**COMPLETION OF WEAKLY SIGN SYMMETRIC PO-MATRIX PROBLEM FOR 5 × 5 MATRICES SPECIFYING DIGRAPHS OF ORDER 5 WITH 4, 5 ARCS WITH POSITIONALLY SYMMETRIC CYCLE.**

**ABSTRACT**
An n × n matrix is a weakly sign symmetric matrix if the off-diagonal elements have the property that if the entry in row i and column j is non-zero, then the entry in row j and column i must have same sign or zero. A digraph D has a Wss Po -matrix completion if every partial weakly sign symmetric Po -matrix that describes D can be extended to a complete weakly sign symmetric Po -matrix. This work extends our earlier investigation into the completion of weakly sign-symmetric Po-matrices, focusing on digraphs of order 5 with up to 5 arcs. In that study, we proved that every acyclic or cyclic digraph of order 5 with up to 5 arcs have completion into weakly sign symmetric Po-matrix. Our research therefore, advances our previous findings by examining digraphs that contain positionally symmetric cycles. Our goal is to further identify and characterize the structural properties of such digraphs that leads to completion or non-completion. Our study established that digraphs of order 5 with 4 or 5 arcs that contain a positionally symmetric cycle do not have completion into a Wss Po-matrix. Moreover, we observed that digraphs with 5 arcs and positionally symmetric cycles inherit the non-completion property from the corresponding 4-arc digraphs with the same cycle structure. These findings can be useful in practical problems such as studying relationships in networks, filling missing data, and solving optimization tasks.

**Keywords:** cyclicdigraphs; acyclic digraphs; digraphs with positionally symmetric cycle; Matrix completion; weakly sign symmetric Po-matrix.

**1. INTRODUCTION**

A Po-matrix A is classified as a weakly sign symmetric Po- matrix if aij aji ≥ 0 for all i and j [1]. A partial matrix is considered partially weakly sign symmetric Po- matrix if the determinant of all fully specified principal sub-matrices are non-negative and aij aji≥0 for all specified entries [2,3]. A cycle is a closed path in a digraph. A digraph with at least one directed cycle is cyclic; if it has no cycles, it is acyclic [4,5]. A symmetric pattern is one where (i, j) is included if and only if (j, i) is also included. A positionally symmetric pattern for n × n matrices includes all diagonal positions that can be represented by a graph G= (V, E) [2]. A completion of a partial matrix is a specific choice of values for the unspecified entries so as that desired matrix has desired type. Completion of a partial matrix is called zero completion if all unspecified entries in the partial matrix are equated to zeros [2,3]. A partial matrix defines a pattern if its specified entries correspond precisely to those positions outlined in the pattern [3]. A pattern has weakly sign symmetric Po- matrix completion if every partial weakly sign symmetric Po- matrix that specifies the pattern can completed to a weakly sign symmetric Po- matrix [6].

**2. PRELIMINARIES**

Basic concepts in linear algebra, group theory and graph theory that are commonly used in Wss Po- matrix completion problems are defined in the section below:

**Definition: 2.1** Graphs and digraphs are used in matrix completion for different matrix types. A graph G= (VG, EG) consists of a finite non-empty set of positive integers as vertices VG, and edges EG are unordered pairs of these vertices. A null graph has no edges [7,8].

**Definition: 2.2** Matrix completion uses patterns, like symmetric pairs and diagonal focus in n × n sub-matrices, to identify possible entries. In Po- matrices. Symmetric properties in Wss Po-matrix aids this process by ensuring specified entries correspond to the outlined patterns [2].

**Definition: 2.3** A pattern D is permutation similar to pattern B if a permutation ϕ exists that maps each pair (i, j) in D to ϕ(i), ϕ(j) to form B [6].

**Lemma: 2.4 Weakly sign symmetric Po-matrices exhibit closure under similarity transformations by permutations.**

A Po- matrix is weakly sign symmetric if permutation matrix P exists such that PAPT becomes sign symmetric. This property supports digraph relabeling due to closure under permutation similarity [4,5].

**Theorem 2.5** A permutation matrix P is obtained by interchanging rows of the identity matrix, and PAPT reflects vertex relabeling on the digraph[9].

**2. Mathematical analysis**

**Consider the digraphs shown below.**

To obtain a partial matrix from the digraphs an arc pointing from one vertex to another, represents a specified entry aij, while for an unspecified entry there is no arc pointing from one vertex to the other and is denoted by xij. We work out the principal minors and apply zero completion by assigning all the unspecified entries xij s to zero and determine if it can be completed to weakly sign symmetric po-matrix.

