**Closed-Form Solutions for Dynamic Finite Deformation of Vulcanised Rubber Cylinders**

**Abstract**

Cylindrical tubes are subjected to internal pressures and, as a result, they undergo finite deformations. Therefore, a good understanding and knowledge of deformation mechanisms of many structures and materials under different loading conditions are to be known because of great importance to materials testing and product development. In this work, the dynamic case of the deformation of an internally pressurised hollow cylinder made of natural vulcanised rubber material is considered. The analysis of the deformation led to a non-linear second-order partial differential equation for the determination of stresses and displacement. Knowing that most of the wave equations are weakly non-linear and as such their solution are time independent. The Monge method was employed, which reduces the equation into a linear second-order ordinary differential equation where the D operator Method of solution was sought, and appropriate boundary conditions were applied for the determination of a closed-form solution of the displacement and stresses at various parts of the cylindrical tubes.

Keywords: Vulcanised rubber, displacement, undeformed radius, stresses and Hollow cylinder.

1. **Introduction**

Cylindrical materials such as tyres, hydraulic hoses, wires, shocks, tubes, seals, vibration absorbers, and their mechanical properties play an important role in their usage. Most Cylindrical tubes are subjected to internal pressures and, as a result, they undergo finite deformations. An important and challenging mechanical behaviour of interest is the extension and torsion of rubber-like cylindrical tubes and solid cylinders (Anssari-Benam & Horgan, 2022; Ahmadinezhad et al., 2023). Therefore, a good understanding and knowledge of deformation mechanisms of many structures and materials under different loading conditions are to be known because of great importance to materials testing and product development. A good understanding has fundamental importance in fulfilling their functional applications and has received increasing interest in recent years (Wang et al., 2021). The analysis of the dynamic case of finite deformation of an internally pressurised hollow cylinder of a vulcanised rubber material is considered. The work aims to determine the displacements and stresses caused by the internal pressure p(t), which is time-dependent, at any cross section of the hollow cylinder. Erumaka (1) worked on internally pressurised vulcanised rubber, where he established a condition for a non-trivial solution of the displacement. Nwagwu (2) also worked on internally pressurized vulcanized rubber and obtained a closed form solution for the displacement in his work it was noted that as the radius of the hollow cylinder made of vulcanized rubber increases the displacement of the material increases and that the maximum displacement may not be possible until collapse of the material. Ejike and Erumaka (3) worked on the deformation of a rotating circular cylinder made of Blatz-Ko material and considered two different cases hollow and a solid cylinder, and were able to obtain an approximate solution for the displacement. Erumaka et. al (4) investigated axial shear waves in an incompressible solid and obtained a closed-form solution for the displacement and stresses across various parts of the cylindrical material. Huang (5) worked on the finite displacement of a hollow sphere under internal and external pressures. Erumaka et. al (6) did work on the title Azimuthal Shear Wave in an Incompressible Hollow Circular Cylinder and achieved a closed-form solution for the displacement and stresses. Their results support that incompressible material does not depend on time.Aani and Rahimi (7) investigated the displacement and stresses of axisymmetric radial deformation of the shell. They employed the use of the Neo-Hookean strain energy function to obtain the behaviour of the material. The analysis and results presented show the effect of the stress generated for an internally pressurised thick-walled cylinder containing an internal radial hole using the finite element method. Darijani and Bahremen (8) employed polynomial hyperelastic models to obtain a closed-form solution for the analysis of a rubbery solid circular cylinder.

