INDICES OF LINE GRAPH OF $\mathcal{R}(\mathcal{G})$

ABSTRACT. The mathematical field of graph theory aids in the study of the chemical, physical, and biological characteristics of chemical substances. The properties and reactivity of chemical compounds can be predicted by examining the graph of chemical structure and its topological indices. Despite the fact that over 140 topological indices have been defined thus far, this number is insufficient to fully characterize all of the features of chemical substances. As a result, new topological index variants are still being established today. In this study, we compute various topological indices of line graphs of triangular graphs of specific families of graphs.

Keywords: Line Graph,Triangle graph, Sombor Index, topological indices

Introduction: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with node set $\mathcal{V}$ and arc set $\mathcal{E}$. The number of nodes in $\mathcal{G}$ is denoted by $|\mathcal{V}| = \eta$ and the number of arcs in $\mathcal{G}$ is denoted by $|\mathcal{E}| = \zeta$. The degree of the node $x \in \mathcal{V}$ is denoted by κ_x, is the number of arcs incident on x.The arc connecting the nodes x and y is denoted by $e = xy$, where $e \in \mathcal{E}$.

The Triangle graph $\mathcal{R}(\mathcal{G})$ is obtained from $\mathcal{G}$ by adding one node corresponding to each arc and is joined to the end nodes of the corresponding arc.
Line graph $\mathcal{L}(\mathcal{G})$ is obtained from $\mathcal{G}$ by taking the arcs of $\mathcal{G}$ as the nodes and is joined by an arc in $\mathcal{L}(\mathcal{G})$ if they have common node in $\mathcal{G}$.

The numerical value that results from a mathematical analysis of a graph is called a topological index.The properties and activities of chemical substances can be predicted with the use of this mathematical research. Finding the topological indices is a crucial field of study. There are three categories for topological indices: degree-based, distance-based, combination of degree-and distance-based, and counting-related topological indices. Degree-based topological indices are among these indicators that are crucial for predicting the characteristics and actions of chemical substances. Known topological indices include the oldest, Zagreb indices, the most recent Gourava indices, Randic´ index, Atom bond index, Harmonic index, Sum connectivity index, and Sombor index.

In this paper, we introduce two variations of Topological indices and calculate certain topological indices of line graph of triangle graph of a regular graph, Path $\mathcal{P}_\eta$, Cycle $\mathcal{C}_\eta$, Complete $\mathcal{K}_\eta$ and Ladder graph $\mathcal{L}_\eta$

Name	Topological Indices
Atom bond Connectivity Index [2]	$\mathcal{ABC}(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \sqrt{\frac{\kappa_x+\kappa_y-2}{\kappa_x\kappa_y}}$
Randic Index[3]	$\mathcal{RA}(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \sqrt{\frac{1}{\kappa_x\kappa_y}}$
sum-Connectivity Index [4]	$\chi(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \sqrt{\frac{1}{\kappa_x+\kappa_y}}$
Geometric Arithmetic Index[5]	$\mathcal{GA}(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \frac{2\sqrt{\kappa_x\kappa_y}}{\kappa_x+\kappa_y}$
Sombor Index [1]	$\mathcal{SO}(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \sqrt{{\kappa_x}^2+{\kappa_y}^2}$
First Zagreb Index [6]	$\mathcal{M}_1(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \kappa_x+\kappa_y$
Second Zagreb Index [6]	$\mathcal{M}_2(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \kappa_x\kappa_y$
First Gourava index [7]	$\mathcal{GO}_1(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \left[(\kappa_x+\kappa_y)+\kappa_x\kappa_y\right]$
second Gourava index [7]	$\mathcal{GO}_2(\mathcal{G}) = \sum_{xy\in\mathcal{E}} \left[(\kappa_x+\kappa_y)\,\kappa_x\kappa_y\right]$

1. Topological indices of Line graph of Triangle graph of a regular graph

Theorem 1.1. *Let $\mathcal{G}$ be a κ - regular graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ containing η and ζ elements respectively. Then the topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{G}))$ is given by*

