AN EXTENSION OF TYMOCZKO CODES TO ROW STRICT YOUNG TABLEAUX

ABSTRACT. In this article, we extend theories of Tymoczko codes on standard young tableaux to row strict tableau of any given shape λ, by investigating the algorithm through which permutations were associated to a set of row-strict tableaux rst. Via this algorithm, we attach a code to each rst and give some combinatorial interpretations of these codes and establish some connections between some existing results on rst and the codes.

Key words and phrases. Partition of integers, Composition of integers, Standard tableau, Schubert Points, Group of permutations

1. Introduction

Let $(\lambda_i)_{i=1}^{k}$ ($n \in \mathbb{N}$) be a sequence of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1+\lambda_2+\cdots+\lambda_k = n$. In other words, we have a partition of n denoted by $\lambda \vdash n$.To this partition λ, there is an associated diagram called Young diagram $\mathcal{Y}_\lambda$, composed of left justified cells (boxes) in such a way that the number of cells in $i^{th}-$row is λ_i. A filling of $\mathcal{Y}_\lambda$ with $a \in [n]$ such that entries strictly increase along rows from left to right is called row-strict Young tableaux, and column-strict if its entries are increasing from top to bottom in each column. We call it standard Young tableau of shape λ if it is both row-strict and column -strict. Readers are referred to [4] for basic information on Young tableaux.
The above mentioned objects are great tools in the hands of authors in [8],[3],[9],[7], [11],[9] and [10] to do exploit in representation theory, algebraic topology, geometry and combinatorics.

In quest of investigating the connections between some combinatorial and geometrical properties of Springer fibers, Tymoczko in [10] introduced an algorithm through which a permutation is attached to each row-strict tableaux (rst) of shape λ. In [6] a code was attached to each standard tableau of any given shape λ via the algorithm in [10] which we call Tymoczko codes. Denoted by cod(T), we study the combinatorial properties of these codes and establish connections between some of the results in [11] and our combinatorial interpretations of these cod(T). In this we extend our results in [6] to row strict tableaux of any given shape.

In section two, we discuss some basic terms in symmetric group (S_n) and partitions of integers as relevant to our discussion. In section three, we give and study some combinatorial properties of Tymoczko code of row-strict tableaux with the dimension pairs of row-strict tableaux as discussed in [11]. Section four contains our main results.

2. Symmetric Group and Integer Partitions

2.1. The symmetric group denoted by S_n generated by the set $S = \{s_1, s_2, \cdots, s_{n-1}\}$ of adjacent transpositions s_i, $(1 \leq i \leq n-1)$ which swaps i and $i+1$ and fixes other elements of $[n]$ subject to the relations:

- $s_i^2 = e$, $\forall\ 1 \leq i \leq n-1$; (involution)
- $s_i s_j = s_j s_i$, if $|i-j| \geq 2$; (commutation)
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $1 \leq i \leq n-1$ (braid relation).

The length $l(w)$ of w, is the smallest integer $k \geq 0$ such that w can be written as a product of k elements of S (i.e. $w = s_{c_1} s_{c_2} \cdots s_{c_k} \in S_n$). This expression is called the reduced decomposition of w and we say k is the length of w and we write $l(w) = k$. The string of subscripts $c_1 c_2 \cdots c_k$ is the word ω of w,(though

2020 *Mathematics Subject Classification.* 14M15, 14N15, 05E05.

not necessarily unique).
A string of integers $a_i > 0$ is said to be a lattice word if in every prefix, we have at least many a_i as $a_i + 1$. It is called a Yamanouchi word if its reversal is a lattice word.
For instance,the string 11122121 is a lattice word, and so 12122111 is a Yamanouchi word.

2.2. An order $\leq$ called the Bruhat order is a partial order defined on S_n, is such that, for any $\sigma, \tau \in S_n$, we say $\sigma \leq \tau$ in Bruhat order if τ can be obtained from σ via a sequence of transpositions. In other words, we say $\sigma \leq \tau$ if and only if the reduced word of σ is a subword of the reduced word of τ.
For instance, if $\tau = s_1s_2s_3s_1s_2$ and $\sigma = s_1s_2$ then $\sigma \leq \tau$ in the Bruhat order, since s_1s_2 is a subword of $s_1s_2s_3s_1s_2$