**Case 1**

The partial matrix that specifies the digraph above is $A=\left(\begin{matrix}d\_{11}&a\_{12}&a\_{13}&x\_{14}&x\_{15}\\a\_{21}&d\_{22}&x\_{23}&x\_{24}&x\_{25}\\a\_{31}&x\_{32}&d\_{33}&x\_{34}&x\_{35}\\x\_{41}&x\_{42}&x\_{43}&d\_{44}&x\_{45}\\x\_{51}&x\_{52}&x\_{53}&x\_{54}&d\_{55}\end{matrix}\right). $

By definition of partial W**ss** Po-matrix d1 ≥ 0, d2 ≥ 0, d3 ≥ 0, d4 ≥ 0, d5 ≥ 0. we compute the principal minors by applying zero completion. All unspecified entries xij of A are set to 0.

$X\_{12}=X\_{13}= X\_{14}=X\_{15}=X\_{21}=X\_{23}=X\_{24}=X\_{25}=X\_{31}=X\_{32}=X\_{34}=X\_{35}=X\_{41}=X\_{42}=X\_{43}=X\_{51}=X\_{52}=X\_{53}=X\_{54} $= 0. We obtain,

Det A (1,2) = d11d22 – a12a21≥ 0, (since (1,2) is fully specified). Similarly, det A (1,3) = d11d33 – a13a31≥ 0, (since (1,3) is fully specified).

Det A (1,4) = d11d44 ≥ 0. Similarly, Det A (1,5), Det A (2,3), Det A (2,4), Det A (2,5), Det A (3,4), Det A (3,5) Det A (4 5) ≥ 0.

Det (1,2,3) = d11 (d22d33 – x23 x32) – a12 (a21 d33 – x23 x31) + a13 (a21 x32 – d22a31). We substitute zero for the unspecified entries i.e. $X\_{12}=X\_{13}= X\_{14}=X\_{15}=X\_{21}=X\_{23}=X\_{24}=X\_{25}=X\_{31}=X\_{32}=X\_{34}=X\_{35}=X\_{41}=X\_{42}=X\_{43}=X\_{51}=X\_{52}=X\_{53}=X\_{54} $= 0. We obtain,

Det A (1,2,3) = d3(d11d22 – a12a21)– a13 a31d22

Det A (1,2,4) = d44 (d11d22 – a12a21)≥ 0. (since (1,2) is fully specified).

Det A (1,2,5) = d55 (d11d22 – a12a21)≥ 0. (since (1,2) is fully specified).

Det A (1,3,4) = d44(d11d33 – a13a31)≥ 0. (since (1,3) is fully specified).

Det A (1,3,5) = d55(d11d33 – a13a31)≥ 0. (since (1,3) is fully specified).

Det A (1,4,5) = d11d44d55 ≥ 0.

Det A (2,3,4) = d22d33d44 ≥ 0.

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Det A (2,4,5) = d22(d44d55) ≥ 0.

Det A (3,4,5) = d33d33d44 ≥ 0.

Det A (1,2,3,4) = d33d44(d11d22 – a12a21) – a13 a31d22 d44.

Det A (1,2,3,5) = d33d55 (d11d22 – a12a21) ─ a13 a31d22 d55

Det (A) = d11d22d33d44d55 – a12 a21d33d44d55 – a13 a31d22 d44 d55

By definition of completion, all determinants must be ≥ 0. however, Det A (1,2,3,), Det A (1,2,3,4), Det A (1,2,3,5) and Det A (1,2,3,4,5) were found to be negative. Since all the determinants are not non-negative then the partial matrix cannot be completed to Wss Po-matrix. Hence it has no zero completion into a Wss Po-matrix.

**Counter example to show non-completion.**

Let the partial matrix specifying the sub digraph (1,2,3) be

M =$ \left[\begin{matrix}2&-2&-2\\-1&2&x\_{23}\\-2&x\_{32}&2\end{matrix}\right]$. After substituting the unspecified entries with zero i.e. $x\_{23}$ =$ x\_{32}$ = 0. Then

|M| = $d\_{11}d\_{22}d\_{33}$ – a12 a21$d\_{33}$ – a13$d\_{22}d\_{33}$

 = 8 – 4 –8 = – 4 < 0. Hence M (1,2,3) has no completion.

**Case2.**

The partial matrix that specifies this digraph is $A= \left(\begin{matrix}d\_{11}&x\_{12}&x\_{13}&a\_{14}&a\_{15}\\x\_{21}&d\_{22}&x\_{23}&x\_{24}&x\_{25}\\x\_{31}&x\_{32}&d\_{33}&x\_{34}&x\_{35}\\a\_{41}&x\_{42}&x\_{43}&d\_{44}&a\_{45}\\a\_{51}&x\_{52}&x\_{53}&x\_{54}&d\_{55}\end{matrix}\right)$.