(1) $\mathcal{ABC}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) = \kappa\zeta\sqrt{\frac{2(3\kappa-2)}{\kappa(2\kappa-1)}}+\frac{\zeta\sqrt{2(2\kappa-1)}}{2}+\frac{(2\kappa^2\eta-\zeta(3\kappa+1))\sqrt{2}\sqrt{4\kappa-3}}{2(2\kappa-1)}$

(2) $\mathcal{RA}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) = \frac{\kappa\zeta}{\sqrt{\kappa(2\kappa-1)}}+\frac{\zeta}{2}+\frac{2\kappa^2\eta-\zeta(3\kappa+1)}{2(2\kappa-1)}$

(3) $\chi(\mathcal{L}(\mathcal{R}(\mathcal{G}))) = \frac{\sqrt{2}\kappa\zeta}{\sqrt{3\kappa-1}}+\frac{\zeta\sqrt{\kappa}}{2}+\frac{2\kappa^2\eta-\zeta(3\kappa+1)}{2\sqrt{2\kappa-1}}$

(4) $\mathcal{GA}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) = \frac{4\kappa\zeta\sqrt{\kappa(2\kappa-1)}}{3\kappa-1}+2\kappa^2\eta-2\kappa\zeta-\zeta$

(5) $\mathcal{SO}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) = 4\kappa\zeta\sqrt{5\kappa^2-4\kappa+1}+2\sqrt{2}\left(\kappa^2\zeta+(2\kappa-1)(2\kappa^2\eta-\zeta(3\kappa+1))\right)$

(6) $\mathcal{M}_1(\mathcal{L}(\mathcal{R}(\mathcal{G})) = 4\kappa\zeta\,(4\kappa-1)+4\,(2\kappa-1)\,(2\kappa^2\eta-\zeta(3\kappa+1))$

(7) $\mathcal{M}_2(\mathcal{L}(\mathcal{R}(\mathcal{G}))) = 8\kappa^2\zeta\,(2\kappa-1)+4\kappa^3\zeta+4\,(2\kappa^2\eta-\zeta(3\kappa+1))\,(2\kappa-1)^2$

(8) $\mathcal{GO}_1(\mathcal{G}) = 4\kappa\left[4\kappa\,(\kappa\eta-\zeta)\,(2\kappa-1)+\zeta\,(\kappa^2+1)\right]$

(9) $\mathcal{GO}_2(\mathcal{L}(\mathcal{R}(\mathcal{G}))) = 16\kappa^2\zeta\,(2\kappa-1)\,(3\kappa-1)+16\kappa^4\zeta+8\,(2\kappa-1)\left[2\kappa^2\eta-\zeta(3\kappa+1)\right]$

Proof. Let $\mathcal{G}$ be a κ - regular graph with η vertices and ζ edges. Then the $\mathcal{R}(\mathcal{G})$ has $\eta+\zeta$ vertices and 3ζ edges with η vertices of degree 2κ and ζ vertices of degree 2. So the graph $\mathcal{L}(\mathcal{R}(\mathcal{G}))$ has 3ζ vertices and $2\kappa^2\eta-\zeta$ edges with 2ζ vertices of degree 2κ and ζ vertices of degree $4\kappa-2$.

The edge partition of $\mathcal{L}(\mathcal{R}(\mathcal{G}))$

Degree of end vertices	No. of edges
$(2\kappa, 2\kappa)$	$\kappa\zeta$
$(4\kappa-2, 2\kappa)$	$2\kappa\zeta$
$(4\kappa-2, 4\kappa-2)$	$2\kappa^2\eta-\zeta(3\kappa+1)$