2.3.
A partition λ of $n \in \mathbb{N}$ denoted by $\lambda \vdash n$, is a sequence $\lambda = (\lambda_i)_{i=1}^k$, $\lambda_i \in \mathbb{N}$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$. Each λ_i is called part of λ. The number of such λ_i called the length of λ denote by $l(\lambda)$, while the sum of parts is the weight of λ denoted by $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_k$. Consider $n = 6$, then $\lambda = (3, 2, 1)$ is one of the partitions of 6, $l(\lambda) = 3$ and $|\lambda| = 6$. We denote the set of all partitions of n by $P(n)$ and the set of partitions by P.
We use indices to record multiplicity of parts in a partition λ to avoid repetition of parts in λ. Hence, we write $\lambda = \lambda_1^{a_1}, \ \lambda_2^{a_2}, \cdots, \lambda_k^{a_k}$, if λ_i, $(1 \leq i \leq k)$ appears in a_i times in λ and we referrer to a_i as the multiplicity of λ_i. For instance, let $n = 5$ and $\lambda = (2, 1, 1, 1) = (2, 1^3)$. Thus $P(5) = \{(5), (4, 1), (3, 2), (3, 1^2), (2^2, 1), (2, 1^3)(1^5)\}$
There are several literatures on theories of partitions such as [1], [2], and [5] which are good for consultation.

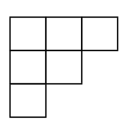
Remark 2.1. Similar to partition of integers, is a sequence $(a_i)_{i=1}^k$ of nonnegative integers such that $\sum_{i=1}^k a_i = n$, called the composition of nonnegative integer n.
For example, let $n = 4$, the following are all compositions of 4

$$(4),\ (3,1),\ (1,3),\ (2,2),\ (2,1,1),\ (1,2,1),\ (1,1,2),\ (1,1,1,1).$$

We consider $(1, 3)$ and $(3, 1)$ as different composition but they are the same as partition.

2.4. Given a partition λ of $n \in \mathbb{N}$, there exists an associated diagram called Young diagram $(\mathcal{Y}_\lambda)$ which gives a graphical way of viewing λ. It is a collection of cells (boxes) arranged in left justified rows in a way that the number of boxes in i^{th} row equals λ_i, and is weakly decreasing from top to bottom.
For example, the Young diagram of shape $\lambda = (3, 2, 1)$ is shown in table 1 .

Table 1. Young diagram of shape $\lambda = 3, 2, 1$



We use matrix notation to label each cell of $\mathcal{Y}_\lambda$, and we write (i, j) to indicate a cell in the i^{th} row and j^{th} columns of $\mathcal{Y}_\lambda$.
The number of cells in each column (column length) denoted by λ'_i is equally a partition of n called the conjugate of λ , denoted by λ'. In a case where $\lambda = \lambda'$ then λ is said to be self conjugate.
Suppose λ and μ are partitions of n and m respectively, such that $n > m$. Then $\mathcal{Y}_\mu$ is said to be a sub-Young diagram of $\mathcal{Y}_\lambda$,and we write $\mathcal{Y}_\mu \subset \mathcal{Y}_\lambda$ if $\mu \subset \lambda$. In other words, $\mathcal{Y}_\mu \subset \mathcal{Y}_\lambda$, if μ is contained in λ. The Fillings of the cells of a Young diagram with numbers from $[n] = \{1, 2, 3, \cdots, n\}$, results to a combinatorial objects called Young tableaux which turn out to be strong tools in representation theory, algebraic combinatorics, geometry and topology.
There are $n!$ Young tableaux of shape λ. For instance, let $n = 2$ and $\lambda = 2, 1$, the list of all possible Young tableaux of the corresponding shape are displayed in table 2.

TABLE 2. All possible Young tableaux of shape $\lambda = 2, 1$

1	2	1	3	2	3	2	1	3	1	3	2
3		2		1		3		2		1	

2	3	6
1	4	
5		

TABLE 3. row strict tableau

The filling of $\mathcal{Y}_\lambda$ is called row strict tableau (rst) if the filling is such that the entries strictly increase from left to right along the row,with no condition on the columns.
We shall denote by $(rst)^\lambda$ the collection of all row strict tableaux of shape λ , the size of $(rst)^\lambda$ is given by the multinomial coefficient. That is,

$$\#(rst)^\lambda = \frac{n!}{\prod_{i=1}^{k} \lambda_i!}$$

For instance, let $n = 5$ and $\lambda = (2, 2, 1)$, $\#(rst)^\lambda = \frac{5!}{2! \times \times 2! \times 1!} = 30$.
If the filling of Young diagram of shape λ is such that the integers from 1 to n appears exactly once and that its entries are increasing across each row and column, such a filling is call standard Young tableaux (SYT). We denote by ST_λ the collection of all standard Young tableaux of shape λ. One of the remarkable results

