A square functional equation on groups

Abstract. This paper examines the square functional equation on groups and compares its solution set with that of a well-known cubic functional equation. The findings reveal that homomorphisms satisfying this equation differ from those in the previously studied case.

Keywords: Functional equation, groups, square, cubic

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1. Introduction and Preliminaries

A functional equation is an equation where the unknowns are functions rather than numbers. The equation expresses a relationship between the values of a function at different points. The goal is to find all functions that satisfy the given equation for all inputs in a specified domain.

A functional equation on groups is an equation involving a function whose domain and/ or codomain is a group. These equations typically express algebraic properties such as additivity, multiplicativity, or symmetry. They are widely used in group theory, functional analysis, and number theory.

Functional equations on groups have been extensively studied, leading to a rich body of literature. For example, Yang [8] studied the functional equation $f(xy) + f(xy^{-1}) = 2f(x) + 2f(y)$, where f maps a group into an abelian group. Using the relationship between its Cauchy kernel and solution of Jensen's functional equation, he deduced many basic reduction formulas and relations, and used them to obtain its general solution on free groups.

Homomorphims from functional equations was studied in [2] while several functional equations defined on groups arising from stochastic distance measures was studied by Heather in [4]. In particular, he considered when the domain is an arbitrary group $G,$ and the range is the field of complex numbers.

In [5], the stability of generalized norm-additive functional equations was studied. This study demonstrated that these equations are Hyers-Ulam stable for surjective function from an arbitrary group G to a real Banach space using the large perturbation method.

Peter and Henrik [6] considered the solutions of the quadratic functional equation. They showed that any solution f is a function on the quotient group G. Consequently, they found sufficient conditions on G for all solution to satisfy Kannappan's condition.

These works collectively enhance the understanding of functional equation within group structures, offering various methodologies and application across different types of groups. The characterization of these groups through the size of their generators was studied by Udeogu and Ndubuisi [7]

The aim of this paper is to investigate a square functional equation defined on groups.

Let ψ be a function on a group G with values in a field F such that ψ satisfies the equation given by

$$\psi^{3}(uv^{-1}) + \psi^{3}(vw^{-1}) + \psi^{3}(wu^{-1}) = 3\psi(uv^{-1})\psi(vw^{-1})\psi(wu^{-1}) \qquad (1)$$

Theorem 1.1 [1] Suppose G is a Klein four group and F a field of characteristic different from 2 or 3, then the solution of (1) are

(a) constant functions

(b) functions ψ such that $\psi(e) = 2k,\ \psi(\pi u) = -k,\ \psi(\pi v) = -k,\ \psi(\pi w) = 2k.$

(c) functions ψ satisfying $\psi(e) = 2k,\ \psi(\pi u) = \psi(\pi v) = \psi(\pi w) = -k$ where $k \epsilon F$ and π is a permutation of $\{u, v, w\}$.

Theorem 1.2 [3] Suppose G is a multiplicative group, and ψ is a formally real field and if $\psi : G \to F$ is a solution to (1) such that $\psi(e) = 0$, e being the identity of G, then we have that

$$w(\psi) = \{u \ : \ \psi(u) = 0\}$$

is a normal subgroup. Also ψ is a homomorphism if e is the only element of order 3 and if also $z(\psi) = \{e\}$, ψ is an isomorphism.

Here we consider the identity.

$$u^2 + v^2 + w^2 - uv - vw - wu = 0 \qquad (2)$$

which is factorizable into the expression

$$\frac{1}{2}[(u-v)^2 + (v-w)^2 + (w-u)^2] = [u + vx + wx][u + v\bar{x} + w\bar{x}]$$

where *x* is the complex number - $\frac{1}{2}\left(1 - i\sqrt{3}\right)$.

Let us observe that the symmetry that holds in $u^3 + v^3 + w^3 = 3uvw$ and subsequently, in (1), is no longer the case. An interesting case occurs, if by way of analogy to the cubic equation in [2], but with no appeal to the symmetry condition that exists there, we consider the functional equation

$$\psi^2(u) + \psi^2(v) + \psi^2(w) = 3\psi(uv)\psi(vw) \qquad (3)$$

where ψ is a function defined on a group G with values on a field F and $\psi^2(t) = \psi(t)\,.\,\psi(t)$. G may be an abelian or non-abelian group.

Now let u, v, w in (3) be replaced respectively by $uv^{-1}, vw^{-1},$ and wu^{-1}.

Thus uv is replaced by $uv^{-1}\,.\,vw^{-1} = uw^{-1}$ and similarly, vw by vu^{-1} and consequently, (3) can be obtained in the form:

$$\psi^2(uv^{-1}) + \psi^2(vw^{-1}) + \psi^2(wu^{-1}) = 3\psi(uw^{-1})\psi(vu^{-1}).$$

If $z = e$, we have that

$$\psi^2(uv^{-1}) + \psi^2(v) + \psi^2(u^{-1}) = 3\psi(u)\psi(vu^{-1}) \qquad (5)$$

If on the converse, we put $u = vw^{-1}, v = uw^{-1}$ then $uv^{-1} = vu^{-1},\ u^{-1} = wv^{-1}$ and $\psi^2(vu^{-1}) + \psi^2(uw^{-1}) + \psi^2(wv^{-1}) = 3\psi(vw^{-1})\psi(vu^{-1})$.

