**Construction of non-homogeneous solution to fractional Differential Equations using Laplace transform Method**

 **Abstract**

This study focuses on fractional differential equations that incorporate Caputo and Mittag-Leffler non-homogeneous fractional differential equations. Utilizing the Laplace transform technique, we formulated a non-homogeneous fractional differential equation and derived an explicit solution expressed in terms of Mittag-Leffler and Caputo functions.

Keywords: fractional differential equation, Mittag-Leffler function, Riemann-Liouville fractional equations, Caputo derivative, Laplace transforms.

1. **Introduction**

“Fractional calculus as a branch of mathematics is used for investigating the properties of derivatives and integrals of non-integer orders called fractional derivatives and integrals. The history of fractional calculus was first mentioned in Leibniz’s letter to L’Hospital in the year 1695, where the idea of semiderivative was suggested. During that time, fractional calculus was built on formal foundations by many famous mathematcians, like Liouville, Grunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, etc. many of them proposed original approaches, which can be found chronologically in” [1]. “Many works have been done on fractional calculus in the derivation of particular solutions of a significantly large number of linear and non-linear ordinary and partial differential equations. The fractional integral could be used for describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as presents in” [2]. “One of the major advantages of fractional calculus is that it can be considered as a super set of integer-order calculus of the second and higher order. For the Analysis of fractional advection equation with improved homotopy analysis method see” [3-5]. “Other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, probability, electrical networks and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing etc” [6]. “Many physical processes appear to exhibit fractional order behaviour that may vary with space or time. The fractional calculus has allowed the operations of differentiation and integration to any fractional order. The order may take on any real or imaginary value. Many applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. Interesting attempts have been made recently to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives proposed. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. It has been believe that many of the great future developments will come from the applications of fractional calculus to differential fields. Some recent contributions to the theory of fractional differential equations are seen in” [7-10].

 “Laplace transform on the other hands is another operational tool of solution that consists of three main steps, such as; the given hard problem is transformed into a simple equation. This simple equation is then solved by purely algebraic manipulations. The solution of the simple equation is transformed back to obtain the solution of the given problem. The Laplace transform method has been applied for solving the fractional ordinary differential equations with constant and variable coefficients. It is a very useful tool for solving linear algebraic equations which can be solved easily” [11-15]. “The solutions are expressed in terms of Mittage-Leffler functions, and then written in a compact simplified form. As special case, when the order of the derivative is two the result is simplified to that of second order equation. To apply Laplace transform method to solve fractional ordinary differential equation with constant coefficients, special formulas of Mittage-Leffler functions are derived and expressed in terms of elementary functions (power, exponential and error functions), instead of an infinite series. Also special formulations of inverse Laplace transformation are obtained in terms of Mittage-Leffler functions which already derived” [16-17]. Our aim of this work is to apply Laplace transform method to construct nonhomogeneous fractional differential equation of the form.

 (1.1)

Where,  is a  continuously differentiable function and  is the fractional Caputo-type derivative of order 

**2. Preliminaries**

We introduce some basic notations of fractional integral calculus

Definition 2.1. (The Riemann-Liouville fractional integral of order  ). The Riemann-Liouville fractional integral of order  for a function is given by

 (2.1.1)

Where, denote the Euler’s Gamma function. The left and right Riemann-Liouville derivative with order  of a given function are respectively given as

 (2.1.2)

and

 (2.1.3)

Where is an integer which satisfies 

Definition 2.2. (The Caputo derivative of fractional order ). The Caputo derivative of fractional order  is defined as

, ,  (2.2.1)

where  denote the integer part of the real number .

Definition 2.3. (Caputo differential operator  ). Caputo differential operator  is given by

 (2.3.1)

Where,

is the Riemann-Liouville differential operator and is the  degree Taylor polynomial for , centered at the origin. Here, is tacitly assumed to be a left-sided differential operator and to have its standing point at, so that one naturally seek solution to the differential equation on an interval of the form  with some 

Definition 2.4. ( Error Function). The Error function can be defined as

 (2.4.1)

Definition 2.5. (The Mittag-Leffler function). The Mittag-Leffler function is defined by

 (2.5.1)

Definition 2.6. (The Wright function). The Wright function is defined by



Definition 2.6. (The Riemann-Liouville fractional derivatives). The Riemann-Liouville fractional derivatives  and  of order  are defined by

 (2.6.1)

and

 (2.6.2)

respectively, where is the integral part of 

Definition 2.7. (The Binomial Coefficients). The Binomial Coefficients are defined by

 (2.7.1)

Where, are integers. Observe that , then , also 

Definition 2.8.( The Gamma Function). The basic interpretation of the Gamma function is simply the genaralization of the factorial for all real numbers. It is defined by

 (2.8.1)