1	2	5
3	4	
6		

TABLE 4. standard tableau

about standard Young tableaux, is the **hook-length formula**. This is useful in counting the number of all possible Standard Young tableaux of any given shape. Let λ be a partition of $n > 0$ and $\mathcal{Y}_\lambda$ a Young diagram of shape λ, then the number $\#ST_\lambda$ of standard Young tableaux of shape λ is obtained by.

$$\#ST_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}. \tag{2.1}$$

Where $h_{i,j}$ is the number of cells directly to the right and directly bellow the cell in $(i, j)^{th}$ position including the cell.

Remark 2.2. Thus far, it is obvious that $ST_\lambda \subset (rst)^\lambda$, hence, we shall be writing $(rst)^\lambda \backslash ST_\lambda$ when our attention is on those row-strict tableaux that are not standard.

3. DIMENSION PAIRS AND TYMOCZKO CODE FOR ROW-STRICT TABLEAUX

In this section, we briefly discuss the algorithm discussed in [11] and [9], where the dimension pairs and permutations were attached to a set of row-strict tableau respectively.
Following [11], we have definition 3.1.

Definition 3.1. Let $\lambda \vdash n$ and $T \in$(rst)$^\lambda$, a pair of entries (a, b) in T is said to be a dimension pair of T if it satisfies all the following conditions;

(1) $a < b$
(2) "b" is below "a" either in the same column, or located anywhere at the left of a
(3) If "a" is immediately bordered on the right by "c" then $b \leq c$.

We denote the set of all such pairs of T by $(DP)^T$.

Example 3.2. Let $n = 6$ with $\lambda = (3, 2, 1)$, consider

$$\mathrm{T} = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}$$

$(\mathrm{DP})^{\mathrm{T}} = \{(1, 2), (1, 3), (5, 6)\}$

Remark 3.3. There is a unique $\mathrm{T} \in (\mathrm{rst})^{\lambda}$ referred to as base filling in [11] . This filling is such that, they decrease from top to bottom for each column. For example, let $n = 6$ and $\lambda = (3, 2, 1)$ then the base filling of shape λ is

$$\mathrm{T} = \begin{array}{|c|c|c|} \hline 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \quad .$$

The set of dimension pair of base filling of any shape λ is usually empty.

Let $\mathrm{T} \in (\mathrm{rst})^{\lambda}$, we denote by $\mathrm{T}^{\mathrm{b}},\ \mathrm{b} \in \mathbb{N}$ a tableau obtained by deleting all entries $c > b$ in T. For instance, let $n = 6$ and $\lambda = (3, 2, 1)$ with

$$\mathrm{T} = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

then

$$\mathrm{T}^3 = \begin{array}{|c|c|} \cline{1-1} 3 \\ \hline 1 & 2 \\ \hline \end{array}$$

Following [9] we have definition 3.4.

Definition 3.4. Let $\mathrm{T} \in (\mathrm{rst})^{\lambda}$, we denote by d_b the number of rows above the row containing b in T^b which are of equal length plus the total number of rows in T^b which are of greater length (either above or below) than the row containing b and w_b denote the increasing product of simple transpositions of length b.
If $d_b = 0$ then $w_b = e$ is the identity. Then the Schubert point associated to T is a permutation in S_n, denoted by w_T and defined as. $w_T = w_n w_{n-1} w_{n-2} \cdots w_2$ [10]

Remark 3.5. If T is a standard Young tableaux, the procedures in the above definition become easier as we only consider the number of rows strictly above b, since it not possible to have any row (either of less, equal of greater length) below b in T^{b}.

Example 3.6. Let $n = 6$, and $\lambda = (3, 2, 1)$ with

$$T = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

$d_1 = 0,\ d_2 = 0,\ d_3 = 1,\ d_4 = 0,\ d_5 = 0,\ d_6 = 2.$

The Shubert point w_T associated to the above row-strict tableau according to Tymoczko and Precup in [9] is $w_T = s_4 s_5 s_2$.
Arranging the values of the $d_b' s,\ 1 \leq b \leq 6$ in example 3.6 in a natural order of $b's$ we have $(d_1, d_2 \cdots d_6) = (0, 0, 1, 0, 0, 2)$. This we call Tymoczko code (denoted by cod(T)) for the Schubert point w_T. We equally attach a numerical value to each row-strict tableau by adding up all the coordinates of cod(T), and call it the weight of T denoted it by wt(T). For instance, the weight of T in example 3.6 is 3.