$$\begin{aligned}
\mathcal{ABC}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}}\sqrt{\frac{\kappa_x+\kappa_y-2}{\kappa_x\kappa_y}}\\
&= 2\kappa\zeta\sqrt{\frac{4\kappa-2+2\kappa-2}{(2\kappa)(4\kappa-2)}}+\kappa\zeta\sqrt{\frac{2\kappa+2\kappa-2}{(2\kappa)(2\kappa)}}+\\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)\sqrt{\frac{4\kappa-2+4\kappa-2-2}{(4\kappa-2)(4\kappa-2)}}\\
&= \kappa\zeta\sqrt{\frac{2(3\kappa-2)}{\kappa(2\kappa-1)}}+\frac{\zeta\sqrt{2(2\kappa-1)}}{2}+\frac{(2\kappa^2\eta-\zeta(3\kappa+1))\sqrt{2}\sqrt{4\kappa-3}}{2(2\kappa-1)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{RA}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}}\sqrt{\frac{1}{\kappa_x\kappa_y}}\\
&= 2\kappa\zeta\sqrt{\frac{1}{(2\kappa)(4\kappa-2)}}+\kappa\zeta\sqrt{\frac{1}{(2\kappa)(2\kappa)}}+\\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)\sqrt{\frac{1}{(4\kappa-2)(4\kappa-2)}}\\
&= \frac{\kappa\zeta}{\sqrt{\kappa(2\kappa-1)}}+\frac{\zeta}{2}+\frac{2\kappa^2\eta-\zeta(3\kappa+1)}{2(2\kappa-1)}
\end{aligned}$$

$$\begin{aligned}
\chi(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}}\sqrt{\frac{1}{\kappa_x+\kappa_y}}\\
&= 2\kappa\zeta\sqrt{\frac{1}{(2\kappa)+(4\kappa-2)}}+\kappa\zeta\sqrt{\frac{1}{(2\kappa)+(2\kappa)}}+\\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)\sqrt{\frac{1}{(4\kappa-2)+(4\kappa-2)}}\\
&= \frac{\sqrt{2}\kappa\zeta}{\sqrt{3\kappa-1}}+\frac{\zeta\sqrt{\kappa}}{2}+\frac{2\kappa^2\eta-\zeta(3\kappa+1)}{2\sqrt{2\kappa-1}}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{GA}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}} 2\frac{\sqrt{\kappa_x\kappa_y}}{\kappa_x+\kappa_y} \\
&= (2\kappa\zeta)\, 2\frac{\sqrt{2\kappa\,(4\kappa-2)}}{(2\kappa)+(4\kappa-2)} + 2\kappa\zeta\frac{\sqrt{(2\kappa)\,(2\kappa)}}{(2\kappa)+(2\kappa)} + \\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right) 2\sqrt{\frac{(4\kappa-2)\,(4\kappa-2)}{(4\kappa-2)+(4\kappa-2)}} \\
&= \frac{4\kappa\zeta\sqrt{\kappa\,(2\kappa-1)}}{3\kappa-1} + 2\kappa^2\eta - 2\kappa\zeta - \zeta \\
\mathcal{SO}(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}} \sqrt{\kappa_x^2+\kappa_y^2} \\
&= 2\kappa\zeta\sqrt{(2\kappa)^2+(4\kappa-2)^2} + \kappa\zeta\sqrt{(2\kappa)^2+(2\kappa)^2} \\
&\qquad + \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)\sqrt{(4\kappa-2)^2+(4\kappa-2)^2} \\
&= 4\kappa\zeta\sqrt{5\kappa^2-4\kappa+1} + 2\sqrt{2}\left(\kappa^2\zeta+(2\kappa-1)(2\kappa^2\eta-\zeta(3\kappa+1))\right) \\
\mathcal{M}_1(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}} \kappa_x+\kappa_y \\
&= 2\kappa\zeta\,(4\kappa-2+2\kappa) + \kappa\zeta\,(2\kappa+2\kappa) + \\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)(4\kappa-2+4\kappa-2) \\
&= 4\kappa\zeta\,(4\kappa-1) + 4\,(2\kappa-1)\left(2\kappa^2\eta-\zeta(3\kappa+1)\right) \\
\mathcal{M}_2(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}} \kappa_x\kappa_y \\
&= 2\kappa\zeta\,[(4\kappa-2)\,(2\kappa)] + \kappa\zeta\,[(2\kappa)\,(2\kappa)] + \\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)[(4\kappa-2)\,(4\kappa-2)] \\
&= 8\kappa^2\zeta\,(2\kappa-1) + 4\kappa^3\zeta + 4\left(2\kappa^2\eta-\zeta(3\kappa+1)\right)(2\kappa-1)^2 \\
\mathcal{GO}_1(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}} [(\kappa_x+\kappa_y)+\kappa_x\kappa_y] \\
&= 2\kappa\zeta\,[(4\kappa-2)+2\kappa+2\kappa\,(4\kappa-2)] + \kappa\zeta\left[(2\kappa)+(2\kappa)+(2\kappa)^2\right] + \\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)\left[(4\kappa-2)+(4\kappa-2)+(4\kappa-2)^2\right] \\
&= 4\kappa\left[4\kappa\,(\kappa\eta-\zeta)\,(2\kappa-1)+\zeta\left(\kappa^2+1\right)\right] \\
\mathcal{GO}_2(\mathcal{L}(\mathcal{R}(\mathcal{G}))) &= \sum_{xy\in\mathcal{E}} \kappa_x\kappa_y\,(\kappa_x+\kappa_y) \\
&= 2\kappa\zeta\left[(4\kappa-2)^2\,(2\kappa)+(4\kappa-2)\,(2\kappa)^2\right] + \kappa\zeta\left[(2\kappa)(2\kappa)^2+(2\kappa)^2(2\kappa)\right] + \\
&\qquad \left(2\kappa^2\eta-\zeta(3\kappa+1)\right)\left[(4\kappa-2)(4\kappa-2)^2+(4\kappa-2)^2(4\kappa-2)\right] \\
&= 16\kappa^2\zeta\,(2\kappa-1)\,(3\kappa-1) + 16\kappa^4\zeta + 8\,(2\kappa-1)\left[2\kappa^2\eta-\zeta(3\kappa+1)\right]
\end{aligned}
$$