Definition 2.9. (The Beta function). The Beta function can be defined in terms of Gamma function as

 (2.9.1)

It can also be defined in term of a definite integral as

 (2.9.2)

Definition 2.10. (The Laplce Transform ). The Laplace transform of a function is defined by

 (2.10.1)

The Laplace transform of the fractional derivative is defined as

 (2.10.2)

Where, 

Definition 2.11. (Convolution Theorem). The theorem states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. If and are the Laplace transforms of and respectively, then

 (2.11.1)

Definition 2.12. (The fractional integral). The fractional integral of of order is defined as

 (2.12.1)

Equation (2.12.1) is actually a convolution integral. So using (2.12.1), we find that

 (2.12.2)

Equation (2.12.2) is the Laplace transform of the fractional integral. We see for that

 (2.12.3)

1. **Main Result**

Preposition 3.1. Let be a continuous function and  for any , . Then for any positive integer , we have

 (3.1.1)

Proof

 (3.1.2)

Observing that the interchange of the order of integration in the above derivation can be justified by applying Fubini’s theorem.

Proposition 3.2. 

Proof

Recall that

 (3.2.1)

 (3.2.2)

Theorem 3.3. 

Proof

We note that

 (3.3.1)



 (3.3.2)

Proposition 3.4. Let  be a real nonnegative number and be piecewise continuous on and integrable on any finite subinterval of . Then for , we defined the Riemann-Liouville fractional integrable of order  as

 (3.4.1)

Proof

Consider the  order differential equation with the given initial conditions:

 (3.4.2)

Using the form of the Cauchy function,

 (3.4.3)

We claim that the unique solution of (3.4.1) is given by

 (3.4.4)

By induction

For , we have

 (3.4.5)

Solving (3.4.5) we obtained



Since , we have

 (3.4.6)

Now, assume that (3.4.4) is true for , we show that the equation is also true for .

Consider

 (3.4.7)

Since . Let . Then (3.4.7) becomes

 (3.4.8)

Using the induction hypothesis, we noticed that



 (3.4.9)

Since , then

 (3.4.10)

So, (3.4.4) is true

Thus, since  in (3.4.2) is the derivative of , we may interpret as the integral of . Therefore,



Lamma 3.5. The one and two parameter representation of Mittag-Leffler function can be define interms of a power series as

 (3.5.1)

 (3.5.2)

The exponential series defined by (3.5.2) is a generalization of (3.5.1). The following relationship holds from the result of the definition of (3.5.2)

and (3.5.3)

 (3.5.4)

 (3.5.5)

So that

 (3.5.6)

Proof of (3.5.3)

By definition (3.5.2), we have that

 (3.5.7)

Observe that . Also for specific values of the Mittag-Leffler function reduces to some familiar functions such as

 (3.5.8)

 (3.5.9)

 (3.5.10)

Proposition 3.6. The Gamma function has some unique properties. By the use of its recursion relations, one can obtain the formulas

 (3.6.1)

 (3.6.2)

From (3.6.1), we observe that . We prove that 

From the definition of Gamma function 

We have

 (3.6.3)

If we let , so that

 (3.6.4)

Equally, we can write (3.6.4) as

 (3.6.5)

Multiplying both (3.6.4) and (3.6.5) together to get

 (3.6.6)

Equation (3.6.6) is a double integral and can be in polar coordinates to get

 (3.6.7)

So that 

Proposition 3.7. 

Proof

recall that in the integer order operations, the Laplace transform of is given by

 (3.7.1)

let so that 

where,



 (3.7.2)

Example 3.8

Consider the second-order fractional differential equation of the form

 (3.8.1)

With the initial condition and 

Let . Then the fractional differential equation (3.8.1) on applying the Laplace Transform method, gives

 (3.8.2)

Applying the initial condition, gives

 (3.8.3)

Using the fact that , gives

 (3.8.4)

 (3.8.5)

  (3.8.6)

Recall that

 (3.8.7)

But,



 (3.8.8)

We recall from the Binomial expansion that

 (3.8.9)

For , we have



 



 (3.8.10)

 (3.8.11)

Substituting this into (3.8.6), gives



  (3.8.12)

We recall that

 (3.8.13)

From (3.8.12), we derived the solution of (3.8.1) by taking the inverse Laplace transform of (3.8.12) i.e

 (3.8.14)

Example. 3.9. (Theorem). Let  . Solve the non-homogeneous two terms fractional differential equations involving Caputo Fractional Differential Equation with 

 (3.9.1)

Taking the Laplace transform of both side, gives

 (3.9.2)

Using the fact that , gives

 (3.9.3)

 (3.9.4)

Applying the condition, gives

 (3.9.5)

 (3.9.6)

 (3.9.7)

taking the inverse Laplace transform, gives

 (3.9.8)

Where,

 (3.9.9)