Remark 3.7. We like to bring to the notice of the reader at this juncture that:

i) For any $\mathrm{T} \in (\mathrm{rst})^{\lambda}$, $l(w_T) = wt(T) = \#(DP)^T$.

ii) If T is a standard Young tableau of shape $\lambda = (\lambda_1,\ \lambda_2,\ \cdots \lambda_k)$, then $\lambda_i = \#\{d_b : d_b = i-1,\ 1 \le i \le k,\ 1 \le b \le n\}$ and we say cod(T) encode at least one of the partitions $\lambda \in P(n)$.

4. Some Combinatorial Properties of Tymoczko Codes for Row-Strict Tableaux

In this section, we itemize our results with their statements of proof.

Proposition 4.1. *For $\lambda = 1^n$ the weight $wt(T)$ of $T \in (\mathrm{rst})^\lambda$ respects the Bruhat order on S_n, hence it preserves the structure the bruhat graph of S_n. In other words, let $w_T,\ w'_T \in S_n$ respectively be the Schubert points of $T,\ T' \in (\mathrm{rst})^\lambda$* with $wt(T)$ and $wt(T')$ the weights of T and T′ then, $wt(T) \le wt(T')$ if and only if $w_T \le w'_T$.

Proof. We know that,for $n \in \mathbb{N}$, if $\lambda = 1^n$, then $\#(rst)^\lambda = n!$ which is the same as the order of S_n.
Now, Suppose $w_T \le w'_T$ in Bruhat order, we need to show that $wt(T) \le wt(T')$.
We recall from remark 3.7 that $l(w_T) = wt(T)$ for any Schubert point $w_T \in S_n$, (where $l(w_T)$ is the length of w_T). By implication $l(w_T) \le l(w'_T)$ implies $wt(T) \le wt(T')$.
Conversely, we assume $wt(T) \le wt(T')$. Since $l(w_T) = wt(T)$, then $l(w_T) \le l(w'_T)$ implies $w_T \le w'_T$. □

Example 4.2. Let $n = 3$ and $\lambda = 1^3$, in this case, there exist six row strict-tableaux which we display in the table below.

$T \in (rst)^\lambda$	cod(T)	wt(T)	$\mathrm{w_T}$
1 2 3	(0,1,2)	3	$s_1s_2s_1$
1 3 2	(0,1,1)	2	s_2s_1
2 1 3	(0,0,2)	2	s_1s_2
2 3 1	(0,0,1)	1	s_2
3 1 2	(0,1,0)	1	s_1
3 2 1	(0,0,0)	0	e

It could be seen from the above table that the length of each Schubert point concise with the weight of the associated tableau. This we display in the figure below

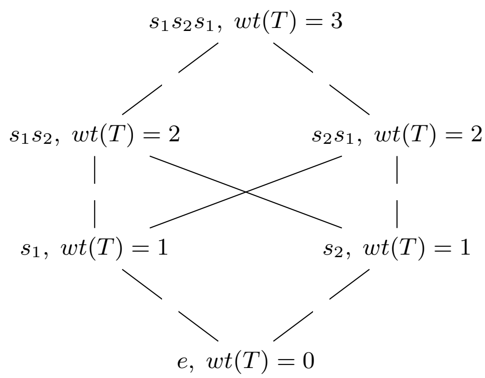


FIGURE 1. Graph of bruhat order of weight of elements in S_3

Proposition 4.3. *Let λ be a partition of the form $\lambda = (n-1, 1)$, There exists only one row-strict tableau* T *which is not a standard tableau and the corresponding code encodes partition $\lambda = n$.*

Proof. For $\lambda = (n-1, 1)$ there are n row-strict tableau out of which $n-1$ of them are standard tableaux. The only one which is not standard is of the form

2	3	$\cdots$	n
1			

In this case, for any $b > 1$ there is no entry in T^b that give non zero coordinate in cod(T), hence cod(T) $= (0, 0, \cdots, 0)$. By condition (ii) of remark 3.7 we say cod(T) encodes one of the partitions λ of n if $\lambda_i = \#\{d_b : d_b = b-1,\ 1 \leq b \leq k,\ 1 \leq b \leq n\}$ and the partition corresponding to the code of such form is $\lambda = n$ ☐

Example 4.4. Let $n = 5,\ \lambda = (4, 1)$. The only non-standard row-strict tableaux of the given shape is

$T =$

2	3	4	5
1			

with cod(T) $= (0, 0, 0, 0, 0)$ and $\lambda_1 = \#\{d_b : d_b = 1-1,\ 1 \leq b \leq 5\} = 5$, this give $\lambda = 5$.