□

2. Topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))$

Theorem 2.1. *Let $\mathcal{P}_\eta$ be a path with η vertices . Then the topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))$ is given by*

(1) $\mathcal{ABC}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = \frac{1}{6}\left[12\sqrt{2} + 3\sqrt{6}\,(2\eta-1) + 4\sqrt{3}\,(2\eta-5) + \sqrt{10}\,(\eta-4)\right]$
(2) $\mathcal{RA}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = \frac{1}{12}\left[3\sqrt{2} + 3\,(2\eta-1) + 2\sqrt{6}\,(2\eta-5) + 2\,(\eta-4)\right]$
(3) $\chi(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = \frac{1}{30}\left[20\sqrt{6} + 30\sqrt{2}\,(2\eta-1) + 6\sqrt{10}\,(2\eta-5) + 5\sqrt{3}\,(\eta-4)\right]$
(4) $\mathcal{GA}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = \frac{1}{15}\left[80\sqrt{3} + 30\sqrt{2}\,(2\eta-1) + 24\sqrt{15}\,(2\eta-5) + 30\sqrt{3}\,(\eta-4)\right]$
(5) $\mathcal{SO}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = \eta\left(14\sqrt{2} + 8\sqrt{13}\right) + 8\sqrt{5} - 28\sqrt{2} - 20\sqrt{13}$
(6) $\mathcal{M}_1(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta)) == 4\,(17\eta - 33)$
(7) $\mathcal{M}_2(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = 4\,(41\eta - 92)$
(8) $\mathcal{GO}_1(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = 4\,(58\eta - 125)$
(9) $\mathcal{GO}_2(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = 16\,(73\eta - 254)$

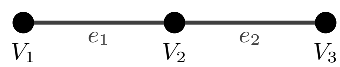

FIGURE 1. $\mathcal{P}_3$

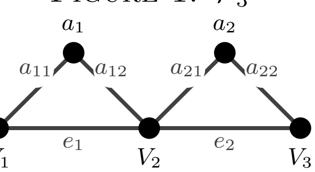

FIGURE 2. $\mathcal{R}(\mathcal{P}_3)$

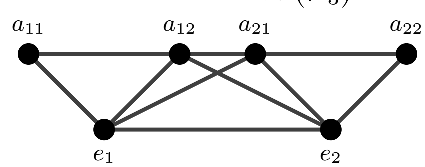

FIGURE 3. $\mathcal{L}(\mathcal{R}(\mathcal{P}_3))$

Proof. Let $\mathcal{P}_\eta$ be a path with η vertices. Then $\mathcal{R}(\mathcal{P}_\eta)$ has $2\eta-1$ vertices and $3(\eta-1)$ edges. Therefore $\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))$ has $3(\eta-1)$ vertices and $7\eta-11$ edges.