Elkholy et. al (9) study on the effect of Finite Element Analysis of Stresses Caused by External Holes in Hydraulic Cylinders. Erumaka et. al (10) worked on the title Combined Axial and Azimuthal Shear Wave in an Incompressible Hollow Circular Cylinder, where they obtained the combined displacements through the principle of superposition**.** Gao(11) analysed the Elasto-plastic analysis of an internally pressurized Elasto-plastic analysis of an internally pressurised thick-walled cylinder using a strain gradient plasticity theory. The numerical data presented demonstrated that the classical plasticity-based solution and the gradient plasticity-based solution predict almost identical results. Fracture mechanics analysis of cylindrical pressure vessels was also carried out. Anani and Gholamhosein(12) worked on spherical material, Stress analysis of a thick pressure vessel composed of incompressible hyperelastic materials, where Neo Hookean strain energy function was used to determine the stress and displacement of the spherical shell that is axisymmetric radially deformed under internal and external pressure. Nabham et. al. (13) study the effect of the stress generated for internally pressurised thick-walled cylinders containing an internal radial hole using the finite element method. Their results show that hoop stress increases due to an increase in the hole parameter, depth and diameter. Furthermore, the characterisations of notch may be used to determine the maximum stress limit. Aani and Rahimi (14) investigated the stability of internally pressurised thick-walled spherical and cylindrical shells made of functionally graded incompressible. Chung et al (15) determined the deformation of internally pressurised hollow cylinders and spheres for the Blatz-Ko type of compressible elastic material. The results presented show that, when the ratio of the outer undeformed radius to the inner undeformed radius is higher than the critical value, the shear bifurcation occurs before the maximum pressure is reached, they also show that the reverse occurs when the ratio is lower than the critical value. In this present paper, knowing that most of the wave equations are weakly non-linear and as such their solution are time independent. We employ the Monge method, which reduces the equation into a linear second-order ordinary differential equation, and we sought a closed solution using the D-operator method of solution of second-order ordinary differential equations for the determination of stresses and displacement across a hollow cylindrical pipe made of vulcanised rubber material.

**2. Governing equations**

Let's consider an open region denote the cross section of a right circular tube with inner radius a and outer radius b in its initial configuration. The cylindrical tube is subjected to a time-dependent internal pressure of magnitude . The resulting deformation is a one to one axisymmetric deformation which maps the point wit cylindrical polar coordinate in the initial configuration to the point in the current region such that

 (1)

where (R,t) is radial displacement, R and r are radii of the cylinder in the initial and current configurations respectively. where is to be obtained

a

b

 (1

The deformation gradient tensor for the given equation (1) is

 (2)

The Left Cauchy-Green deformation gradient tensor associated with the given (1) is

 (3)

 (4) is the transpose of

Then equation (4) becomes

 (5)

=

Where the three principal strain invariants , and have their usual formulas as

 = (6.1)

 (6.2)

 (6.3)

 is the strain energy function and =

Here we consider compressible isotropic elastic vulcanized rubber material characterized by the elastic potential

 (7)

where is shear modulus, I1 and I3 are the first and third principal invariants respectively.

 (8)

Evaluating in (8) using (6), we obtain the following:

 (9)

 (10)

 (11)

**3 Stress Tensor T:** Considering the stress tensor T for compressible material as

 (12)

where

 (13.1)

 (13.2)

 (13.3)

Since

 (14a)

Evaluating (14a) using (5) and (13), we obtain

)

 (14b)

The stress tensor T in cylindrical polar form is given as:

 (15)

By comparing (14b) with (15), we obtain the components of stress tensor as

 (16a)

 (16b)

 (16c)

 (16d)

Where

**4 Equations of motion:** The equations of motion in cylindrical polar co-ordinates is given as:

 (17a)

 (17b)

 (17c)

Here, , and bz are the components of the body forces,, and az are the components of the acceleration and is the material mass density. Based on the deformation equations in (3.1), the motions are on radial direction, that is, ar ≠ 0 and

The non-zero components of the equations of motion is given as

 (18a)

Substituting (16) in (18), we have

+

+

where

 (18b)

1. **MONGE METHOD OF SOLUTION**

The standard form of the Monge equation is given as

 (19)

where .

where

comparing (18) and (19), we have

 ,

Monge Subsidary equations

 (21)

Substituting the values of in (20) and (21) we have (22) and (23)

 dRdt=0 (22)

Equation becomes

 (24)

Substituting (24) into (22) we have (25a) and (25a)

 dRdt=0 (25a) dRdt=0 (25b)

 (25a)+(25b) becomes

 )dRdt=0 (26)