Corollary 4.5. *For any partition $\lambda \in P(n)$ there exists a unique $T \in$*(rst)$^\lambda$ (called base filling in [9]) with cod(T) $= (0, 0, \cdots, 0)$, which encodes partition $\lambda = n$.

Proof. Let T $\in$ (rst)$^\lambda$ be a base filling, since its entries increase from bottom to the top, then there does not exists entry a in T with any row directly above a or any entry either above or below the cell containing a which is of greater length than the length of the row containing a. Therefore cod(T) $= (0, 0, \cdots, 0)$ ☐

Proposition 4.6. *Let* T *be a row-strict tableaux. The number of time b occurs in the dimension pair(s) (a, b) of* T *determines the value in the b^{th} coordinate of* cod(T).

Proof. Suppose there are two entries a and a' above b in the same column or b is located anywhere at the left of a and a', in addition if a and a' are not bordered at the right then, by condition 2 of definition 3.1, we have $(a, b),\ (a', b)$ as the dimension pairs of T. Hence the direct implication of this is that $d_b = 2$ (i.e there are two rows strictly above b in T).
Suppose either a or a' is bordered immediately at the right by c or c', if $b \leq c$ and $b \leq c'$ and b is below a and a' or any where at the left then c and c' are deleted from T^b. Since the entries in the right neighbourhood of a and a' and by condition 2 of dimension pair of T^b, $(DP)^{T^b}$ are (a, b) and (a', b). Therefore $d_b = 2$.
In general, since we are interested in T^b and all $c > b$ are deleted from T^b, then the number of time b occurs in the pair $(\cdot, b)$ will be equal to the number of rows directly above b plus the number of rows which are of greater length than the row containing b either above or below. ☐

Example 4.7. Let $n = 5,\ \lambda = (3, 2, 1)$ with

$$\begin{array}{|c|c|c|}\hline 3 & 4 & 6 \\ \hline 1 & 2 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}$$

$DP^T = \{(2, 5),\ (4, 5)\}$ and $\text{cod(T)} = (0, 0, 0, 0, 2, 0)$. It could be obviously seen that 5 occurs twice in the dimension pair of T and we have 2 at the 5^{th} coordinate of cod(T).

Remark 4.8. From the above result, it could be seen that given a set $(DP)^T$ of dimension pairs of any $T \in (\text{rst})^\lambda$ it is possible to obtain the code of the associated tableau from $(DP)^T$.

Corollary 4.9. *Given any* $\text{T} \in (\text{rst})^\lambda$, *the weight* $wt(T)$ *of* cod(T) *gives the dimension of* T.

Proof. It has been shown in proposition 4.6 that the number of time b appear in the pair $(\cdot, b)$ indicates the numerical value of b^{th} coordinate, and $wt(T)$ is the sum of non-zero coordinate of cod(T), hence the result. ☐

Proposition 4.10. *Let* $\text{T} \in (\text{rst})^\lambda \backslash \text{ST}_\lambda$, *the word of* cod(T) *is not always a lattice word.*

Proof. We shall proof this with counter example. Let $n = 6$ and $\lambda = (3, 2, 1)$ with

$$T = \begin{array}{|c|c|c|}\hline 3 & 5 & 6 \\ \hline 2 & 4 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array}$$

, then, $\text{cod(T)} = (0, 0, 0, 0, 0,)$ and $\omega(T) = 000000$. It is seen here that there is only one integer 0 in any subword which contradicts the definition of lattice word. Hence the result. ☐

4.1. Characterization of Schubert Points Associated to Row-Strict tableaux. We consider the composition structure of the reduced word of Schubert points w_T and give its standard form for any $\text{T} \in (\text{rst})^\lambda \backslash \text{ST}_\lambda$.