The edge partition of $\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))$

Degree of end vertices	No. of edges
$(2,2)$	4
$(4,4)$	$2\eta-1$
$(4,6)$	$4\eta-10$
$(6,6)$	$\eta-4$

$$
\begin{aligned}
\mathcal{ABC}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\sqrt{\frac{2+4-2}{(2)\,(4)}} + (2\eta-1)\sqrt{\frac{4+4-2}{(4)\,(4)}} + (4\eta-10)\sqrt{\frac{4+6-2}{(4)\,(6)}} \\
&\qquad + (\eta-4)\sqrt{\frac{6+6-2}{(6)\,(6)}} \\
&= \frac{1}{6}\left[12\sqrt{2} + 3\sqrt{6}\,(2\eta-1) + 4\sqrt{3}\,(2\eta-5) + \sqrt{10}\,(\eta-4)\right] \\
\mathcal{RA}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\sqrt{\frac{1}{(2)\,(4)}} + (2\eta-1)\sqrt{\frac{1}{(4)\,(4)}} + (4\eta-10)\sqrt{\frac{1}{(4)\,(6)}} + (\eta-4)\sqrt{\frac{1}{(6)\,(6)}} \\
&= \frac{1}{12}\left[3\sqrt{2} + 3\,(2\eta-1) + 2\sqrt{6}\,(2\eta-5) + 2\,(\eta-4)\right] \\
\chi(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\sqrt{\frac{1}{2+4}} + (2\eta-1)\sqrt{\frac{1}{4+4}} + (4\eta-10)\sqrt{\frac{1}{4+6}} + (\eta-4)\sqrt{\frac{1}{6+6}} \\
&= \frac{1}{30}\left[20\sqrt{6} + 30\sqrt{2}\,(2\eta-1) + 6\sqrt{10}\,(2\eta-5) + 5\sqrt{3}\,(\eta-4)\right] \\
\mathcal{GA}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4(2)\sqrt{\frac{(2)\,(4)}{2+4}} + (2\eta-1)\,(2)\sqrt{\frac{(4)\,(4)}{4+4}} + (4\eta-10)\,(2)\sqrt{\frac{(4)\,(6)}{4+6}} \\
&\qquad + (\eta-4)\,(2)\sqrt{\frac{(6)\,(6)}{6+6}} \\
&= \frac{1}{15}\left[80\sqrt{3} + 30\sqrt{2}\,(2\eta-1) + 24\sqrt{15}\,(2\eta-5) + 30\sqrt{3}\,(\eta-4)\right] \\
\mathcal{SO}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\sqrt{2^2+4^2} + (2\eta-1)\sqrt{4^2+4^2} + (4\eta-10)\sqrt{4^2+6^2} + (\eta-4)\sqrt{6^2+6^2} \\
&= \eta\left(14\sqrt{2} + 8\sqrt{13}\right) + 8\sqrt{5} - 28\sqrt{2} - 20\sqrt{13} \\
\mathcal{M}_1(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\,(2+4) + (2\eta-1)\,(4+4) + (4\eta-10)\,(4+6) + (\eta-4)\,(6+6) \\
&= 4\,(17\eta-33) \\
\mathcal{M}_2(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\,(2.4) + (2\eta-1)\,(4.4) + (4\eta-10)\,(4.6) + (\eta-4)\,(6.6) \\
&= 4\,(41\eta-92) \\
\mathcal{GO}_1(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\,(2+4+2.4) + (2\eta-1)\,(4+4+4.4) + (4\eta-10)\,(4+6+4.6) \\
&\qquad + (\eta-4)\,(6+6+6.6)) \\
&= 4\,(58\eta-125) \\
\mathcal{GO}_2(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) &= 4\,(2.4)\,(2+4) + (2\eta-1)\,(4.4)\,(4+4) + (4\eta-10)\,(4.6)\,(4+6) \\
&\qquad + (\eta-4)\,(6.6)\,(6+6) \\
&= 16\,(73\eta-254)
\end{aligned}
$$

□

3. Topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))$

Theorem 3.1. *Let $\mathcal{C}_\eta$ be a cycle with η vertices . Then the topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))$ is given by*

(1) $\mathcal{ABC}(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))) = \frac{\eta}{6}\left(3\sqrt{6} + 8\sqrt{3} + \sqrt{10}\right)$