Equation (26) reduces to

 where

Let

where

Let

 (28)

Therefore

Using (14) and (15), (13) becomes

Let

Then (7) becomes

 (30)

The auxillary equation of equation (30) is given as

 (31)

1. **Boundary conditions**

Using boundary conditions in terms of the displacement of the form

It is important to note that the above boundary conditions must satisfy

Then we obtain the values of A and B in equation (31) as

 and

substituting the value of and B in equation (31) we have

 (32)

For steady state then equation (32) becomes

Considering the case of constant velocity where becomes

 (33)

1. **Results and Analysis**

Table 1: Table of values for displacement, undeformed radius and time generated from (33).

|  |  |  |
| --- | --- | --- |
| Time | Undeformed radius, R | Displacement, r(R, t) |
| 0 | 10 | 0.0000 |
| 1 | 12 | 0.8098 |
| 2 | 14 | 1.3191 |
| 3 | 16 | 1.6072 |
| 4 | 18 | 1.7223 |
| 5 | 20 | 1.6953 |
| 6 | 22 | 1.5474 |
| 7 | 24 | 1.2938 |
| 8 | 26 | 0.9461 |
| 9 | 28 | 0.5126 |
| 10 | 30 | 0.0001 |

Figure 1: A graph of displacement against undeformed radius for a hollow cylindrical material.

The graph shows the mathematical description of a travelling wave

Figure 2: A graph of displacement against time

Mathematical description of travelling waves moving lines in time.

Table 2: Table of values for displacement, undeformed radius and time for the case of constant velocity. Note that in Table 1, stress is undefined at t=0, which can be due to pre-stress in the material, even though the initial displacement is zero and vice versa

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| time, t | undeformed radius, R (mm) | Displacementr(R, t) |  |  |
| **1** | **10** | **0.9999** | **-0.1087** | **0.4993** |
| **2** | **12** | **1.6192** | **-0.1674** | **0.4993** |
| **3** | **14** | **1.9779** | **-0.2031** | **0.4993** |
| **4** | **16** | **2.1420** | **-0.2270** | **0.4992** |
| **5** | **18** | **2.1515** | **-0.2443** | **0.4991** |
| **6** | **20** | **2.0325** | **-0.2578** | **0.4990** |
| **7** | **22** | **1.8030** | **-0.2977** | **0.4987** |
| **8** | **24** | **1.4759** | **-0.2785** | **0.4985** |
| **9** | **26** | **1.0609** | **-0.2871** | **0.4979** |
| **10** | **28** | **0.5655** | **-0.2950** | **0.4961** |
| **11** | **30** | **0.0044** | **-0.3025** | **0.0000** |

Figure 3: A graph of stress against displacement

The graph shows that vulcanised rubber is a highly compressible material. It can undergo large deformation without reaching its breaking point.

Figure 4: A graph of stress against undeformed radius

The graph shows that stress will become zero irrespective of the input variable

**Figure 5: A graph of stress against time**

The graph shows that as time t progresses, the stress tends to zero

1. **Conclusion**

In this work, we were able to establish a closed solution for the displacement and stresses for the dynamic case of an internally pressurised hollow cylindrical pipe made of vulcanised rubber material. Table 1 showed that at the initial time, the stress can become undefined due to pre-stress in the material even when the displacement is zero. The graph of Figure 3 shows that vulcanised rubber is a highly compressible material which can undergo large deformation without reaching its breaking point. The graph of Figure 5 shows that as time t, progresses, the stress must become zero. Figure 2 describes the Mathematical description of travelling waves moving lines in time. A graph of undeformed radius against displacement is plotted as shown in Figure 1. Equations (33) and (16) give the displacement and components of the stress, and Table 2 represents values of displacement and stresses at a certain cross section of the hollow cylindrical pipe made of vulcanised rubber material. It was observed that as the radius of the hollow cylinder made of vulcanised rubber increases, the deformation gradient decreases. The result shows that the radius of the hollow cylinder made of vulcanised rubber increases, and the displacement of the material increases.

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