By interchanging u with v, it is obvious that (4) can be obtained from (5).

2. Main Result

In this section, we present the main result of this paper.

Theorem 2.1. Suppose G be a group, F a field of characteristic $m, m \neq 2$ or $m \neq 3$ and ψ is a function $\psi : G \to F$ such that

$$\psi^2(u) + \psi^2(v) + \psi^2(w) = 3\psi(uv)\psi(vw) \tag{3}$$

If G is a Klein four group then $\psi = \psi_1$ is a solution of (3) if and only if either

(i) ψ_1 is a constant function

or (ii) ψ_1 is a function such that $\psi_1(\pi v) = \psi_1(\pi u) = 2t,\ \psi_1(\pi w) = t$

or (iii) ψ_1 is a function such that $\psi_1(\pi v) = \psi_1(\pi w) = t,\ \psi_1(\pi u) = 2t,$ where $t = \psi_1(e)$ is the zero element on F, in each case.

Proof. Let $G = \{e, u, v, w\}$ be a Klein four group. Then

$$u^2 = v^2 = w^2 = e,\ \ uv = vw = w,\ \ vw = wv = u,\ \ uw = wu = v.$$

Suppose that $u = v \neq e$. Then

$$2\psi^2(u) + \psi^2(e) = 3\psi(u)\psi(e) \tag{6}$$

which on factorization yields

$[\psi(u) - \psi(e)][2\psi(u) - \psi(e)] = 0$ so that $\psi(u) = \psi(e)$ or $\psi(u) = \frac{3}{2}\psi(e)$.

We consider the following cases. First, let π be a parameter of $\{u, v, w\}$

(i) If $\psi = \psi_1$ is a function such that $\psi_1(u) = \psi_1(e)$ then

$\psi_1(\pi u) = \psi_1(\pi v) = \psi_1(u) = \psi_1(\pi w) = \psi_1(u) = \psi_1(e) = t$ is obviously a solution.

(ii) If $\psi_1(\pi u) = \psi_1(\pi v) = \psi_1(e),\ \psi_1(\pi w) = \frac{1}{2}\psi(e),$

it turns out that

$\psi_1(\pi u)$ is a solution if and only if $\psi_1(e) = 0,$

since if (ii) holds then

$$\left(1 + 1 + \frac{1}{4} - \frac{3}{2}\right)\psi_1{}^2(e) = 0 \text{ if and only if } \psi_1(e) = 0.$$

Observe of course that $u = v,$ then $uv = e,\ vw = u$ and $\psi_1(u) = \psi_1(e)$

(iii) Let $\psi_1(\pi u) = \psi_1(e)$, $\psi_1(\pi v) = \psi_1(\pi w) = \frac{1}{2}\psi_1(e)$. Then, as in (ii) ψ_1 is a solution if and only if $\psi_1(e) = 0$.

Theorem 2.2. Let G be a multiplicative group, and F, a real field. If the function $\psi : G \to F$ satisfies $\psi(e) = 0$, where $e \epsilon G$ is the identity, then $\psi(u) = 0$ is a homomorphism of G into F. The set given by $K = \{\mathrm{u} : \psi(u) = 0\}$ is not a normal subgroup.

Proof. Let $\psi(e) = 0$, then $\psi(u) = 0$. Unlike what obtain for the solution of the cubic functional equation (1), the set of elements of G with $\psi(u) = 0$ does not form a normal subgroup. To establish this, we let G to be a multiplicative group and F, a real field.

Consider a function $\psi : G \to F$ such that $\psi(e) = 0$. Let us put $K = \{\mathrm{u} : \psi(u) = 0\}$. Then

$\psi^2(uwu^{-1}) + \psi^2(uw^{-1}) + \psi^2(u^{-1}) = 3\psi(u)\psi(w^{-1})$ follows from (5).

If $\psi(w) = 0$ then $\psi(w^{-1}) = 0$. Hence

$$\psi^2(uwu^{-1}) + \psi^2(uw^{-1}) + \psi^2(u^{-1}) = 0.$$

Since F is a real field,

$$\psi^2(uwu^{-1}) = \psi^2(uw^{-1}) = \psi^2(u^{-1}) = 0,$$

so that $uwu^{-1} \epsilon K$. Although $w \epsilon K \Rightarrow uwu^{-1} \epsilon K$ as shown above, yet substituting $v = e$ and w into (3) it is clear that $\psi^2(u) = 0$, hence $\psi(u) = 0$ for any arbitrary element of G.

Thus, $\psi = 0$ on G if $\psi(e) = 0$. A check on the equation reveals that with $\psi(u) = 0, \psi$ is a homomorphism. This concludes the proof.

3. Conclusion

This study highlights the distinct nature of homomorphisms satisfying the square functional equation compared to those associated with the cubic functional equation. The comparison underscores fundamental differences in their solution sets, contributing to a deeper understanding of functional equation on groups. These findings open awareness for further exploration of related equation and their implication on algebraic structure.

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