(2) $\mathcal{RA}(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))) = \frac{\eta}{3}\left(2 + \sqrt{6}\right)$

(3) $\chi(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))) = \frac{\eta}{30}\left(30\sqrt{2} + 12\sqrt{10} + 5\sqrt{3}\right)$

(4) $\mathcal{GA}(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))) = \eta\left(4\sqrt{2} + 16\sqrt{15} + 2\sqrt{3}\right)$

(5) $\mathcal{SO}(\mathcal{L}(\mathcal{R}(\mathcal{P}_\eta))) = \eta\left(14\sqrt{2} + 8\sqrt{13}\right)$

(6) $\mathcal{M}_1(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta)) = 68\eta$

(7) $\mathcal{M}_2(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))) = 164\eta$

(8) $\mathcal{GO}_1(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))) = 232\eta$

(9) $\mathcal{GO}_2(\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))) = 1648\eta$

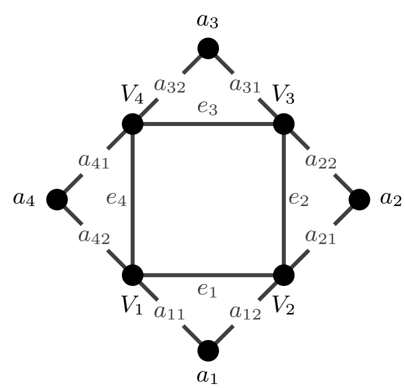

FIGURE 4. $\mathcal{R}(\mathcal{C}_4)$

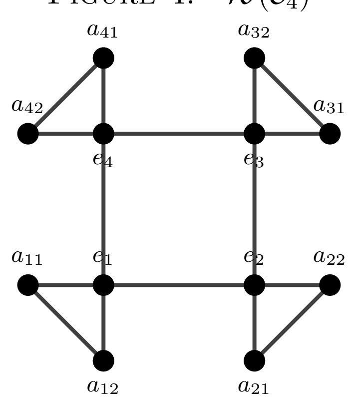

FIGURE 5. $\mathcal{L}(\mathcal{R}(\mathcal{C}_4))$

Proof. Let $\mathcal{C}_\eta$ be a cycle with η vertices. Then $\mathcal{R}(\mathcal{C}_\eta)$ has 2η vertices and 3η edges. Therefore $\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))$ has 3η vertices and 7η edges.

The edge partition of $\mathcal{L}(\mathcal{R}(\mathcal{C}_\eta))$

Degree of end vertices	No. of edges
$(4,4)$	2η
$(4,4)$	4η
$(6,6)$	η

The proof is similar as above.

□

4. Topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))$

Theorem 4.1. *Let $\mathcal{K}_\eta$ be a complete with η vertices . Then the topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))$ is given by*

(1) $\mathcal{ABC}(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = \eta C_2 \left[\frac{\sqrt{4\eta-6}}{2} + \sqrt{\frac{2(3\eta-5)(\eta-1)}{(2\eta-3)}} + \frac{\eta-2}{(2\eta-3)}\sqrt{2(4\eta-7)}\right]$

(2) $\mathcal{RA}(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = \eta C_2 \left[\frac{1}{2} + \sqrt{\frac{\eta-1}{2\eta-3}} + \frac{\eta-2}{2(\eta-3)}\right]$

(3) $\chi(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = \eta C_2 \left[\frac{\sqrt{\eta-1}}{2} + \frac{\sqrt{2}(\eta-1)}{\sqrt{(3\eta-4)}} + \frac{\eta-2}{\sqrt{(2\eta-3)}}\right]$

(4) $\mathcal{GA}(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = 2(\eta C_2) \left[\frac{2\eta-3}{2} + 2\frac{\eta-1}{3\eta-4}\sqrt{(\eta-1)(\eta-3)}\right]$

(5) $\mathcal{SO}(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = (2\sqrt{2})\eta C_2 \left[(3\eta^2-9\eta+7) + \sqrt{2}(\eta-1)\sqrt{5\eta^2-14\eta+10}\right]$

(6) $\mathcal{M}_1(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta)) = 4(\eta C_2)(6\eta^2-16\eta+11)$