The reduced word of $w_T = s_{c_1} s_{c_2} \cdots s_{c_4}$ is the string of subscript $c_1 c_2 \cdots c_k$. For example, let $n = 5,\ \lambda = (2, 2, 2)$. Consider

$$T = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline 4 & 6 \\ \hline \end{array},\ w_T = s_4 s_5 s_2 s_3 s_1, \omega(T) = 45231.$$

On breaking the reduced word into blocks in a way that, string of integers in each block increase in a natural order from left to right. taken into consideration, the number of integers in each block results into composition structure of w_T.

For $w_T = s_4 s_5 s_2 s_3 s_1,\ \omega(T) = 45|23|1$ and $(2, 2, 1)$ as its composition structure. If we arrange the composition structure of w_T such that they are weakly decreasing,then we have a partition of integer, this we denote by α_{c_T}.

Remark 4.11. It is noteworthy that the composition structure of the reduced word of w_T, $T \in (rst)^\lambda \backslash ST_\lambda$ are not always the same and that each d_b determines a block.

Proposition 4.12. *Let* w_T *be the Schubert point associated to* $\text{T} \in (\text{rst})^\lambda(\text{n}) \backslash \text{ST}_\lambda$ *of any shape. Then, the canonical form for the composition structure of the reduced word of* w_T *is given as*
$x_1(x_1 + 1)(x_1 + 2) \cdots (x_1 + k_1)|x_2(x_2 + 1)(x_2 + 2) \cdots (x_2 + k_2)| \cdots |x_r(x_r + 1)(x_r + 2)$
$\cdots (x_r + k_r)|$
Where $x_j = (b - d_b),\ k_j = d_b - 1$ *and* $j = n - b + 1,\ 1 \leq j \leq r$, r *is the number of* d_i *such that* $d_b \neq 0,\ 1 \leq b \leq n$.

Proof. Let $\text{w}_\text{T} \in (\text{rst})^\lambda \backslash \text{ST}_\lambda(\text{n})$ such that T is of any shape λ.

Let $j = n - b + 1$. Suppose $d_b = 0$, then there is nothing to proof since $w_b,\ (2 \leq b \leq n)$ is always an identity (from the definition of w_b).

Now, suppose $d_b \neq 0$ and $b = n$. Then $j = n - n + 1$ which implies that $x_1 = (n - d_n)$.
Since $d_n \neq 0$, let's assume $d_n = q,\ 1 \leq q \leq l(\lambda) - 1$.
From the definition of w_b in [10],

$$w_n = s_{n-q}s_{n-q+1}s_{n-q+2}\cdots s_{n-2}s_{n-1}$$

Then the first block from the left is written as

$$|(n-q)(n-q+1)(n-q+2)\cdots(n-2)(n-1)|$$

By replacing n with b and q with d_b in the above, we have

$$|(b-d_b)(b-d_b+1)(b-d_b+2)\cdots(b-d_b+d_b-2)(b-d_b+d_b-1)|$$

with $a_j = (b - d_b)$ and $k_j = d_b - 1$ then the above equation becomes

$$|x_j(x_j+1)(x_j+2)\cdots(x_j+k_j-1)(x_j+k_j)|$$

Also, we have from the theorem that $j = n - b + 1$ which implies that $j = 1$ (since $b = n$ by hypothesis).
Hence, we have

$$|x_1(x_1+1)(x_2+2)\cdots(x_1+k_1-1)(x+k_1)|$$

This gives the first block of the composition structure of w_T provided $d_n \neq 0$.
Mimicking the proof of the first block we obtain the structure of the remaining blocks. $\square$

Example 4.13. Let T be an arbitrary row-strict tableaux with cod(T) $= (0,1,0,0,0,2)$, be a code of a certain Schubert point . It is easy to see that $n = 6, d_1 = 0, d_6 = 2$. From the statement of the theorem, we have that;
$j = n - b + 1,\ 1 \leq j \leq 2,\ x_j = (b - d_b),\ k_j = d_b - 1.$
Now, when $b = n = 6$, then $j = 1 \implies a_1 = 4$ also, $k_1 = 1$. Therefore we have

$$x_1(x_1+k_1)| = 45|$$

This give our first block. For the second, we consider $b = 2$ and neglect other b for which $d_b = 0$. In this case, we have $x_5 = 2 - 1$

Hence $x_1(x_1+1)|x_5 = 45|1$ is the composition structure of the given code.

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