UNDER PEER REVIEW

Generalized “Reich” Fixed Point Theorem on “C-M-S” Depend on Another Function

Abstract In the present paper, we obtained a unique common fixed point theorem of Reich of mappings on “C - M – S” (Complete - Metric - Space) depends on another function. Our results generalizes, improves some of existing results in the literature.

Keywords: Contractive mapping, fixed point, sequentially convergent, and sub sequentially convergent.

Mathematics Subject Classification: 46J10, 46L15, 47H10.

1. Introduction and Preliminaries

The Fixed point theory is an important topic in non linear analysis. The Banch fixed point theorem is considered as first theorem. Later, many authors have been developed in different forms(see for e.g.[1-67]). In 1968 Kannan [4] established a fixed point result. In 2000 Branciari [3] introduced a class of generalized metric spaces and prove some theorems. Recently S. Moradi[5] established a Kannan fixed point theorem on C-M-S and generalized metric spaces depend on another function. in this paper, we obtained a fixed point theorem for generalized Reich contraction mapping on C-M-S depend on another function.

For our main results we need some of the following definitions.

1.1.Definition [2] Let (X, ρ) be a metric space. A mapping $A : X \to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ay_n\}$ is convergence then $\{y_n\}$ is also convergence. A is said sub sequentially convergent if we have , for every sequence $\{y_n\}$, if $\{Ay_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

1.2.Definition [1] Let X be a non- empty set . Suppose the $A : X \to X$ satisfies the following

(a). $\rho(x,y) \geq 0$ for all $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$.

(b). $\rho(x,y) = \rho(y, x)$, for all $x, y \in X$.

(c). $\rho(x,y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$, for all $x,y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$ [Rectangular Property]. Then ρ is called a generalized metric and (X, ρ) is a generalized metric space.

2. Main Results

In this section, we obtain our main result.

2.1. Theorem. Let (X, ρ) be C-M-Space and A, B: $X \rightarrow X$ be mappings such that A is continuous one to one and sub sequentially convergent.

$$\rho(ABx, ABy) \leq \alpha_1 \rho(Ax, ABx) + \alpha_2 \rho(Ay, ABy) + \alpha_3 \rho(Ax, Ay) . \qquad \ldots \qquad (1)$$

For all $x, y \in X$, where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + 2\alpha_2 + \alpha_3 < 1$.

Then B has a unique fixed point. Also if A is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{B^n x_0\}$ converges to this point.

Proof: let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Bx_n$ (equal to $x_n = B^n x_0$) , n= 1,2,3..… By (1) we have

$$\begin{aligned}
\rho(Ax_n , Ax_{n+1}) &\leq \alpha_1 \rho(ABx_{n-1} , ABx_n) \\
&\leq \alpha_1 \rho(Ax_{n-1}, ABx_{n-1}) + \alpha_2 \rho(Ax_n , ABx_n) + \alpha_3 \rho(Ax_{n-1}, Ax_n) \\
&\leq (\alpha_1 + \alpha_3) \rho(Ax_{n-1}, Ax_n) + \alpha_2 \rho(Ax_{n-1}, Ax_n) + \alpha_2 \rho(Ax_{n+1}, Ax_n) \\
&\leq (\alpha_1 + \alpha_2 + \alpha_3) \rho(Ax_{n-1}, Ax_n) + \alpha_2 \rho(Ax_{n+1}, Ax_n) \\
&\leq (\alpha_1 + \alpha_2 + \alpha_3) / (1- \alpha_2) \rho(Ax_{n-1}, Ax_n) \\
&\leq h \rho(Ax_{n-1}, Ax_n). \qquad \ldots \qquad (3)
\end{aligned}$$

Where, $h = (\alpha_1 + \alpha_2 + \alpha_3) / (1- \alpha_2) < 1$.

Since $\alpha_1 + 2\alpha_2 + \alpha_3 < 1$.

By the same argument

$$\rho(Ax_n, Ax_{n+1}) \leq h \rho(Ax_{n-1} , Ax_n) \leq h^2 \rho(Ax_{n-2} ,Ax_{n-1}) \leq \ldots \leq h^n \rho(Ax_0, Ax_1). \quad \ldots \qquad (4)$$

By (4) for every $m, n \in N$ such that $m > n$ we have

$$\begin{aligned}
\rho(Ax_m , Ax_n) &\leq \rho(Ax_m , Ax_{m-1}) + \rho(Ax_{m-1} , Ax_{m-2}) + \ldots + \rho(Ax_{n+1} , Ax_n) , \\
&\leq [h^{m-1} + h^{m-2} + \ldots + h^n] \rho(Ax_0 , Ax_1) , \\
&\leq [h^n + h^{n+1} + \ldots] \rho(Ax_0 , Ax_1).
\end{aligned}$$

