge which contain a positionally symmetric cycle do not have completion into a Wss Po-matrix. Moreover, we observed that digraphs with 5 arcs and positionally symmetric cycles inherit the non-completion property from the corresponding 4-arc digraphs with the same cycle structure. These findings could be applied in practical problems such as studying relationships in networks, filling missing data, and solving optimization tasks.

**Keywords:** cyclicdigraphs; acyclic digraphs; digraphs with positionally symmetric cycle; Matrix completion; weakly sign symmetric Po-matrix.

**1. INTRODUCTION**

A Po-matrix A is considered a Wss Po -matrix if aij aji ≥0 for all indices i and j [1]. A partially filled matrix is considered partially Wss Po -matrix if the principal minor of every completely defined principal sub-matrices are non-negative and aij aji≥0 for all completely defined entries [2,3]. A cycle is a path that starts and ends at the same vertex. A directed graph that contains at least one directed cycle is considered to be cyclic; if it has no cycles, it is acyclic [4,5]. A symmetric pattern is one where (i, j) is included if and only if (j, i) is also included. A positionally symmetric pattern for a square matrix includes all diagonal elements that can be represented by a graph G= (V, E) [2]. A completion of a partial matrix refers to selecting specific values for the missing entries in such a way that the resulting matrix meets a particular desired property or type. Filling in the missing entries of a partial matrix by assigning zeros to all unspecified positions is known as zero completion [2,3]. A partially filled matrix defines a pattern if its known entries correspond precisely to those positions outlined in the pattern [3]. A pattern has Wss Po -matrix completion if every partial weakly sign symmetric Po- matrix that specifies the pattern can completed to a Wss Po -matrix [6].

**2. PRELIMINARIES**

The section below defines fundamental concepts from linear algebra, group theory, and graph theory that are frequently applied in Wss Po -matrix completion problems.

**Definition: 2.1** Graphs and digraphs are used in matrix completion for different matrix types. A graph G= (VG, EG) consists of a non-empty, countable group of positive integers as vertices VG, and edges EG are unordered pairs of these vertices. A null graph has no edges [7,8].

**Definition: 2.2** Matrix completion uses patterns, like symmetric pairs and diagonal focus in n × n sub-matrices, to identify possible entries. In Po- matrices. Symmetric properties in Wss Po-matrix aids this process by ensuring specified entries correspond to the outlined patterns [2].

**Definition: 2.3** A pattern D is permutation similar to pattern B if a permutation ϕ exists that maps each pair (i, j) in D to ϕ(i), ϕ(j) to form B [6].

**Lemma: 2.4 Weakly sign symmetric Po-matrices exhibit closure under similarity transformations by permutations.**

A Po- matrix is weakly sign symmetric if permutation matrix P exists such that PAPT becomes sign symmetric. This property supports digraph relabeling due to closure under permutation similarity [4,5].

**Theorem 2.5** A permutation matrix P is obtained by interchanging rows of the identity matrix, and PAPT reflects vertex relabeling on the digraph[9,10].

**2. Mathematical breakdown**

**Consider the digraphs shown below.**

To construct a partial matrix from a directed graph, each arc from one vertex to another corresponds to a specified entry aij ​. If there is no arc between two vertices, the corresponding entry xij is unspecified. We then compute the principal minors and perform zero completion by setting all unspecified entries xij to zero. This process helps us determine whether the matrix can be completed into Wss Po-matrix.

**Case 1**

The matrix that partially outlines this digraph is

As defined for partial Wss Po-matrix d1 ≥ 0, d2 ≥ 0, d3 ≥ 0, d4 ≥ 0, d5 ≥ 0. we compute the principal minors by applying zero completion. All unspecified entries xij of A are set to 0.

= 0. We obtain principal minors of each sub-matrix.

Det A (1,2) = d11d22 – a12a21≥ 0, (since (1,2) is completely defined). Similarly, det A (1,3) = d11d33 – a13a31≥ 0, (since (1,3) is completely defined).