By definition of partial W**ss** Po-matrix d1 ≥ 0, d2 ≥ 0, d3 ≥ 0, d4 ≥ 0, d5 ≥ 0.

we compute the principal minors by applying zero completion. All unspecified entries xij of A are set to 0. $X\_{12}=X\_{13}= X\_{14}=X\_{15}=X\_{21}=X\_{23}=X\_{24}=X\_{25}=X\_{31}=X\_{32}=X\_{34}=X\_{35}=X\_{41}=X\_{42}=X\_{43}=X\_{51}=X\_{52}=X\_{53}=X\_{54} $= 0. We obtain,

Det (1,2) = d11d22 – x12x21. Setting the unspecified entries to zero, i.e. x12 = 0, x21 = 0. Then det (1,2) = d11d22 ≥ 0. Similarly, A (1,3), A (2,3), A (2,4), A (2,5), A (3,4), A (3,5), A (4,5) ≥ 0.

Det A (1,4) = (d11d44) – (a14a41)≥ 0. Since (1,4) is fully specified.

Det A (1,5) = (d11d55) – (a15a51)≥ 0. Since (1,5) is fully specified.

Det (1,2,3) = d11 (d22d33 – x23 x32) – x12 (x21 d33 – x23 x31) + x13 (x21 x32 – d22x31). We substitute zero for the unspecified entries i.e. $X\_{12}=X\_{13}= X\_{14}=X\_{15}=X\_{21}=X\_{23}=X\_{24}=X\_{25}=X\_{31}=X\_{32}=X\_{34}=X\_{35}=X\_{41}=X\_{42}=X\_{43}=X\_{51}=X\_{52}=X\_{53}=X\_{54} $= 0. We obtain,

Det A (1,2,3) = d11d22d33 ≥ 0.

Det A (1,2,4) = d22(d11d44 – a14a41)≥ 0. Since (1,4) is fully specified.

Det A (1,2,5) = d22(d11d55 – a15a51)≥ 0. Since (1,5) is fully specified.

Det A (1,3,4) = d33(d11d44 – a14a41)≥ 0. Since (1,4) is fully specified.

Det A (1,3,5) = d33(d11d55 – a15a51)≥ 0. Since (1,5) is fully specified.

Det A (1,4,5) = d11d44d55 – a14a41d55 +a14a45 a51 – a15a51d44.

Det A (2,3,4) = d22d33d44 ≥ 0.

Det A (2,3,5) =d22d33d55 ≥ 0.

Det A (2,4,5) = d22d44d55 ≥ 0.

Det A (3,4,5) = d33d44d55 ≥ 0.

Det A (1,2,3,4) = d22d33 (d11d44– a14a41)≥ 0. Since (1,4) is fully specified.

Det A (1,2,3,5) = d22d33 (d11d55 – a15a51)≥ 0. Since (1,5) is fully specified.

Det A (1,2,4,5) = d11d22d44d55 – a14a41 d22d55 +a14a45d22a51 – a15a51d22d44.

Det A (1,3,4,5) = d11d33d44d55 – a14a41d33d55 +a14a45d33a51 – a15a51d33d44.

Det A (2,3,4,5) = d22d33d44d55 ≥ 0.

Det (A) = d11d22d33d44d55 + a14a41 d22 d33d55 – a14a51 a45d22d33 ≥ 0.

By definition of completion, all determinants must be ≥ 0. however, Det A (1,4,5), Det A (1,2,4,5) and Det A (1,3,4,5) were found to be negative. Since all the determinants are not non-negative then the partial matrix cannot be completed to W**ss** Po-matrix. Hence it has no zero completion into a W**ss** Po-matrix.

**Counter example to show non-completion.**

Let the partial matrix specifying the sub digraph (1,4,5) be

A =$ \left[\begin{matrix}2&-3&3\\-2&4&1\\4&x\_{54}&6\end{matrix}\right]$. After substituting the unspecified entries with zero i.e. $x\_{54}$ = 0. Then

|A| = $d\_{11}d\_{44}d\_{55}$ – a14 a41$d\_{55}$ + a14 a45 a51 – a15$d\_{44}$a51

 = 48 – 36 – 12 – 48 = – 48 < 0. Hence A (1,4,5) has no completion.

**3. CONCLUSION AND RECCOMENDATION.**

Hence, we concluded that all digraphs of order 5 with 4 and 5 arcs which possess positionally symmetric cycle were discovered to have no completion. Future research could be done on other digraphs characteristics which lead to non-completion.

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