(7) $\mathcal{M}_2(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = 4(\eta C_2)(9\eta^3-37\eta^2+52\eta-25)$

(8) $\mathcal{GO}_1(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = 4(\eta-1)(\eta C_2)(9\eta^2-22\eta+14)$

(9) $\mathcal{GO}_2(\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))) = 16(\eta C_2)(15\eta^4-85\eta^3+184\eta^2-168\eta+67)$

Proof. Let $\mathcal{K}_\eta$ be a complete graph with η vertices. Then $\mathcal{R}(\mathcal{K}_\eta)$ has $\eta + \eta C_2$ vertices and $3(\eta C_2)$ edges. Therefore $\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))$ has $3(\eta C_2)$ vertices and $\eta C_2(4\eta-5)$ edges

The edge partition of $\mathcal{L}(\mathcal{R}(\mathcal{K}_\eta))$

Degree of end vertices	No. of edges
$(2\eta-2, 2\eta-2)$	$(\eta-1)\eta C_2$
$(2\eta-2, 4\eta-6)$	$2(\eta-1)\eta C_2$
$(4\eta-6, 4\eta-6)$	$(\eta-2)\eta C_2$

The proof is similar as above. ∎

5. Topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))$

Theorem 5.1. *Let $\mathcal{L}_\eta$ be a Ladder graph. Then the topological indices of $\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))$ is given by*

(1)
$$\mathcal{ABC}(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = \eta\left[\frac{3\sqrt{10}}{2}+\frac{3\sqrt{210}}{5}+\frac{9\sqrt{2}}{5}\right]+8+3\sqrt{6}+ 4\sqrt{3}+\frac{18\sqrt{5}}{5}-\frac{10\sqrt{10}}{3}-\frac{8\sqrt{210}}{5}-6\sqrt{2}.$$

(2) $\mathcal{RA}(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = \frac{\eta}{10}\left[6\sqrt{15}+21\right]+\sqrt{2}+\frac{4}{3}\sqrt{3}+\frac{2}{5}\sqrt{5}+\sqrt{6}-\frac{8}{5}\sqrt{15}-\frac{23}{6}.$

(3) $\chi(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = \frac{\sqrt{5}\eta}{10}\left[6+9\sqrt{5}+3\sqrt{15}\right]+\frac{17}{6}\sqrt{2}+\frac{6}{5}\sqrt{10}+\frac{8}{7}\sqrt{14}-2\sqrt{3}-2\sqrt{5}-12.$

(4)
$$\mathcal{GA}(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = \frac{3\eta}{2}\left[10+3\sqrt{5}\right]+\frac{16}{3}\sqrt{2}+\frac{64}{7}\sqrt{3}+\frac{32}{9}\sqrt{5} +\frac{24}{5}\sqrt{6}-12\sqrt{15}-37.$$

(5) $\mathcal{SO}(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = 6\eta\left[19\sqrt{2}+6\sqrt{34}\right]+8\left[4\sqrt{5}+3\sqrt{13}+2\sqrt{41}-12\sqrt{34}-56\sqrt{2}\right]$

(6) $\mathcal{M}_1(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta)) = 4\eta\,(129\eta-204)$

(7) $\mathcal{M}_2(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = 4\eta\,(501\eta-888)$

(8) $\mathcal{GO}_1(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = 8\,(315\eta-313)$

(9) $\mathcal{GO}_2(\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))) = 16\,(2073\eta-4108)$

Proof. The ladder graph $\mathcal{L}_\eta$ has 2η vertices and $3\eta-2$ edges.Thus $\mathcal{R}(\mathcal{L}_\eta)$ has $5\eta-2$ vertices and $3(3\eta-2)$ edges. Therefore $\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))$ has $3(3\eta-2)$ vertices and $33\eta-38$ edges.