Det A (1,4) = d11d44 ≥ 0. correspondingly, Det A (1,5), Det A (2,3), Det A (2,4), Det A (2,5), Det A (3,4), Det A (3,5) Det A (4 5) ≥ 0.

Det (1,2,3) = d11 (d22d33 – x23 x32) – a12 (a21 d33 – x23 x31) + a13 (a21 x32 – d22a31). We substitute zero for the unspecified entries i.e. = 0. We obtain principal minors (PM) of each sub-matrix,

Det A (1,2,3) = d3(d11d22 – a12a21)– a13 a31d22

Det A (1,2,4) = d44 (d11d22 – a12a21)≥ 0. (since (1,2) is completely defined).).

Det A (1,2,5) = d55 (d11d22 – a12a21)≥ 0. (since (1,2) is completely defined).

Det A (1,3,4) = d44(d11d33 – a13a31)≥ 0. (since (1,3) is completely defined).

Det A (1,3,5) = d55(d11d33 – a13a31)≥ 0. (since (1,3) is completely defined).

Det A (1,4,5) = d11d44d55 ≥ 0.

Det A (2,3,4) = d22d33d44 ≥ 0.

Det A (2,3,5) = d22d33d55 ≥ 0.

Det A (2,4,5) = d22(d44d55) ≥ 0.

Det A (3,4,5) = d33d33d44 ≥ 0.

Det A (1,2,3,4) = d33d44(d11d22 – a12a21) – a13 a31d22 d44.

Det A (1,2,3,5) = d33d55 (d11d22 – a12a21) ─ a13 a31d22 d55

Det (A) = d11d22d33d44d55 – a12 a21d33d44d55 – a13 a31d22 d44 d55

By definition of completion, all determinants must be ≥ 0. however, Det A (1,2,3,), Det A (1,2,3,4), Det A (1,2,3,5) and Det A (1,2,3,4,5) were found to be negative. Since not all the determinants are non-negative, the partial matrix cannot be completed into a Wss P₀-matrix. Therefore, it does not admit a zero completion into a Wss P₀-matrix.

**Counter example to show non-completion.**

Let the partial matrix specifying the sub digraph (1,2,3) be

M =. After substituting the unspecified entries with zero i.e. = = 0. Then

|M| = – a12 a21 – a13

= 8 – 4 –8 = – 4 < 0. Hence M (1,2,3) has no completion.

**Case 2.**

The matrix that partially outlines this digraph is .

As defined for partial Wss Po-matrix d1 ≥ 0, d2 ≥ 0, d3 ≥ 0, d4 ≥ 0, d5 ≥ 0.

we compute the principal minors by applying zero completion. All unspecified entries xij of A are set to 0. = 0. We obtain principal minors of each sub-matrix,

Det (1,2) = d11d22 – x12x21. Assigning zeros to the unspecified entries, i.e. x12 = 0, x21 = 0. Then det (1,2) = d11d22 ≥ 0. Similarly, A (1,3), A (2,3), A (2,4), A (2,5), A (3,4), A (3,5), A (4,5) ≥ 0.

Det A (1,4) = (d11d44) – (a14a41)≥ 0. Since (1,4) is completely defined.

Det A (1,5) = (d11d55) – (a15a51)≥ 0. Since (1,5) is fully specified.

Det (1,2,3) = d11 (d22d33 – x23 x32) – x12 (x21 d33 – x23 x31) + x13 (x21 x32 – d22x31). We substitute zero for the unspecified entries i.e. = 0. We obtain the principal minors,

Det A (1,2,3) = d11d22d33 ≥ 0.

Det A (1,2,4) = d22(d11d44 – a14a41)≥ 0. Since (1,4) is completely defined.

Det A (1,2,5) = d22(d11d55 – a15a51)≥ 0. Since (1,5) is completely defined.

Det A (1,3,4) = d33(d11d44 – a14a41)≥ 0. Since (1,4) is completely defined.

Det A (1,3,5) = d33(d11d55 – a15a51)≥ 0. Since (1,5) is completely defined.

Det A (1,4,5) = d11d44d55 – a14a41d55 +a14a45 a51 – a15a51d44.