 (3.9.10)

We recall that

 (3.9.11)

Comparing (3.9.10) and (3.9.11), we see that



Substituting this, gives

 (3.9.12)

Substituting into (3.9.8), gives

 (3.9.13)

Where,

 and 

Convolution of two functions , gives

 (3.9.14)

 (3.9.15)

From (3.9.13),

 (3.9.16)

Using the definition of delta function, we have

 (3.9.17)

And using the definition of , gives

 (3.9.18)

1. **Conclusion**

This work is focused on understanding the properties of fractional derivatives and their effectiveness in certain Differential equations called Caputo and Mittage-Leffler non-homogeneous fractional differential equation. We proposed an adapted Laplace transform method that gives the solution of a non-homogeneous fractional differential equation and obtained explicit solution in terms of Mittag-Leffler and Caputo functions.

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

**References**

1. Agarwal, R. P., Lakshmikantham, V., Nieto, J.J. (2010). On the concept of solution for fractional differential equations with uncertainty. Nonlinear Analysis 72 (2010), 2859-2862.
2. Alkan, A., Aktürk, T., & Bulut, H. (2024). The Traveling Wave Solutions of the Conformable Time-Fractional Zoomeron Equation by Using the Modified Exponential Function Method. *Eskişehir Technical University Journal of Science and Technology A-Applied Sciences and Engineering*, *25*(1), 108-114.
3. Alkan, A. (2024). Analysis of fractional advection equation with improved homotopy analysis method. *Osmaniye Korkut Ata Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, *7*(3), 1215-1229.
4. Alkan, A., & Anaç, H. (2024). The novel numerical solutions for time-fractional Fornberg-Whitham equation by using fractional natural transform decomposition method. *AIMS Mathematics*, *9*(9), 25333-25359.
5. Alkan, A. (2022). Improving homotopy analysis method with an optimal parameter for time-fractional Burgers equation. *Karamanoğlu Mehmetbey Üniversitesi Mühendislik ve Doğa Bilimleri Dergisi*, *4*(2), 117-134.
6. Araya, D., Lizama, C. (2007). Almost automorphic mild solutions to fractional differential equations. Nonlinear Anal. 69. P. 3692-3705.
7. Avit, Ö., & Anaç, H. (2024). The Efficient Robust Conformable Methods for Solving the Conformable Fractional Cahn-Allen Equation. *Bilecik Şeyh Edebali Üniversitesi Fen Bilimleri Dergisi*, *11*(2), 422-436.
8. . Avit, Ö., & Anaç, H. (2024). THE NOVEL CONFORMABLE METHODS TO SOLVE CONFORMABLE TIME-FRACTIONAL COUPLED JAULENT-MIODEK SYSTEM. *Eskişehir Technical University Journal of Science and Technology A-Applied Sciences and Engineering*, *25*(1), 123-140
9. Alkan, A., Kayalar, M., & Bulut, H. (2025). The analysis of the traveling wave solutions of the Hirota-Ramani equation via the modified Kudryashov method. *AIMS Mathematics*, *10*(2), 3291-3305.
10. Erol, A. S., Anaç, H., & Olgun, A. (2023). Numerical solutions of conformable time-fractional Swift-Hohenberg equation with proportional delay by the novel methods. *Karamanoğlu Mehmetbey Üniversitesi Mühendislik ve Doğa Bilimleri Dergisi*, *5*(1), 1-24.
11. Bashir, A., Mohamed, M. M. (2017). Existence of solutions and Ulam Stability for Caputo Type Sequential Fractional Differential Equations of order . International Journal of Analysis and Applications.
12. Belmekki, M., Nieto, J.J, Rodriguez-Lopes, R. (2009). Existence of periodic solutions for a nonlinear fractional differential equation. Boundary Value Problem. (2009) Art. ID. 324561.
13. Benson, D.A., Wheatcraft, S. W., Meerschaert, M.M. (2000). The fractional-order Governing Equation of Levy Motion. Vol. 36, no. 6, p. 1413-1423.
14. Gazizov R.K., Kasatkin A.A. (2013). Construction of exact solution for fractional order differential equations by the invariant subspace method.
15. Kalu,Uchenna, (2024). Solution to Fourier Transforms Method to Complex Variable of Non-homogeneous Fractional Differential Equations. Advanced Journal of Science, Engineering 4(1), 52-66. DOI: 10.52589/AJSTE-3EV5SPX.
16. Kilbas, A. A., Srivastava, H.M., Trujillo, J.J. (2006). Theory and Applications of Fractional Differential Equations. John Van Mill. Netherlands: Elsevier, 523p. ISBN 978-0-12-558840-2.
17. Lakshmikantham, V., Vatsala, A.S. (2008). Basic theory of fractional differential equations. Nonlinear Anal. 69, p. 2677-2682.