The edge partition of $\mathcal{L}(\mathcal{R}(\mathcal{L}_\eta))$

Degree of end vertices	No. of edges
$(4,4)$	6
$(4,6)$	12
$(4,8)$	8
$(6,6)$	$9\eta-20$
$(6,8)$	16
$(6,10)$	$18\eta-48$
$(8,10)$	8
$(10,10)$	$(6\eta-20)$

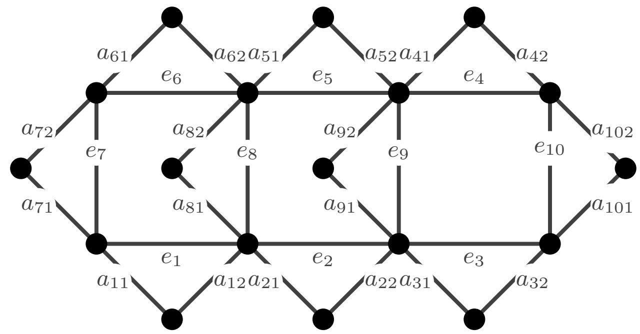

FIGURE 6. $\mathcal{R}(\mathcal{L}_4)$

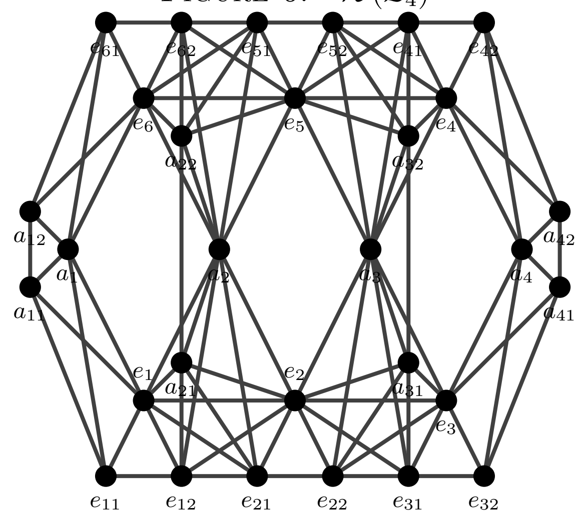

FIGURE 7. $\mathcal{L}(\mathcal{R}(\mathcal{L}_4))$

The proof is similar as above.

□

References

1. Gutman, I. (2021). Geometric approach to degree-based topological indices: Sombor indices. MATCH Commun. Math. Comput. Chem, 86(1), 11-16.
2. Estrada, E., Torres, L., Rodriguez, L., & Gutman, I. (1998). An atom-bond connectivity index: modelling the enthalpy of formation of alkanes.
3. Randic, M. (1975). Characterization of molecular branching. Journal of the American Chemical Society, 97(23), 6609-6615.
4. Zhou, B., and Trinajstić, N. (2009). On a novel connectivity index. Journal of mathematical chemistry, 46, 1252-1270.
5. Vukičević, D.and Furtula, B. (2009). Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges. Journal of mathematical chemistry, 46, 1369-1376.
6. Gutman, I.and Trinajstić, N. (1972). Graph theory and molecular orbitals. Total ϕ-electron energy of alternant hydrocarbons. Chemical physics letters, 17(4), 535-538.
7. Kulli, V. R. (2017). The Gourava indices and coindices of graphs. Annals of Pure and Applied Mathematics, 14(1), 33-38.

8. Adnan, M., Bokhary, S. A. U. H., Abbas, G., & Iqbal, T. (2022). Degree-Based Topological Indices and QSPR Analysis of Antituberculosis Drugs. Journal of Chemistry, 2022(1), 5748626.
9. Bokhary, S. A. U. H., Adnan, Siddiqui, M. K., & Cancan, M. (2022). On topological indices and QSPR analysis of drugs used for the treatment of breast cancer. Polycyclic Aromatic Compounds, 42(9), 6233-6253.
10. Mondal, S., De, N., & Pal, A. (2022). Topological indices of some chemical structures applied for the treatment of COVID-19 patients. Polycyclic Aromatic Compounds, 42(4), 1220-1234.
11. Khalaf, A. J. M., Hanif, M. F., Siddiqui, M. K., & Farahani, M. R. (2020). On degree based topological indices of bridge graphs. Journal of Discrete Mathematical Sciences and Cryptography, 23(6), 1139-1156.
12. Nasir, S., Farooq, F. B., & Parveen, S. (2022). Topological indices of novel drugs used in blood cancer treatment and its QSPR modeling. AIMS Mathematics, 7(7), 11829-11850.