$\leq h^n$ [1/ 1- h] $\rho(Ax_0, Ax_1)$. … (5)

Note that where h < 1.Taking limit in (5) we get that $\rho(Ax_m, Ax_n) \to 0$ as m,n→∞. Therefore $\{Ax_n\}$ is a Cauchy sequence and since X is a complete metric space, there exists p∊X such that

$\lim_{n\to\infty} Ax_n$ = p. … (6)

Since A is a sequentially convergent, $\{x_n\}$ has a convergent subsequence. So there exists q∊X and

$\{x_{n(r)}\}$such that $\lim_{n\to\infty} x_{n(r)}$ = q. Since A is continuous and $\lim_{n\to\infty} Ax_{n(r)}$. By (6) we conclude that Aq = p. So

$$\rho(ABq, Aq) \leq \rho(ABq, AB^{n(r)} x_0) + \rho(AB^{n(r)} x_0, AB^{n(r)+1} x_0) + \rho(AB^{n(r)+1} x_0, Aq),$$

$$\leq \rho(ABq, AB^{n(r)} x_0) + \rho(AB^{n(r)} x_0, AB^{n(r)+1} x_0) + \rho(AB^{n(r)+1} x_0, Aq),$$

$$\leq \rho(ABq, AB(B^{n(r)-1} x_0)) + \rho(AB^{n(r)} x_0, AB^{n(r)+1} x_0) + \rho(AB^{n(r)+1} x_0, Aq),$$

$$\leq \alpha_1\rho(Aq, ABq) + \alpha_2\rho(AB^{n(r)-1} x_0, AB^{n(r)} x_0) + \alpha_3\rho(Aq, AB^{n(r)-1} x_0) +$$

$$\rho(AB^{n(r)}x_0, AB^{n(r)+1} x_0)+\rho(AB^{n(r)+1} x_0, Aq),$$

$$\leq \alpha_1\rho(Aq, ABq)+\alpha_2\rho(AB^{n(r)-1} x_0, AB^{n(r)} x_0) + \alpha_3\rho(Aq, AB^{n(r)-1} x_0)$$

$$+ (h / 1-h)^{n(r)} \rho(AB^{n(r)} x_0, Ax_0) +\rho(AB^{n(r)+1} x_0, Aq),$$

$$\leq \alpha_1\rho(Aq, ABq)+\alpha_2\rho(AB^{n(r)-1} x_0, AB B^{n(r)-1}x_0) + \alpha_3\rho(Aq, AB^{n(r)-1} x_0)$$

$$+ \rho(AB^{n(r)}x_0, AB^{n(r)+1} x_0)+\rho(AB^{n(r)+1} x_0, Aq)$$

$$\leq \alpha_2 /(1-\alpha_1)\rho(Ax_{n(r)-1}, Ax_{n(r)})+ \alpha_3 /(1-\alpha_1)\rho(Aq, Ax_{n(r)-1})$$

$$+ (h / 1-h)^{n(r)} /(1- \alpha_1)\, \rho(Ax_{n(r)}, Ax_0)$$

$$+ 1/(1- \alpha_1)\, \rho(Ax_{n(r)-1}, Aq) \to 0, \text{ as } r \to\infty.$$

Therefore ABq = Aq . Since A is one –to –one Bq = q , so B has a fixed point. Now if A is sequentially convergent , by replacing {n } by {n(r)} , we conclude that

$\lim_{n\to\infty} x_n$ = q and this shows that $\{x_n\}$ converges to the fixed point of B.

2.2.Remark . If we take $\alpha_1 = \alpha_2 = \lambda$ and $\alpha_3 = 0$ in the above Theorem 2.1 , the we get the Theorem 2.1 of [5].

Conclusion: In this paper, our results are more general than the results of [5].

References

[1] A. Azam and M. Arshad, Kannan fixed point theorem on generalized metric spaces, The J. Nonlinear .Sci., no.1(2008) ,45-48.
[2] A. Beiravand, S.Moradi, M.Omidand H. Pazandeh,Two fixed- point theorems for special mappings, arXiv:0903.504v1[math; FA](2009), 1-6.
[3] A. Branciari, A fixed point theorem of Banach - Caccippoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57/1-2 (2000), 31-37.
[4] R. Kannan, Some results on fixed points, Bull. Calcutta Math.Soc.,60 (1968), 71-76.
[5] S. Moradi, kannanfixed point theorem on complete metric spaces and on generalized metric spaces depended an another function, arXiv:0903.1577 v1[math;FA] (2009), 1-6.

[6]K.Prudhvi, Unique Fixed Points on OWC Self-Maps with Generalized Contractive Type Conditions in CMS, American Journal of Applied Mathematics and Statistics, Vol. 11, No.2, (2023), 61-62.