Det A (2,3,4) = d22d33d44 ≥ 0.

Det A (2,3,5) =d22d33d55 ≥ 0.

Det A (2,4,5) = d22d44d55 ≥ 0.

Det A (3,4,5) = d33d44d55 ≥ 0.

Det A (1,2,3,4) = d22d33 (d11d44– a14a41)≥ 0. Since (1,4) is completely defined.

Det A (1,2,3,5) = d22d33 (d11d55 – a15a51)≥ 0. Since (1,5) is completely defined.

Det A (1,2,4,5) = d11d22d44d55 – a14a41 d22d55 +a14a45d22a51 – a15a51d22d44.

Det A (1,3,4,5) = d11d33d44d55 – a14a41d33d55 +a14a45d33a51 – a15a51d33d44.

Det A (2,3,4,5) = d22d33d44d55 ≥ 0.

Det (A) = d11d22d33d44d55 + a14a41 d22 d33d55 – a14a51 a45d22d33 ≥ 0.

By definition of completion, all determinants must be ≥ 0. however, Det A (1,4,5), Det A (1,2,4,5) and Det A (1,3,4,5) were found to be negative. Since not all the determinants are non-negative, the partial matrix cannot be completed into a Wss P₀-matrix. Therefore, it does not admit a zero completion into a Wss P₀-matrix.

**Counter example to show non-completion.**

Let the partial matrix specifying the sub digraph (1,4,5) be

A =. After substituting the unspecified entries with zero i.e. = 0. Then

|A| = – a14 a41 + a14 a45 a51 – a15a51

= 48 – 36 – 12 – 48 = – 48 < 0. Hence A (1,4,5) has no completion.

**Case 3.**

The matrix that partially outlines this digraph is .

As defined for partial Wss Po-matrix d1 ≥ 0, d2 ≥ 0, d3 ≥ 0, d4 ≥ 0, d5 ≥ 0. We compute the principal minors by applying zero completion. All unspecified entries xij of A are set to 0. = 0. We obtain principal minors of each sub-matrix,

Det (1,2) = d11d22 – x12x21. Assigning zeros to the unspecified entries, i.e. x12 = 0, x21 = 0. Then det (1,2) = d11d22 ≥ 0. Similarly, A (1,3), A (1,4), A (1,5), A (2,3), A (2,4), A (3,4), A (4,5) ≥ 0.

Det A (2,5) = (d22d55) – (a25a52)≥ 0. Since (2,5) is completely defined.

Det A (3,5) = (d33d55) – (a35a53)≥ 0. Since (3,5) is fully specified.

Det (1,2,3) = d11 (d22d33 – x23 x32) – x12 (x21 d33 – x23 x31) + x13 (x21 x32 – d22x31).We substitute zero for the unspecified entries i.e. = 0. We obtain,

Det A (1,2,3) = d11d22d33 ≥ 0.

Det A (1,2,4) = d11d22d44 ≥ 0.

Det A (1,2,5) = d11d22d55 ≥ 0.

Det A (1,3,4) = d11d33d44 ≥ 0.

Det A (1,3,5) = d11(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (1,4,5) = d11d44d55 ≥ 0.

Det A (2,3,4) = d22d33d44 ≥ 0.

Det A (2,3,5) = d22(d33d55 – a35a53)– a25d33a52

Det A (2,4,5) = d44(d22d55 – a25a52)≥ 0. Since (2,5) is completely defined.

Det A (3,4,5) = d44(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (1,2,3,4) = d11d22d33d44 ≥ 0.

Det A (1,2,3,5) = d11d22(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (1,2,4,5) = d11d44(d22d55 – a25a52)≥ 0. Since (2,5) is completely defined.

Det A (1,3,4,5) = d11d44(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (2,3,4,5) = d22d44(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (1,2,3,4,5) = d11d22d44(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

By definition of completion, all determinants must be ≥ 0. however, Det A (2,3,5) was found to be negative. Since not all the determinants are non-negative, the partial matrix cannot be completed into a Wss P₀-matrix. Therefore, it was found not to have zero completion into a Wss P₀-matrix.

**Counter example to show non-completion.**

Let the partial matrix specifying the sub digraph (2,3,5) be

A =. After substituting the unspecified entries with zero i.e. and = 0. Then

|A| = – a35 a53 – a25a53

= 10 – 8 – 9 = – 7 < 0. Hence A (2,3,5) has no completion.

**Case 4**

The matrix that partially outlines this digraph is .

As defined for partial Wss Po-matrix d1 ≥ 0, d2 ≥ 0, d3 ≥ 0, d4 ≥ 0, d5 ≥ 0. We compute the principal minors by applying zero completion. All unspecified entries xij of A are set to 0. = 0. We obtain principal minors of each sub-matrix,

Det (1,2) = d11d22 – x12x21. Assigning zeros to the unspecified entries, i.e. x12 = 0, x21 = 0. Then det (1,2) = d11d22 ≥ 0. Similarly, A (1,3), A (1,4), A (1,5), A (2,3), A (2,4), A (2,5), A (4,5) ≥ 0.

Det A (3,5) = (d33d55) – (a35a53)≥ 0. Since (3,5) is completely defined.

Det A (4,5) = (d44d55) – (a4a54)≥ 0. Since (4,5) is completely defined.

Det (1,2,3) = d11 (d22d33 – x23 x32) – x12 (x21 d33 – x23 x31) + x13 (x21 x32 – d22x31).We substitute zero for the unspecified entries i.e. = 0. We obtain,

Det A (1,2,3) = d11d22d33 ≥ 0.

Det A (1,2,4) = d11d22d44 ≥ 0.

Det A (1,3,4) = d11(d33d44 – a34a43)≥ 0. Since (3,4) is completely defined.

Det A (1,3,5) = d11(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (1,3,5) = d11d33d55 ≥ 0.

Det A (1,4,5) = d11d44d55 ≥ 0.

Det A (2,3,4) = d22(d33d44 – a34a43)≥ 0. Since (3,4) is completely defined.

Det A (2,3,5) = d22(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (2,4,5) = d22d44d55 ≥ 0.

Det A (3,4,5) = d33d44d55 – a34a43 d55 + a34a45 a53 – a35a53 d44

Det A (1,2,3,4) = d11d22(d33d44 – a34a43)≥ 0. Since (3,4) is completely defined.

Det A (1,2,3,5) = d11d22(d33d55 – a35a53)≥ 0. Since (3,5) is completely defined.

Det A (1,2,4,5) = d11d22d44d55 ≥ 0.

Det A (1,3,4,5) = d11d33d44 d55 – a34 a53 d33 d55 + a34a45d33 a53 – a35 a53 d33 d44.

Det A (2,3,4,5) = d22d33d44 d55 – a34 a43 d33 d55 + a34a45d33 a53 – a35 a53 d33 d44.

Det A (1,2,3,4,5) = d11d22d33d44 d55 – d11a34 a43 d22 d55 + d11a34a45d22 a53 – d11a35 a53 d22 d44.

By definition of completion, all determinants must be ≥ 0. however, Det A (3,4,5), Det A (1,3,4,5), Det A (2,3,4,5), Det A (1,2,3,4,5) was found to be negative. Since not all the determinants are non-negative, the partial matrix cannot be completed into a Wss P₀-matrix. Therefore, it was found not to admit zero completion into a Wss P₀-matrix.

**Counter example to show non-completion.**

Let the partial matrix specifying the sub digraph (3,4,5) be

A =. After substituting the unspecified entries with zero i.e. = 0. Then

|A| = d33d44d55 – a34a43d55 + a34a45 a53 – a35a53 d44

= 30 – 24 – 3– 10 = – 7 < 0. Hence A (3,4,5) has no completion.

**3. CONCLUSION AND RECCOMENDATION.**

Hence, we concluded that all Directed graphs with 5 vertices and either 4 or 5 directed edges which possess positionally symmetric cycle were discovered to have no completion. Future research could be done on other digraphs characteristics which lead to non-completion.

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**1.**

**2.**

**3.**

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